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Bounds for the second Hankel determinant of certain univalent functions

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Abstract

Let A denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic on the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. This paper investigates the analytic functions $f \in A$ for which either $\frac{2zf'(z)}{f(z) - f(-z)}$ or $\frac{(2zf'(z))'}{(f(z) - f(-z))'}$ is subordinate to certain analytic function. In particular, the estimates for the second Hankel determinant $a_2a_4 - a_3^2$ of the two classes are obtained.

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1. Introduction

Let A denote the class of all analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

defined on the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. The qth Hankel determinant of f for $q \geq 1$ and $n \geq 1$ is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \quad (a_1 = 1).$$

In the recent years, several authors have investigated bounds for the Hankel determinant of functions belonging to various subclasses of univalent and multivalent functions. Observe that the well-known Fekete and Szegö functional is the Hankel determinant $H_2(1) = a_3 - a_2^2$. A classical theorem of Fekete and Szegö functional was introduced by Fekete and Szegö as early as 1933, see [3]. Other results related to this functional, see [[1], [2], [5], [8], [9]].

For our discussion, the Hankel determinant for the case $q = 2$ and $n = 2$ is being considered i.e. $H_2(2) = a_2a_4 - a_3^2$. In [6] and [7], Janteng et al. obtained the bounds for the functional $|a_2a_4 - a_3^2|$ for functions belonging to the classes of functions starlike, convex, starlike with respect to symmetric points and convex with respect to symmetric points.

In this paper, we seek the bounds for the functional $|a_2a_4 - a_3^2|$ for functions belonging to the two classes defined by subordination. For two functions f and g, analytic in D, we say that f is subordinate to g in D (written $f \prec g$) if there exists a Schwarz function $w(z)$, analytic in D with $w(0) = 0$ and $|w(z)| < 1$, $z \in D$ such that $f(z) = g(w(z))$ for all $z \in D$. Specially, if g is univalent in D, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(D) \subseteq g(D)$.

Definition 1.1. Let $\varphi\colon D \to C$ be analytic and let the Maclaurin series of φ be given by

$$\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \dots (B_1, B_2 \in R,\ B_1 > 0). \tag{1.2}$$

The class $S_s^*(\varphi)$ consists of functions $f \in A$ satisfying the subordination

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \varphi(z).$$

Definition 1.2. Let $\varphi\colon D \to C$ be analytic and be given as in (1.2). The class $C_s(\varphi)$ consists of functions $f \in A$ satisfying the subordination

$$\frac{2(zf'(z))'}{(f(z) - f(-z))'} \prec \varphi(z).$$

2. Preliminary Results

Let P be the family of all functions p analytic in D for which $Re\ p(z) > 0$ and

$$p(z) = 1 + c_1z + c_2z^2 + \dots \tag{2.1}$$

for $z \in D$.

Lemma 2.1. [10] If $p \in P$ then $|c_k| \le 2$ for each k.

Lemma 2.2. [4] The power series for $p(z)$ given by (2.1) converges in D to a function in P if and only if the Toeplitz determinants

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \dots & c_n \\ c_{-1} & 2 & c_1 & \dots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \dots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \dots \tag{2.2}$$

and $c_{-k} = \bar{c}_k$, are all nonnegative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$ for $k \neq j$; in this case $D_n > 0$ for $n < m - 1$ and $D_n = 0$ for $n \geq m$.

This necessary and sufficient condition is due to Carathéodory and Toeplitz and can be found in [4].

3. Main Results

Theorem 3.1. Let the function $f \in S_s^*(\varphi)$ be given by (1.2).

1. If B_1, B_2 and B_3 satisfy the conditions

   $$B_1^2 + 4|B_2| - 6B_1 \leq 0, \quad |B_1^2 B_2 + 2B_1 B_3 - 4B_2^2| - 4B_1^2 \leq 0$$

   then the second Hankel determinant satisfies

   $$|a_2 a_4 - a_3^2| \leq \frac{B_1^2}{4}.$$

2. If B_1, B_2 and B_3 satisfy the conditions

   $$B_1^2 + 4|B_2| - 6B_1 \geq 0, \quad 2|B_1^2 B_2 + 2B_1 B_3 - 4B_2^2| - B_1^3 - 4B_1|B_2| - 2B_1^2 \geq 0$$

   or the conditions

   $$B_1^2 + 4|B_2| - 6B_1 \leq 0, \quad |B_1^2 B_2 + 2B_1 B_3 - 4B_2^2| - 4B_1^2 \geq 0$$

   then the second Hankel determinant satisfies

   $$|a_2 a_4 - a_3^2| \leq \frac{1}{16}|B_1^2 B_2 + 2B_1 B_3 - 4B_2^2|.$$

3. If B_1, B_2 and B_3 satisfy the conditions

   $$B_1^2 + 4|B_2| - 6B_1 > 0, \quad 2|B_1^2 B_2 + 2B_1 B_3 - 4B_2^2| - B_1^3 - 4B_1|B_2| - 2B_1^2 \leq 0$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{64}\left(\frac{M}{N}\right).$$

where

$$M = 16|B_1^2B_2 + 2B_1B_3 - 4B_2^2| - 4B_1^3 - 16B_1|B_2| \\ - 4B_1^2 - B_1^4 - 8B_1^2|B_2| - 16|B_2|^2$$

and

$$N = |B_1^2B_2 + 2B_1B_3 - 4B_2^2| - B_1^3 - 4B_1|B_2| + 2B_1^2.$$

Proof. Since $f \in S_s^*(\varphi)$, there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in D such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = \varphi(w(z)). \tag{3.1}$$

Define the function p by

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \dots,$$

or equivalently,

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2}\left(c_1z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \dots\right). \tag{3.2}$$

Then p is analytic in D with $p(0) = 1$ and has a positive real part in D. By using (3.2) together with (1.2), it is evident that

$$\varphi\left(\frac{p(z) - 1}{p(z) + 1}\right) = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2 + \dots. \tag{3.3}$$

It follows from (3.1), (3.2) and (3.3) that

$$a_2 = \frac{1}{4}B_1c_1$$
$$a_3 = \frac{1}{4}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{8}B_2c_1^2$$
$$a_4 = \frac{1}{64}\left[\left(2B_1^2 - 8B_1 + 8B_2\right)c_1c_2 + \left(B_1B_2 - B_1^2 + 2B_1 - 4B_2 + 2B_3\right)c_1^3 + 8B_1c_3\right]$$

Therefore

$$a_2a_4 - a_3^2 = \frac{B_1}{256}\Bigg[\left(2B_1^2 - 8B_2 + 8B_1\right)c_1^2c_2 + \left(2B_3 + 4B_2 - B_1^2 + B_1B_2 - 2B_1 - \frac{4B_2^2}{B_1}\right)c_1^4 + 8B_1c_1c_3 - 16B_1c_2^2\Bigg]$$

Let

$$\begin{aligned} d_1 &= 2B_1^2 - 8B_2 + 8B_1, \qquad & d_2 &= 2B_3 + 4B_2 - B_1^2 + B_1B_2 - 2B_1 - \frac{4B_2^2}{B_1}, \\ d_3 &= 8B_1, & d_4 &= -16B_1, \\ T &= \frac{B_1}{256}. \end{aligned} \tag{3.4}$$

Then

$$\left|a_2a_4 - a_3^2\right| = T\left|d_1c_1^2c_2 + d_2c_1^4 + d_3c_1c_3 + d_4c_2^2\right| \tag{3.5}$$

We make use of Lemma 2.2 to obtain the proper bound on (3.5). We begin by rewriting (2.2) for the cases $n = 2$ and $n = 3$,

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ c_1 & 2 & c_1 \\ \bar{c}_2 & c_1 & 2 \end{vmatrix} = 8 + 2Re\left(c_1^2c_2\right) - 2|c_2|^2 - 4c_1^2 \geq 0,$$

which is equivalent to

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{3.6}$$

for some x, $|x| \leq 1$.

Further, $D_3 \geq 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|2c_2 - c_1^2|^2 \tag{3.7}$$

and from (3.7) and (3.6), we have

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{3.8}$$

for some value of z, $|z| \leq 1$.

Suppose $c_1 = c$ and $c \in [0, 2]$. Using (3.6) and (3.8) in (3.5), we obtain

$$\left|a_2a_4 - a_3^2\right| = \frac{T}{4}\Bigg| c^4\left(2d_1 + 4d_2 + d_3 + d_4\right) + 2xc^2\left(4 - c^2\right)\left(d_1 + d_3 + d_4\right) - \left(4 - c^2\right)x^2d_3c^2 + \left(4 - c^2\right)^2x^2d_4 + 2d_3c\left(4 - c^2\right)\left(1 - |x|^2\right)z\Bigg|$$

Replacing $|x|$ by μ and substituting the values of d_1, d_2, d_3 and d_4 from (3.4) yield

$$\begin{aligned}
\left|a_2a_4 - a_3^2\right| \leq \frac{T}{4} \Bigg[& c^4 \left|4B_1B_2 + 8B_3 - 16\frac{B_2^2}{B_1}\right| \\
& + 4\left(B_1^2 + 4|B_2|\right)\mu c^2\left(4-c^2\right) + 8B_1\mu^2c^2\left(4-c^2\right) \\
& + 16B_1\mu^2\left(4-c^2\right)^2 + 16B_1c\left(4-c^2\right)\left(1-\mu^2\right) \Bigg] \\
= T \Bigg[& c^4 \left|B_1B_2 + 2B_3 - 4\frac{B_2^2}{B_1}\right| + 4B_1c\left(4-c^2\right) \\
& + \left(B_1^2 + 4|B_2|\right)\mu c^2\left(4-c^2\right) + 2B_1\mu^2\left(4-c^2\right)(2-c)(c+4) \Bigg] \\
\equiv F & (c, \mu)
\end{aligned} \tag{3.9}$$

Note that for $(c, \mu) \in [0, 2] \times [0, 1]$, differentiating $F(c, \mu)$ in (3.9) partially with respect to μ yields

$$\frac{\partial F}{\partial \mu} = T\left[\left(B_1^2 + 4|B_2|\right)c^2\left(4-c^2\right) + 4B_1\mu\left(4-c^2\right)(2-c)(c+4)\right] \tag{3.10}$$

Then, for $0 < \mu < 1$ and for any fixed c with $0 < c < 2$, it is clear from (3.10) that $\frac{\partial F}{\partial \mu} > 0$, that is, $F(c, \mu)$ occurs at $\mu = 1$ and

$$max\ F(c, \mu) = F(c, 1) \equiv G(c).$$

Now, note that

$$\begin{aligned}
G(c) = \frac{B_1}{256} \Bigg[& c^4 \left(\left|B_1B_2 + 2B_3 - 4\frac{B_2^2}{B_1}\right| - \left(B_1^2 + 4\,|B_2|\right) + 2B_1 \right) \\
& + 4c^2\left(\left(B_1^2 + 4\,|B_2|\right) - 6B_1\right) + 64B_1 \Bigg]
\end{aligned}$$

Let

$$\begin{aligned}
P &= \left|B_1B_2 + 2B_3 - 4\frac{B_2^2}{B_1}\right| - \left(B_1^2 + 4\,|B_2|\right) + 2B_1 \\
Q &= 4\left(\left(B_1^2 + 4\,|B_2|\right) - 6B_1\right) \\
R &= 64B_1
\end{aligned} \tag{3.11}$$

Since

$$\max_{0\le t\le 4}\left(Pt^2+Qt+R\right)=\begin{cases} R, & Q\le 0,\ P\le -\dfrac{Q}{4};\\ 16P+4Q+R, & Q\ge 0,\ P\ge -\dfrac{Q}{8} \text{ or } Q\le 0,\ P\ge -\dfrac{Q}{4};\\ \dfrac{4PR-Q^2}{4P}, & Q>0,\ P\le -\dfrac{Q}{8};\end{cases}$$

we have

$$\left|a_2a_4-a_3^2\right|\le \frac{B_1}{256}\begin{cases} R, & Q\le 0,\ P\le -\dfrac{Q}{4};\\ 16P+4Q+R, & Q\ge 0,\ P\ge -\dfrac{Q}{8} \text{ or } Q\le 0,\ P\ge -\dfrac{Q}{4};\\ \dfrac{4PR-Q^2}{4P}, & Q>0,\ P\le -\dfrac{Q}{8};\end{cases}$$

where P, Q, R are given by (3.11). This completes the proof of theorem. ■

Remark 3.2. When $B_1 = B_2 = B_3 = 2$, Theorem 3.1 reduces to [[7], Theorem 3.1].

Theorem 3.3. Let the function $f \in C_s(\varphi)$ be given by (1.2).

1. If B_1, B_2 and B_3 satisfy the conditions

$$9B_1^2+28|B_2|-46B_1\le 0,\quad |9B_1^2B_2+18B_1B_3-32B_2^2|-32B_1^2\le 0$$

then the second Hankel determinant satisfies

$$|a_2a_4-a_3^2|\le \frac{B_1^2}{36}.$$

2. If B_1, B_2 and B_3 satisfy the conditions

$$9B_1^2+28|B_2|-46B_1\ge 0,\ 2|9B_1^2B_2+18B_1B_3-32B_2^2|-9B_1^3-28B_1|B_2|-18B_1^2\ge 0$$

or the conditions

$$9B_1^2+28|B_2|-46B_1\le 0,\quad |9B_1^2B_2+18B_1B_3-32B_2^2|-32B_1^2\ge 0$$

then the second Hankel determinant satisfies

$$|a_2a_4-a_3^2|\le \frac{1}{1152}|9B_1^2B_2+18B_1B_3-32B_2^2|.$$

3. If B_1, B_2 and B_3 satisfy the conditions

   $$9B_1^2+28|B_2|-46B_1 > 0,\ 2|9B_1^2B_2+18B_1B_3-32B_2^2|-9B_1^3-28B_1|B_2|-18B_1^2 \leq 0$$

   then the second Hankel determinant satisfies then the second Hankel determinant satisfies
   $$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{4608}\left(\frac{U}{V}\right).$$

   where

   $$\begin{aligned} U = &\ 128|9B_1^2B_2 + 18B_1B_3 - 32B_2^2| - 324B_1^3 - 1008B_1|B_2| \\ &- 324B_1^2 - 81B_1^4 - 504B_1^2|B_2| - 784|B_2|^2 \end{aligned}$$

   and
   $$V = |9B_1^2B_2 + 18B_1B_3 - 32B_2^2| - 9B_1^3 - 28B_1|B_2| + 14B_1^2.$$

Proof. Since $f \in C_s(\varphi)$, there exists an analytic function w with $w(0) = 0$ and $|w(z) < 1|$ in D such that
$$\frac{2(zf'(z))'}{(f(z) - f(-z))'} = \varphi(w(z)). \tag{3.12}$$

Since

$$\frac{2(zf'(z))'}{(f(z) - f(-z))'} = 1 + 4a_2z + 6a_3z^2 + (16a_4 - 12a_2a_3)z^3 + \dots, \tag{3.13}$$

equations (3.3), (3.12) and (3.13) yield

$$\begin{aligned} a_2 &= \frac{1}{8}B_1c_1 \\ a_3 &= \frac{1}{24}\left[(B_2 - B_1)c_1^2 + 2B_1c_2\right] \\ a_4 &= \frac{1}{256}\left[\left(2B_1^2 - 8B_1 + 8B_2\right)c_1c_2 + \left(B_1B_2 - B_1^2 + 2B_1 - 4B_2 + 2B_3\right)c_1^3 + 8B_1c_3\right] \end{aligned}$$

Therefore

$$\begin{aligned} a_2a_4 - a_3^2 = \frac{B_1}{2048}\Bigg[&\left(2B_1^2 - \frac{56}{9}B_2 + \frac{56}{9}B_1\right)c_1^2c_2 \\ &+ \Bigg(2B_3 + \frac{28}{9}B_2 - B_1^2 + B_1B_2 - \frac{14}{9}B_1 \\ &- \frac{32}{9}\frac{B_2^2}{B_1} \Bigg) c_1^4 + 8B_1c_1c_3 - \frac{128}{9}B_1c_2^2 \Bigg] \end{aligned}$$

By writing

$$\begin{aligned} d_1 &= 8B_1, & d_2 &= 2B_1^2 - \frac{56}{9}B_2 + \frac{56}{9}B_1, \\ d_3 &= -\frac{128}{9}B_1, & d_4 &= 2B_3 + \frac{28}{9}B_2 - B_1^2 + B_1B_2 - \frac{14}{9}B_1 - \frac{32}{9}\frac{B_2^2}{B_1}, \\ T &= \frac{B_1}{2048}, \end{aligned} \tag{3.14}$$

we have

$$\left|a_2a_4 - a_3^2\right| = T\left|d_2c_1^2c_{2+}d_4c_1^4 + d_1c_1c_3 + d_3c_2^2\right| \tag{3.15}$$

Similar as in Theorem 3.1, it follows from (3.6) and (3.8) that

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| = \frac{T}{4}\Bigg| & c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2\left(4 - c^2\right)(d_1 + d_2 + d_3) \\ & - \left(4 - c^2\right)x^2d_1c^2 + \left(4 - c^2\right)^2 x^2d_3 + 2d_1c\left(4 - c^2\right)\left(1 - |x|^2\right)z\Bigg| \end{aligned}$$

Replacing $|x|$ by μ and substituting the values of d_1, d_2, d_3 and d_4 from (3.14) yield

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| &\le \frac{T}{4}\Bigg[c^4\left|4B_1B_2 + 8B_3 - \frac{128}{9}\frac{B_2^2}{B_1}\right| + 2\mu c^2\left(4 - c^2\right)\left(2B_1^2 + \frac{56}{9}|B_2|\right) \\ &\quad + \mu^2\left(4 - c^2\right)\left(-\frac{56}{9}B_1c^2 + \frac{512}{9}B_1\right) + 16B_1c\left(4 - c^2\right)\left(1 - \mu^2\right)\Bigg] \\ &= T\Bigg[c^4\left|B_1B_2 + 2B_3 - \frac{32}{9}\frac{B_2^2}{B_1}\right| + 4B_1c\left(4 - c^2\right) \\ &\quad + \frac{1}{2}\mu c^2\left(4 - c^2\right)\left(2B_1^2 + \frac{56}{9}|B_2|\right) + \frac{1}{9}B_1\mu^2\left(4 - c^2\right)(2 - c)(14c + 64)\Bigg] \\ &\equiv F(c, \mu) \end{aligned} \tag{3.16}$$

Again, differentiating $F(c, \mu)$ in (3.16) partially with respect to μ yields

$$\frac{\partial F}{\partial \mu} = T\left[\left(2B_1^2 + \frac{56}{9}|B_2|\right)\frac{c^2}{2}\left(4 - c^2\right) + \frac{2}{9}B_1\mu\left(4 - c^2\right)(2 - c)(14c + 64)\right] \tag{3.17}$$

It is clear from (3.17) that $\frac{\partial F}{\partial \mu} > 0$. Thus $F(c, \mu)$ is an increasing function of μ for $0 < \mu < 1$ and for any fixed c with $0 < c < 2$. So the maximum of $F(c, \mu)$ occurs at $\mu = 1$ and

$$max\ F(c, \mu) = F(c, 1) \equiv G(c).$$

Now, note that

$$G(c) = T\left[\frac{c^4}{9}\left(\left|9B_1B_2 + 18B_3 - 32\frac{B_2^2}{B_1}\right| - 9B_1^2 - 28|B_2| + 14B_1\right)\right.$$
$$\left. + \frac{4}{9}c^2\left(9B_1^2 + 28|B_2| - 46B_1\right) + \frac{512}{9}B_1\right]$$

Let

$$\begin{aligned} P &= \frac{1}{9}\left(\left|9B_1B_2 + 18B_3 - 32\frac{B_2^2}{B_1}\right| - 9B_1^2 - 28|B_2| + 14B_1\right) \\ Q &= \frac{4}{9}\left(9B_1^2 + 28|B_2| - 46B_1\right) \\ R &= \frac{512}{9}B_1 \end{aligned} \tag{3.18}$$

Since

$$\max_{0\le t\le 4}\left(Pt^2 + Qt + R\right) = \begin{cases} R, & Q \le 0,\, P \le -\dfrac{Q}{4}; \\ 16P + 4Q + R, & Q \ge 0,\, P \ge -\dfrac{Q}{8} \text{ or } Q \le 0,\, P \ge -\dfrac{Q}{4}; \\ \dfrac{4PR - Q^2}{4P}, & Q > 0,\, P \le -\dfrac{Q}{8}; \end{cases}$$

we have

$$\left|a_2a_4 - a_3^2\right| \le \frac{B_1}{2048}\begin{cases} R, & Q \le 0,\, P \le -\dfrac{Q}{4}; \\ 16P + 4Q + R, & Q \ge 0,\, P \ge -\dfrac{Q}{8} \text{ or } Q \le 0,\, P \ge -\dfrac{Q}{4}; \\ \dfrac{4PR - Q^2}{4P}, & Q > 0,\, P \le -\dfrac{Q}{8}; \end{cases}$$

where P, Q, R are given by (3.18). This completes the proof of theorem. ■

Remark 3.4. For the choice of $\varphi(z) = \dfrac{1+z}{1-z}$, Theorem 3.3 reduces to [[7], Theorem 3.2].

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