Modified Variational Iteration Method for Solving Fourth Order Parabolic PDEs With Variable Coefficients

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Abstract

In this paper, we apply a new Modified Variational Iteration Method (MVIM) to solve one dimensional fourth order parabolic linear partial differential equations with variable coefficients. This method is a combination of the two initial conditions.

Keywords: Modified Variational Iteration Method, linear partial differential equation, exact solution.

Introduction

Many problems of physical interest are described by linear partial differential equations with initial and boundary conditions. One of them is fourth order parabolic partial differential equations with variable coefficients; these equations emerge on the transverse vibration issue [1]. In recent years, many research workers have paid attention to find the solution of these equations by using various methods. Among these are the variational iteration method [Biazar and Ghazvini (2007)], Adomian decomposition method [Wazwaz (2001) and Biazar etal (2007)], homotopy perturbation method [Mehdi Dehghan and Jalil Manafian (2008)], homotopy analysis method [Najeeb Alam Khan (2010)] and Laplace decomposition algorithm [Majid Khan, Muhammad Asif Gondal and Yasir Khan (2011)]. In this paper, we use the Modified Variational Iteration Method. This method is a useful technique for solving linear and nonlinear differential equations. The principle point of this paper is to integrate initial conditions for solving higher order linear partial differential equations with variable coefficients. This method gives the solution as convergent series leads to the exact solution.

Modified Variational Iteration Method:
Consider a one dimensional linear, non-homogeneous fourth order parabolic partial differential equation with variable coefficients of the form,

$$\frac{\partial^2 u}{\partial t^2} + \psi(x)\frac{\partial^4 u}{\partial x^4} = \phi(x,t), \tag{1}$$

Where, $\psi(x)$ is a variable coefficient, with the following initial conditions,

$$u(x,0) = f(x), \text{ and } \frac{\partial u}{\partial t}(x,0) = h(x), \tag{2}$$

and the boundary conditions are,

$$u(a,t) = \beta_1(t), u(b,t) = \beta_2(t), \frac{\partial^4 u}{\partial x^4}(a,t) = \beta_3(t), \frac{\partial^4 u}{\partial x^4}(b,t) = \beta_4(t), \tag{3}$$

Apply modified variational iteration method of Eq. (1),

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} + \psi(x)\frac{\partial^4 \tilde{u}_n}{\partial x^4}(x,\xi) - \phi(x,\xi)\right]d\xi, \tag{4}$$

Where λ is a Lagrange multiplier $(\lambda = \xi - t)$, the subscripts n denote the n th approximation, $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta \tilde{u}_n = 0$. Equation (4) is called a correction functional.

The successive approximation u_{n+1} of the solution u will be readily obtained by using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by,

$$u = \lim_{n\to\infty} u_n$$

Applications
To show that the method is effected, we have solved homogeneous and non-homogeneous one dimensional fourth order linear partial differential equations with the initial and boundary condition.

Example 1.
Consider the fourth order homogenous partial differential equations,

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 u}{\partial t^4} = 0, \quad \frac{1}{2} < x < 1, t > 0 \tag{5}$$

With the following initial conditions,

$$u(x,0) = 0, \frac{\partial u}{\partial t}(x,0) = 1 + \frac{x^5}{120}, \tag{6}$$

and the boundary conditions,

$$u(0.5,t) = \left(1 + \frac{(0.5)^5}{120}\right)\sin t, u(1,t) = \frac{121}{120}\sin t,$$

$$\frac{\partial^2 u}{\partial t^2}(0.5,t) = 0.02084\sin t, \frac{\partial^2 u}{\partial t^2}(1,t) = \frac{1}{6}\sin t.$$

(7)

Appling Modified Variational Iteration Method to Eq.(5), we get:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 \tilde{u}_n}{\partial x^4}(x,\xi)\right]d\xi,$$

(8)

take $\lambda = \xi - t$, then:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 u_n}{\partial x^4}(x,\xi)\right]d\xi,$$

take, $u_0(x,t) = u(x,0) = \left(1 + \frac{x^5}{120}\right)t$, then:

$$u_1(x,t) = \left(1 + \frac{x^5}{120}\right)t + \int_0^t (\xi - t)\left[\left(\frac{1}{x} - \frac{x^4}{120}\right)x\xi\right]d\xi$$

$$= \left(1 + \frac{x^5}{120}\right)t - \left(1 + \frac{x^5}{120}\right)\frac{t^3}{3!}$$

$$u_2(x,t) = \left(1 + \frac{x^5}{120}\right)t - \left(1 + \frac{x^5}{120}\right)\frac{t^3}{3!} + \int_0^t (\xi - t)\left[-\left(1 + \frac{x^5}{120}\right)t + \left(\frac{1}{x} - \frac{x^4}{120}\right)\left(x\xi - x\frac{\xi^3}{3!}\right)\right]d\xi$$

$$= \left(1 + \frac{x^5}{120}\right)t - \left(1 + \frac{x^5}{120}\right)\frac{t^3}{3!} + \left(1 + \frac{x^5}{120}\right)\frac{t^5}{5!}$$

$$u_3(x,t) = \left(1 + \frac{x^5}{120}\right)t - \left(1 + \frac{x^5}{120}\right)\frac{t^3}{3!} + \left(1 + \frac{x^5}{120}\right)\frac{t^5}{5!} +$$

$$\int_0^t (\xi - t)\left[\left(1 + \frac{x^5}{120}\right)\left(-t + \frac{t^3}{6}\right) + \left(\frac{1}{x} - \frac{x^4}{120}\right)\left(x\xi - x\frac{\xi^3}{3!} + x\frac{\xi^5}{5!}\right)\right]d\xi$$

$$= \left(1 + \frac{x^5}{120}\right)t - \left(1 + \frac{x^5}{120}\right)\frac{t^3}{3!} + \left(1 + \frac{x^5}{120}\right)\frac{t^5}{5!} - \left(1 + \frac{x^5}{120}\right)\frac{t^7}{7!}$$

:

$$u_n(x,t) = \left(1 + \frac{x^5}{120}\right)t - \left(1 + \frac{x^5}{120}\right)\frac{t^3}{3!} + \left(1 + \frac{x^5}{120}\right)\frac{t^5}{5!} - \left(1 + \frac{x^5}{120}\right)\frac{t^7}{7!} + \ldots$$

$$= \left(1 + \frac{x^5}{120}\right)\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots\right)$$

The exact solution is:

$$u(x,t) = \left(1 + \frac{x^5}{120}\right)\sin t \tag{9}$$

Example 2.

Consider fourth order homogenous partial differential equation,

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{x}{\sin x} - 1\right)\frac{\partial^4 u}{\partial x^4} = 0, \quad 0 < x < 1, t > 0 \tag{10}$$

with the following initial conditions,

$$u(x,0) = x - \sin x, \frac{\partial u}{\partial t}(x,0) = -x + \sin x \tag{11}$$

And the boundary conditions,

$$u(0,t) = 0, u(1,t) = e^{-t}(1 - \sin 1),$$

$$\frac{\partial^2 u}{\partial t^2}(0,t) = 0, \frac{\partial^2 u}{\partial t^2}(1,t) = e^{-t}\sin 1. \tag{12}$$

Solution:

Appling Modified Variational Iteration Method to Eq.(10), we get:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} + \left(\frac{x}{\sin x} - 1\right)\frac{\partial^4 \tilde{u}_n(x,\xi)}{\partial x^4}\right]d\xi$$

take, $\lambda = \xi - t$, then:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} + \left(\frac{x}{\sin x} - 1\right)\frac{\partial^4 u_n(x,\xi)}{\partial x^4}\right]d\xi$$

Using initial conditions from Eq. (11), we get:

$$u_0(x,t) = (x - \sin x) + (-x + \sin x)t$$

$$u_1(x,t) = u_0(x,t) + \int_0^t (\xi - t)\left[\frac{\partial^2 u_0(x,\xi)}{\partial \xi^2} + \left(\frac{x}{\sin x} - 1\right)\frac{\partial^4 u_0(x,\xi)}{\partial x^4}\right]d\xi$$

$$= (x - \sin x) + (-x + \sin x)t + (x - \sin x)\left(\frac{t^2}{2!} - \frac{t^3}{3!}\right)$$

$$u_2(x,t) = u_1(x,t) + \int_0^t (\xi - t)\left[\frac{\partial^2 u_1(x,\xi)}{\partial \xi^2} + \left(\frac{x}{\sin x} - 1\right)\frac{\partial^4 u_1(x,\xi)}{\partial x^4}\right]d\xi$$

$$= (x - \sin x) + (-x + \sin x)t + (x - \sin x)\left(\frac{t^2}{2!} - \frac{t^3}{3!}\right) + (x - \sin x)\left(\frac{t^4}{4!} - \frac{t^5}{5!}\right)$$

$$\vdots$$

$$u_n(x,t) = (x - \sin x) + (-x + \sin x)t + (x - \sin x)\left(\frac{t^2}{2!} - \frac{t^3}{3!}\right) + (x - \sin x)\left(\frac{t^4}{4!} - \frac{t^5}{5!}\right) + \dots$$

$$= (x - \sin x)\left[1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \dots\right]$$

The exact solution:

$$u(x,t) = (x - \sin x)e^{-t}$$

Example 3.

Consider fourth order homogenous partial differential equation,

$$\frac{\partial^2 u}{\partial t^2} + (1+x)\frac{\partial^4 u}{\partial x^4} = \left(x^4 + x^3 - \frac{6}{7!}x^7\right)\cos t, \;\; 0 < x < 1, t > 0 \tag{13}$$

with the following initial conditions,

$$u(x,0) = \frac{6}{7!}x^7, \frac{\partial u}{\partial t}(x,0) = 0. \tag{14}$$

And the boundary conditions,

$$u(0,t) = 0, u(1,t) = \frac{6}{7!}\cos t,$$

$$\frac{\partial^2 u}{\partial t^2}(0,t) = 0, \frac{\partial^2 u}{\partial t^2}(1,t) = \frac{1}{20}\cos t. \tag{15}$$

Solution:

Appling Modified Variational Iteration Method to Eq. (13), to find:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} + (1+x)\frac{\partial^4 \tilde{u}_n(x,\xi)}{\partial x^4} - \left(x^4 + x^3 - \frac{6}{7!}x^7\right)\cos\xi\right]d\xi$$

take, $\lambda = \xi - t$, then:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} + (1+x)\frac{\partial^4 u_n(x,\xi)}{\partial x^4} - \left(x^4 + x^3 - \frac{6}{7!}x^7\right)\cos\xi\right]d\xi$$

Using initial conditions from Eq. (14), to get:

$$u_0(x,t) = \frac{6}{7!}x^7,$$

$$u_1(x,t) = u_0(x,t) + \int_0^t (\xi - t)\left[\frac{\partial^2 u_0(x,\xi)}{\partial \xi^2} + (1+x)\frac{\partial^4 u_0(x,\xi)}{\partial x^4} - \left(x^4 + x^3 - \frac{6}{7!}x^7\right)\cos\xi\right]d\xi$$

$$= \frac{6}{7!}x^7\cos t - (x^4 + x^3)\frac{t^2}{2!} + \left(x^4 + x^3\right)(1 - \cos t)$$

$$= \frac{6}{7!}x^7\cos t + \left(x^4 + x^3\right)(1 - \cos t) + noiseterms$$

$$u_2(x,t) = u_1(x,t) + \int_0^t (\xi - t)\left[\frac{\partial^2 u_1(x,\xi)}{\partial \xi^2} + (1+x)\frac{\partial^4 u_1(x,\xi)}{\partial x^4} - \left(x^4 + x^3 - \frac{6}{7!}x^7\right)\cos\xi\right]d\xi$$

$$= \frac{6}{7!}x^7\cos t - (1+x)\frac{t^2}{2!}$$

$$= \frac{6}{7!}x^7\cos t + noiseterms$$

:

$$u_n(x,t) = \frac{6}{7!}x^7\cos t$$

This is the approximate solution.

Conclusion

In this paper, we applied modified variational iteration method for solving one dimensional fourth order homogenous and non-homogenous linear partial differential equations with variable coefficients. The modified variational iteration method is successfully implemented by using the initial conditions only. And give the exact or approximate solution to the equation.

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