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Application of Contraction Mapping in Menger Spaces

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Abstract

In this paper we shall establish some coincidence theorems on an arbitrary set with values in generalized Menger space and derive fixed-point theorems for mappings commuting only at coincidence point. The results of this paper is an application of well-known results of Piyush Tripathi, *et al*[8].

Introduction

In 1932, Menger [127] generalized the metric axioms by associating a distribution function with each pair of points of an abstract set X. (A distribution functions is a mapping $f : R \to R^+$ which is non-decreasing, left continuous, with inf f = 0 and sup f = 1). Thus for any ordered pair of points p, q of X, we associate a distribution function denoted by $F_{p,q}$ and, for any positive number x, we interpret $F_{p,q}(x)$ as the probability that the distance between p and q is less than x. This gives rise to a new theory of 'probabilistic metric spaces' which started developing rapidly after the publication of the paper of Schweizer and Sklar [177].

For the further basic works in this direction, refer to Constantin and Istrătescu [43], Schweizer [172]-[175], Schweitzer *et al*. [176].

Probabilistic Metric Spaces [8]

Definition 2.1. A mapping $f : R \to R^+$ is called a distribution function if it is non decreasing, left continuous and inf $f(x) = 0$, sup $f(x) = 1$.

We shall denote by L the set of all distribution functions. The specific distribution function $H \in L$ is defined by

$$H(x) = \left.\begin{aligned} &0, \quad x \le 0 \\ &1, \quad x > 0 \end{aligned}\right\}$$

Definition 2.2. A probabilistic metric space (PM space) is an ordered pair (*X,F*), *X* is a nonempty set and $F : X \times X \to L$ is mapping such that, by denoting $F(p,q)$ by $F_{p,q}$ for all *p*, *q* in *X*, we have

(I) $F_{p,q}(x) = 1 \ \forall\ x > 0$ iff $p = q$

(II) $F_{p,q}(0) = 0$

(III) $F_{p,q} = F_{q,p}$

(IV) $F_{p,q}(x) = 1$, $F_{q,r}(y) = 1 \Rightarrow F_{p,r}(x + y) = 1$.

We note that $F_{p,q}(x)$ is value of the distribution function $F_{p,q} = F(p,q) \in L$ at $x \in R$.

Definition 2.3. A mapping $t : [0,1] \times [0,1] \to [0,1]$ is called t-norm if it is non-decreasing (by non-decreasing, we mean $a \le c, b \le d \Rightarrow t(a,b) \le t(c,d)$), commutative, associative and $t(a,1) = a$ for all *a* in [0, 1], $t(0,0) = 0$.

Definition 2.4. A Menger PM space is a triple (*X*, *F*; *t*) where (*X*, *F*) is a PM space and *t* is t-norm such that,

$$F_{p,r}(x + y) \ge t\left(F_{p,q}(x), F_{q,r}(y)\right) \ \forall\ x, y \ge 0 .$$

If (*X*, *F*; *t*) is Menger Probabilistic metric space with $\sup t(x,x) = 1,\ 0 < x < 1$, then (*X, F; t*) is a Hausdorff topological space in the topology *T* induced by the family of (ε, λ) neighborhoods $\{U_p(\varepsilon, \lambda) : p \in X, \in > 0, \lambda > 0\}$ where $U_p(\varepsilon, \lambda) = \{x \in X : F_{x,p}(\varepsilon) > 1 - \lambda\}$ ([8]).

Singh and Jain [191] defined a class of functions Φ of all real continuous functions $\phi : [0,1]^4 \to R$, (where *R* is the set of real numbers) with the property,

(*i*) for $u, v \ge 0$, $\phi(u,v,v,u) \ge 0$ or $\phi(u,v,u,v) \ge 0$ implies $u \ge v$.

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(*ii*) $\phi(u,v,1,1) \ge 0$ implies $u \ge 1$.

(*ii*) $\phi(u,v,1,1) \ge 0$ implies $u \ge 1$.

In 2010 Piyush Tripathi and Manisha Gupta [ ] Proved the following theorems.

Theorem 3.1. Let (*X*, *F*; *T*) be a generalized Menger space under a continuous t-norm *T* in (*a*, 1) $\forall a \in (0,1)$. Suppose $k \in (0,1)$, $\phi \in \Phi$ and $f, g : Y \to X$ are mappings such that,

(i) $\phi\left(F_{fp,fq}(kx), F_{gp,gq}(x), F_{fp,gp}(x), F_{fq,gq}(kx)\right) \geq 0 \ \ \forall p, q \in Y, \forall x > 0,$

(ii) $f(Y) \subset g(Y),$

and (iii) $\exists p_0, p_1$ in Y such that $fp_0 = gp_1$ and $\lim_{n\to\infty} T_{i=n}^{\infty} F_{fp_0, fp_1}(r^i) = 1,$ for $r > 1$.
Then *f* and g have a coincidence point.

Theorem 3.2: Let (*X*, *F*; *T*) be a generalized Menger space under a continuous t-norm T in (*a*, 1) $\forall a \in (0,1)$. Suppose $k \in (0,1)$, $\phi \in \Phi$ and $f, g : X \to X$ are mappings such that,

(i) $\phi\left(F_{fp,fq}(kx), F_{gp,gq}(x), F_{fp,gp}(x), F_{fq,gq}(kx)\right) \geq 0 \ \ \forall p, q \in Y, \forall x > 0,$

(ii) $f(X) \subset g(X),$

(iii) $\exists p_0, p_1$ such that $fp_0 = gp_1$ and $\lim_{n\to\infty} T_{i=n}^{\infty} F_{fp_0, fp_1}(r^i) = 1,$ for $r > 1,$

(iv) Either $f(X)$ or $g(X)$ is $F-$ complete,

and (v) f and g are commuting at their coincidence point.
Then *f* and g have a unique common fixed point.

Corollary 3.1. Let (*X*, *F*; *T*) be a generalized Menger space under a continuous t-norm $T \in H$. Suppose $k \in (0,1)$, $\phi \in \Phi$ and $f, g : X \to X$ are mappings such that,

(i) $\phi\left(F_{fp,fq}(kx), F_{gp,gq}(x), F_{fp,gp}(x), F_{fq,gq}(kx)\right) \geq 0 \ \ \forall p, q \in X, \forall x > 0,$

(ii) $f(X) \subset g(X),$

(iii) $\exists p_0, p_1$ such that $fp_0 = gp_1$ for which $F_{fp_0, fp_1} \in D_+,$

(iv) Either $f(X)$ or $g(X)$ is $F-$ complete,

and (v) f and g are commuting at their coincidence point.
Then *f* and g have coincidence point as well as unique fixed point.

Application

Now as an application of theorem 3.1, in this section we prove coincidence and common fixed point theorems for three mappings.

Theorem 4.1. Let (X, F; T) be a generalized Menger space under a continuous t-norm T in (a, 1) $\forall a \in (0,1)$ and Y an arbitrary set. Suppose $k \in (0,1)$, $\phi \in \Phi$ and $f, g, h, : Y \to X$ are mappings such that,

$(i)\ \phi\left(F_{fp,gq}(kx), F_{hp,hq}(x), F_{fp,hp}(x), F_{gq,hq}(kx)\right) \geq 0 \ \ \forall p, q \in Y, \forall x > 0,$

$(ii)\ f(Y) \cup g(Y) \subset h(Y),$

$(iii)\ \exists p_1, p_2$ and $r > 1$ for which $T_{i=1}^{\infty} F_{hp_1, hp_2}(r^i) = 1,$

and (iv) One of $f(Y)$, $g(Y), h(Y)$ is $F-$ complete.

Then *f*, g and *h* have coincidence point.

Proof. For $p_0 \in Y$ there exist $p_1, p_2 \in Y$ such that $fp_0 = hp_1, gp_1 = hp_2$ (because $f(Y) \cup g(Y) \subset h(Y)$). Inductively we can construct a sequence $\{p_n\}$ such that $fp_{2n} = hp_{2n+1}$, $gp_{2n+1} = hp_{2n+2}$.

Putting $p = p_{2n}$ and $q = p_{2n+1}$ in (i), we have,

$$\phi\left(F_{fp_{2n}, gp_{2n+1}}(kx), F_{hp_{2n}, hp_{2n+1}}(x), F_{fp_{2n}, hp_{2n}}(x) F_{gp_{2n+1}, hp_{2n+1}}(kx)\right) \geq 0,$$

i.e. $\phi\left(F_{hp_{2n+1}, hp_{2n+2}}(kx), F_{hp_{2n}, hp_{2n+1}}(x), F_{hp_{2n+1}, hp_{2n}}(x), F_{hp_{2n+2}, hp_{2n+1}}(kx)\right) \geq 0.$

From the property of ϕ, we have,

$F_{hp_{2n+1}, hp_{2n+2}}(kx) \geq F_{hp_{2n}, hp_{2n+1}}(x)$, $\forall x > 0$.

Again putting $p = p_{2n+2}$ and $q = p_{2n+1}$ in (i), we get,

$F_{hp_{2n+3}, hp_{2n+2}}(kx) \geq F_{hp_{2n+2}, hp_{2n+1}}(x)$, $\forall x > 0$.

Therefore by Lemma 2.1 $\{hp_n\}$ is a Cauchy sequence. Suppose $h(Y)$ is F – complete. Then $\{hp_n\} \to p \in h(Y)$, also then there exists $u \in Y$ such that $hu = p$.

Putting $p = u, q = p_{2n+1}$ in (i), we get,

$$\phi\left(F_{fu, gp_{2n+1}}(kx), F_{hu, hp_{2n+1}}(x), F_{fu, hu}(x) F_{gp_{2n+1}, hp_{2n+1}}(kx)\right) \geq 0,$$

i.e. $\phi\left(F_{fu,p}(kx), F_{p,p}(x), F_{fu,hu}(x), F_{p,p}(kx)\right) \geq 0.$

Again from the property of ϕ, we have, $fu = p = hu$.
Lastly, putting $q = u, p = p_{2n+1}$ in (i), we obtain

$$\phi\left(F_{fp_{2n+2},gu}(kx), F_{hp_{2n+2},hu}(x), F_{fp_{2n+2},hp_{2n+2}}(x)F_{gu,hu}(kx)\right) \geq 0,$$

i.e. $$\phi\left(F_{p,gu}(kx), F_{p,p}(x), F_{p,p}(x), F_{p,p}(kx)\right) \geq 0.$$

Hence as above we have, $gu = p = hu = fu$. Therefore p is the coincidence point of f, g and h.

Theorem 4.2. Let (X, F; T) be a generalized Menger space under a continuous t-norm T in (a, 1) $\forall a \in (0,1)$. Suppose $k \in (0,1)$, $\phi \in \Phi$ and $f, g, h,: X \to X$ are mappings such that,

(i) $\phi\left(F_{fp,gq}(kx), F_{hp,hq}(x), F_{fp,hp}(x), F_{gq,hq}(kx)\right) \geq 0 \quad \forall p, q \in X, \forall x > 0,$

(ii) $f(X) \cup g(X) \subset h(X),$

(iii) $\exists p_1, p_2$ and $r > 1$ for which $T_{i=1}^{\infty} F_{hp_1,hp_2}(r^i) = 1,$

(iv) If one of $f(X)$, $g(X), h(X)$ is $F-$ complete,

and (v) f and h are coincidently commuting.
Then f, g and h have a unique fixed point.

Proof. In the Theorem 4.1 if we take $Y = X$ then we get $gu = p = hu = fu$. Since f and g are coincidently commuting, hence $fhu = hfu \Rightarrow fp = hp$.
Putting $p = fu, q = p_{2n+1}$ in (i), we have

$$\phi\left(F_{ffu,gp_{2n+1}}(kx), F_{hu,hp_{2n+1}}(x), F_{fu,hu}(x)F_{ghp_{2n+1},hhp_{2n+1}}(kx)\right) \geq 0,$$

i.e. $$\phi\left(F_{fp,p}(kx), F_{hp,p}(x), F_{fp,hp}(x), F_{p,p}(kx)\right) \geq 0.$$

Hence from the property of ϕ we have, $fp = p = hp$.
Again putting $q = p, p = p_{2n}$ in (i), we have,

$$\phi\left(F_{fp_{2n},gp}(kx), F_{hp_{2n},hp}(x), F_{fp_{2n},hp_{2n}}(x)F_{gp,hp}(kx)\right) \geq 0,$$

i.e. $$\phi\left(F_{p,gp}(kx), F_{p,p}(x), F_{p,p}(x), F_{gp,p}(kx)\right) \geq 0.$$

Using the property of ϕ, we have, $gp = p = fp = hp$. Therefore *p* is a common fixed point of *f*, g and *h*.

For uniqueness suppose p' and q' are common fixed point of *f*, g and *h*. Then by putting $p = p'$ and $q = q'$ in (*i*), and using the property of ϕ, we have $p' = q'$. Therefore *f*, g and *h* have unique common fixed point.

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