Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 2 (2016), pp. 1635-1646

http://www.ripublication.com

Implementing an Order Six Implicit Block Multistep Method for Third Order ODEs Using Variable Step Size Approach

Jimevwo G. Oghonyon[1*], Nicholas A. Omoregbe[2], Sheila A. Bishop[1]

[1]*Department of Mathematics, College of Science and Technology, Covenant University, Ota, Ogun State, Nigeria.*

[2]*Department of Computer and Information Sciences, College of Science and Technology, Covenant University, Ota, Ogun State, Nigeria.*

Abstract

In this paper, an order six implicit block multistep method is implemented for third order ordinary differential equations using variable step size approach. The idea which originated from Milne's possess a number of computational vantages when equated with existing methods. They include; designing a suitable step size/changing the step size, convergence criteria (tolerance level) and error control/minimization. The approach employs the estimates of the principal local truncation error on a pair of explicit and implicit of Adams type formulas which are implemented in $P(CE)^m$ mode. Gauss Seidel method is adopted for the execution of the suggested method. Numerical examples are given to examine the efficiency of the method and will be compared with subsisting methods.

Keywords and phrase: Variable step size approach• Implicit block multistep method • convergence criteria (tolerance level)• Gauss Seidel Method • Principal Local Truncation Error• Adams type formulas

1. Introduction

Consider the initial value problem of the form

$$y'''(x) = f(x, y, y', y''), \quad y(a) = \alpha, \quad y'(a) = \beta, \quad y''(a) = \psi, \quad x \in [a,b] \quad \text{and} \quad f : R \times R^m \to R^m \quad . (1)$$

The solution to (1) is generally, written as

$$\sum_{i=1}^{j}\alpha_i y_{n+i} = h^3 \sum_{i=1}^{j}\beta_i f_{n+i} \quad (2)$$

where the step size is h, $\alpha_j = 1, \alpha_i, \ i = 1,...j, \beta_j$ are unknown constants which are uniquely specified such that the formula is of order j as discussed in [1].
We assume that $f \in R$ is sufficiently differentiable on $x \in [a,b]$ and satisfies a global Lipchitz condition, i.e., there is a constant $L \geq 0$ such that

$$\left| f(x,y) - f\left(x,\bar{y}\right)\right| \leq L\left|y-\bar{y}\right|, \ \forall y, \bar{y} \in R.$$

Under this presumption, equation (1) assured the existence and uniqueness defined on $x \in [a,b]$ as discussed in [11, 20].

Where a and b are finite and $y^{(i)}[y_1^{(i)}, y_2^{(i)}, \ldots, y_n^{(i)}]^T$ for $i = 0(1)3$ and $f = [f_1, f_1, \ldots, f_n]^T$, originate in real life applications for problems in science and engineering such as fluid dynamics and motion of rocket as presented by [15].Researchers have proposed that the reduction of (1) to the system of first-order equationwill lead to computationalburden and wastage of human effort (see[2 3, 4, 13, 15, 18]).
This paper presents optional method to design directly solution of (1) founded on variable step size approach. This approach has many computational advantages as stated in the abstract. Several authors in recent past suggested direct method to solve (1) such as [2, 3, 4, 13, 15, 18, 21] just to mention a few have applied and discovered a more appropriate method for solving (1) directly. [15] implemented directly variable step size block multistep method for solving general third order ODEs. The method which combined a pair of predictor and corrector of Adams type formula implemented in $PE(CE)^m$ mode. [13] developed two-point four step direct implicit block method in simple form of Adams-Moulton method for solving directly the general third order (ODEs) applying variable step size. [2]implemented block algorithm for the solution of general third order initial value problems of ODEs via the method of interpolation and collocation of the power series as the approximate solution. [4] constructed a five step P-stable method for the numerical integration of third order ODEs using fixed step size approach. [18] developed an accurate implicit block method for numerically integrating third order ODEs based on the idea of interpolation and collocation of power series approximate solution.

Definition 1.1 According to [1]. A block-by-block method is a method for computing vectors $Y_0, Y_1, ...$ in sequence. Let the r-vector (r is the number of points within the block) $Y_\mu, F_\mu,$ and G_μ, for n=mr, m=0, 1,. . . be given as $Y_w = \left(y_{n+1}, \ldots, y_{n+r}\right)^T, F_w = \left(f_{n+1}, \ldots, f_{n+r}\right)^T$, then the l-block r-point methods for (1) are given by

$$Y_w = \sum_{i=1}^{j} A^{(i)} Y_{w-i} + h\sum_{i=1}^{j} B^{(i)} F_{w-i}$$

where $A^{(i)}$, $B^{(i)}$, $i = 0,..., j$ are r by r matrices as introduced by [8].
Thus, from the above definition, a block method has the advantage that in each application, the solution is approximated at more than one point simultaneously. The number of points depends on the structure of the block method. Therefore applying these methods can give quicker and faster solutions to the problem which can be managed to produce a desired accuracy. See [12, 14]. The main purpose of this paper is to propose an order six implicit block multistep method for solving directly (1) by implementing an order six implicit block multistep method for third order ODEs applying variable step size method. This approach possess some computational advantages like designing a suitable step size/changing the step size, stating the convergence criteria (tolerance level) and error control/minimization and help addressed the gaps stated above.
The block algorithm proposed in this paper is based on interpolation and collocation. The continuous representation of the algorithm generates a main discrete collocation method to render the approximate solution Y_{n+i} to the solution of (1) at points $x_{n+i}, i = 1,...k$ as discussed in [1].
The residual of this paper is discussed as follows: in Section 2 the introductory idea behind the computational method is discussed and a continuous representation $Y(x)$ for the exact solution $y(x)$ which is used to generate a main discrete block method for solving (1) is derived. In Section 3 the order of accuracy of the method is introduced. In Section 4 the stability regions of the order six implicit block multistep method is discussed. In Section 5 we show the accuracy of the method. In conclusion, Section 6 presents some final remarks as seen in [1].

2. Formulation of the Method

Following [1, 18] in this section, the main aim is to derive the principal implicit block method of the form (2). We proceed forward by seeking an approximation of the exact solution $y(x)$ by assuming a continuous solution $Y(x)$ of the form

$$Y(x) = \sum_{i=0}^{q+k-1} m_i \vartheta_i(x) \tag{3}$$

such that $x \in [a,b]$, m_i are unknown coefficients and $\vartheta_i(x)$ are polynomial basis functions of degree $q + k - 1$, where q is the number of interpolation point and the collocation points k are respectively chosen to satisfy $q = j \geq 3$ and $k > 1$. The integer $j \geq 1$ denotes the step number of the method. Thus, weconstruct a j-step implicit block multistep method with $\vartheta_i(x) = \left(\frac{x - x_i}{h}\right)^i$ by imposing the following conditions

$$\sum_{i=0}^{q} m_i \left(\frac{x-x_i}{h}\right)^i = y_{n-i}, \; i = 0,\dots,q-1 \tag{4}$$

$$\sum_{i=0}^{q} m_i i(i-1)(i-2)\left(\frac{x-x_i}{h}\right)^{i-3} = f_{n+i}, \; i \in Z, \tag{5}$$

where y_{n+i} is he approximation for the exact solution $y(x_{n+i}), f_{n+i} = f(x_{n+i}, y_{n+i})$, n is the grid index and $x_{n+i} = x_n + ih$. It should be observed that equations (4) and (5) leads to a system of $q+1$ equations of the AX=B where

$$A = \begin{vmatrix} x_n^0 & x_n^1 & x_n^2 & x_n^3 & x_n^4 & \cdot & \cdot & \cdot & x_n^q \\ & & \dots & \dots & & & & & \\ x_{n-k}^0 & x_{n-k}^1 & x_{n-k}^2 & x_{n-k}^3 & x_{n-k}^4 & & & & x_{n-k}^q \\ 0 & 0 & 0 & k(k-1)(k-2)x_{n-k}^3 & 0 & 0 & 0 & 0 & 0 \\ & & & \dots & \dots & & & & \\ 0 & 0 & 0 & k(k-1)(k-2)x_{n-k}^3 & k(k-1)(k-2)x_{n-k}^4 & \cdot & \cdot & \cdot & k(k-1)(k-2)x_{n-k}^q \\ 0 & 0 & 0 & k(k-1)(k-2)x_{n+1}^3 & k(k-1)(k-2)x_{n+1}^4 & \cdot & \cdot & \cdot & k(k-1)(k-2)x_{n+1}^q \\ & & & \dots & \dots & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & k(k-1)(k-2)x_{n+k}^3 & k(k-1)(k-2)x_{n+k}^4 & \cdot & \cdot & \cdot & k(k-1)(k-2)x_{n+k}^q \end{vmatrix}$$

$$X = [x_0, x_1, x_2, x_3 \dots, x_k]^T$$
$$U = [f_n, f_{n-1}, \dots, f_{n-k-1}, f_{n+1}, f_{n+2}, \dots, f_{n+k}, y_n, y_{n-1}, \dots, y_{n-k-1}]^T \tag{6}$$

Solving equation (6) using Mathematica, we get the coefficients of m_i and substituting the values of m_i into (4) and after some algebraic computation, the implicit block multistep method is obtain as

$$\sum_{i=0}^{q-1} \alpha_i y_{n-i} = h^3\left[\sum_{i=0}^{q-1} \beta_i f_{n-i} + \sum_{i=0}^{q-1} \beta_i f_{n+i}\right] \tag{7}$$

where α_i and β_i are continuous coefficients.

Differentiating (7) once and twice, we arrive at a block of first and second order derivatives which can be used to evaluate the derivative term in the initial value problem (1) as sited by [6].

$$\sum_{i=1}^{k} y'_{n+i} = \frac{1}{h}\sum_{i=0}^{q-1} \alpha_i y_{n-i} + h^2\left[\sum_{i=0}^{q-1} \beta_i f_{n-i} + \sum_{i=0}^{q-1} \beta_i f_{n+i}\right] \tag{8}$$

$$\sum_{i=1}^{k} y''_{n+i} = \frac{1}{h^2}\sum_{i=0}^{q-1} \alpha_i y_{n-i} + h\left[\sum_{i=0}^{q-1} \beta_i f_{n-i} + \sum_{i=0}^{q-1} \beta_i f_{n+i}\right] \tag{9}$$

3. Analysis of Some Theoretical properties

3.1 Order of Accuracy of the Method

Following [1, 4, 11], we define the associated linear multistep method (7)) and the difference operator as

$$L[y(x);h] = \sum_{i=0}^{j}\left[\alpha_i y(x+ih) + h^3 \beta_i y'''(x+ih)\right]. \tag{10}$$

Presuming that $y(x)$ is sufficiently and continuously differentiable on an interval $[a,b]$ and that $y(x)$ has as many higher derivatives as needed then, we write the terms in (10) as a Taylor series expression of $y(x_{n+i})$ and $f(x_{n+i}) \equiv y'''(x_{n+i})$ as

$$y(x_{n+i}) = \sum_{k=0}^{\infty}\frac{(ih)}{k!} y^{(k)}(x_n) \text{ and } y'''(x_{n+i}) = \sum_{k=0}^{\infty}\frac{(ih)}{k!} y^{(k+1)}(x_n). \tag{11}$$

Substituting (10) and (11) into (7) we obtain the following expression

$$L[y(x);h] = c_0 y(x) + c_1 h y^{(1)}(x) + \ldots + c_{p+2} h^{p+2} y^{(p+2)}(x) + \ldots, \tag{12}$$

Thus, we noticed that the implicit block multistep method of (7) has order p, if $c_{p+2}, p = 0,1,2,\ldots, i = 1,2,\ldots, j$, are given as follows:

$$c_0 = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k,$$
$$c_1 = \alpha_0 + 2\alpha_1 + \ldots + k\alpha_k,$$
$$c_2 = \frac{1}{2!}(\alpha_0 + \alpha_1 + \alpha_2 + \ldots + k\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \ldots + \beta_k), `- q = 4,3,\ldots$$

Hence, themethod (7) has order $p \geq 1$ and error constants given by the vector, $C_{p+3} \neq 0$.

Agreeing with [11], we say that the method (2) has order p if

$$L[y(x);h] = O(h^{p+3}), C_0 = C_1 = \ldots = C_p = C_{p+1} = C_{p+2} = 0, C_{p+3} \neq 0. \tag{13}$$

Therefore, C_{p+3} is the error constant and $C_{p+3} h^{p+3} y^{(p+3)}(x_n)$ is the principal local truncation error at the point x_n. Subsequently, this definition stated above is true for first and second order ODEs according to [11] then it is true for higher order ODEs.

3.2 Stability Analysis of the Method

To analyze the method for stability, (7) is normalize and written as a block method given by the matrix finite difference equations as seen in[1, 10, 16]

$$A^{(0)} Y_m = A^{(1)} Y_{m-1} + h^3 (B^{(0)} F_m + B^{(1)} F_{m-1}), \tag{14}$$

where

$$Y_m=\begin{bmatrix} y_{n+1} \\ y_{n+1} \\ . \\ . \\ . \\ y_{n+r} \end{bmatrix},\ Y_{m-1}=\begin{bmatrix} y_{n-r+1} \\ y_{n-r+2} \\ . \\ . \\ . \\ y_{n} \end{bmatrix},\ F_m=\begin{bmatrix} f_{n+1} \\ f_{n+1} \\ . \\ . \\ . \\ f_{n+r} \end{bmatrix},\ F_{m-1}=\begin{bmatrix} f_{n-r+1} \\ f_{n-r+2} \\ . \\ . \\ . \\ f_{n} \end{bmatrix}.$$

The matrices $A^{(0)}, A^{(1)}, B^{(0)}, B^{(1)}$ are r by r matrices with real entries while $Y_m, Y_{m-1}, F_m, F_{m-1}$ are r-vectors specified above.

Adopting[10, 11], we stick to the boundary locus method to determine the region of absolute stability of the block method and to obtain the roots of absolute stability. Substituting the test equation $y' = -\lambda y$ and $h' = h\lambda$ into the block (14) to obtain

$$\rho(r) = \det\left[r\left(A^{(0)} + B^{(0)} h^3 \lambda^3\right) - \left(A^{(1)} - B^{(1)} h^3 \lambda^3\right)\right] = 0 \qquad (15)$$

Replacing $h = 0$ in (15), we obtain all the roots of the derived equation to be $r \leq 1$. Therefore, according to [11], the implicit block multistep method is absolutely stable. So, as seen in [11], the boundary of the region of absolute stability can be obtained by filling (7) into

$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)} \qquad (16)$$

and permit $r = e^{i\theta} = \cos\theta + i\sin\theta$ then after reduction together with evaluating (16) within $[0^0, 180^0]$. Consequently, the boundary of the region of absolute stability rests on the real axis.

Note- fig. 1 is free handing.

4. Implementation of the Method

Embracing [5, 11], afterward this is implemented in the $P(EC)^m$ mode then it becomes important if the explicit (predictor) and the implicit (corrector) methods are individually of the same order, and this prerequisite makes it necessary for the stepnumber of the explicit (predictor) method to be greater than that of the implicit (corrector) method. Consequently, the mode $P(EC)^m$ can be formally determined as follows for $m = 1, 2, \ldots$:

$P(EC)^m$:

$$y_{n+j}^{[0]} + \sum_{i=0}^{j-1} \alpha_i^{\bullet} y_{n+i}^{[m]} = h^3 \sum_{i=0}^{j-1} \beta_i^{\bullet} f_{n+i}^{[m]},$$

$$f_{n+j}^{[s]} \equiv f(x_{n+j}, y_{n+j}^{[s]}),$$

$$y_{n+j}^{[s+1]} + \sum_{i=0}^{j-1} \alpha_i y_{n+i}^{[m]} = h^3 \beta_j f_{n+j}^{[s]} + h^3 \sum_{i=0}^{j-1} \beta_i f_{n+i}^{[m-1]}, \} s = 0,1,\ldots,m-1, \qquad (17)$$

Note that as $m \to \infty$, the result of calculating with the above mode will incline to those given by the mode of correcting to convergence.

Moreover, predictor-corrector pair based on (1) can be applied. The mode $P(EC)^m$ specified by (17), where h^3 is the step size. Since the predictor and corrector both have the same order p, Milne's device is applicable and relevant.

According to [7, 11], Milne's device proposes that it possible to estimate the principal local truncation error of the explicit-implicit (predictor-corrector) method without estimating higher derivatives of $y(x)$. Assume that $p = p^*$, where p^* and p represents the order of the explicit (predictor) and implicit (corrector) method with the same order. Now for a method of order p, the principal local truncation errors can be written as

$$C_{p+3}^{*} h^{p+3} y^{(p+3)}(x_n) = y(x_{n+j}) - W_{n+j} + O(h^{p+4}) \qquad (18)$$

Also,

$$C_{p+3} h^{p+3} y^{(p+3)}(x_n) = y(x_{n+j}) - C_{n+j} + O(h^{p+4}) \qquad (19)$$

where W_{n+j} and C_{n+j} are called the predicted and corrected approximations given by method of order p while C_{p+3}^{*} and C_{p+3} are independent of h.

Neglecting terms of degree $p+4$ and above, it is easy to make estimates of the principal local truncation error of the method as

$$C_{p+3} h^{p+3} y^{(p+3)}(x_n) \quad \frac{C_{p+3}}{C_{p+3}^{*} - C_{p+3}} \left| W_{n+j} - C_{n+j} \right| < \varepsilon \qquad (20)$$

Noting the fact that $C_{p+3} \neq C_{p+3}^{*}$ and $W_{n+j} \neq C_{n+j}$.

Furthermore, the estimate of the principal local truncation error (20) is used to determine whether to accept the results of the current step or to reconstruct the step with a smaller step size. The step is accepted based on a test as prescribed by (20) as in [19]. Equation (20) is the convergence criteria otherwise called Milne's estimate for correcting to convergence

Furthermore, equation (20) ensures the convergence criterion of the method during the test evaluation. ε is called the convergence criteria.

5. Numerical Examples

The performance of the implicit block multistep method was executed on nonstiff problems as discussed below.

Example 5.1 The first example to be considered is sited by[13] which was gotten from [3]. Moreover, [3] designed a P-stable linear multistep method for solving third order ODEs using fixed step size. On the other hand, [13] constructed a two-point four step block method for solving third order ODEs employing variable step size method. Furthermore, the newly intendedorder six implicit block multistep method is developed to numerically integrate third order ODEs applying variable step size approach.

The problem is given as follows:

$$y'''(x) + 2y''(x) - 9y'(x) - 18y(x) = -18x^2 - 18x + 22, \quad y(0) = -2, \ y'(0) = -8, \ y''(0) = -12,$$

$0 \le x \le b$,

with theoretical solution

$$y(x) = -2e^{-3x} + e^{-2x} + x^2 - 1.$$

Examples 5.2 Example 5.2 is situated in [17]and afterward, [4]. [17]executed example 5.2 on a new block method for special third order ODEs applying fixed step size. While, [4]used example 5.2 on five-step P-stable method for the numerically integrating third order ODEs with the same fixed step size method. Moreover, the newly proposed order six implicit block multistep method is designed to compute nonstiff third order ODEs employing variable step size approach.

The experiment is given as follows:

$$y'''(x) - e^{x} = 0, \ y(0) = 3, \ y'(0) = 1, \ y''(0) = 5, \ 0 \le x \le 1,$$

with exact solution

$$y(x) = 2 + 2x^2 + e^x.$$

Example 5.3

Nutrient Flow in an Aquarium

Example 5.3 is extracted from www.math.edu/~gustafso/2250systems-de.pdf and results are offered using first order method for solving ODEs analytically. The newly proposed method transform the first order systems of first order ODEs into third order ODEs before they were successfully implemented applying variable step size approach.

Consider a vessel of water containing a radioactive isotope, to be used as a tracer for the food chain, which consists of aquatic plankton varieties A and B.

Let

$w(x) =$ Isotope concentration in A,

$y(x) =$ Isotope concentration in B,

$z(x) =$ Isotope concentration in water,

Typical biological model is

$$w'(x) = 2w(x) - 3y(x),$$

$$y'(x) = w(x) + 6y(x) - 5z(x),$$

$$z'(x) = -3w(x) + 6y(x) + 5z(x).$$

The initial radioactive isotope concentrations is given by $w(0) = y(0) = 0, z(0) = z_0$ (assuming $z_0 = 1$).

However, we convert from systems of first order ODEs to third order ODEs. The conversion to third order ODEs is expressed below:

$$w'''(x) - 13w''(x) + 94 = 0, \quad w(0) = w'(0) = 0, \quad w''(0) = 1,$$

with exact solution

$$w(x) = 69 - \frac{\dfrac{69e^{\frac{13x}{2}}\cos 3\sqrt{23x}}{2} + \dfrac{13\sqrt{23x}\, e^{\frac{13x}{2}}\sin 3\sqrt{23x}}{2}}{6486}.$$

The following notational system are used on Tables 1, 2 and 3.

TOL- Tolerance Level

MAXE NPM- Magnitude of the Maximum Errors of the Implicit Block Multistep Method of Order Six (IBMMOS)

MAXE [17] - Magnitude of the Maximum Errors of [17]

MAXE [13] - Magnitude of the Maximum Errors of [13]

MAXE [4] - Magnitude of the Maximum Errors of [4]

Table 1: Comparing the Maximum Errors in the Implicit Block Multistep Method of Order Six to Maximum Errors in [13] for example 5.1.

B	MAXE [13]	TOL	MAXE IBMMOS
1.0	9.33(-7)	10^{-6}	8.55792(-7)
4.0	2.26(-6)	10^{-6}	
1.0	7.82(-8)	10^{-8}	1.3373(-8)
4.0	7.82(-8)	10^{-8}	
1.0	8.16(-10)	10^{-10}	2.08949(-10)
4.0	1.07(-9)	10^{-10}	

Table 2: Comparing the Maximum Errors in the Implicit Block Multistep Method of Order Six to [4, 17] for example 2.

MAXE [17]	MAXE [4]	TOL	MAXE IBMMOS
1.65922(-10)	0.0000	10^{-10}	1.10437(-10)
4.76275(-10)	2.8422(-13)		4.0499(-10)
6.23182(-10)	1.6729(-12)		1.44484(-09)
2.91345(-10)	2.9983(-11)		
8.71118(-10)	3.1673(-11)		
3.92904(-09)	9.1899(-11)		

9.55347(-09)	8.9531(-11)		
1.80415(-08)	1.9168(-10)		
3.03120(-08)	2.1110(-10)		
4.73044(-08)	4.9398(-10)		
7.00367(-08)	8.6728(-10)		
9.96300(-08)	2.3764(-09)		
		10^{-12}	1.61426(-12)
			5.95746(-12)
			2.10232(-11)

Table 3: gives the Maximums Errors of the Implicit Block Multistep Method of Order Six Implemented Using Variable Step Size Approach.

TOL	MAXE IBMMOS
10^{-6}	1.99839(-6)
	7.58587(-6)
	1.08127(-4)
10^{-8}	3.4862(-8)
	1.28809(-7)
	6.57593(-7)
10^{-10}	5.47059(-10)
	2.00965(-9)
	7.81941(-9)

6. Conclusion

Table 1 stated that[13] constructed two-point four step block method for solving general third order ODEs directly. The method which is particularly projected to solve stiff ODEs, but instead, solved problems is focused on nonstiff ODEs extracted from [3]. Furthermore, the implicit block multistep method of order six is formulated to solve directly general third order ODEs with particular interest on nonstiff problems. Hence, in comparison with the maximum errors at all tolerance levels of 10^{-6}, 10^{-8} and 10^{-10}, the implicit block multistep method of order six performs better than [13]. Table 2 presents the results of [4, 17] apart from the implicit block multistep method of order six. [17] develop a method which integrates special third order ODEs using fixed step size, while [4] constructed a five step P-stable method to integrate third order ODEs directlyadopting fixed step size. Nevertheless, both results cannot be compared with the implicit block multistep method of order six which is implemented employing variable step size approach. Furthermore, comparing the results of both maximum errors with the implicit block multistep method of order six, the is found to be more efficient and perform better at all tolerance levels of 10^{-10} and 10^{-12}.

Table 3 shows the tolerance levels of 10^{-8}, 10^{-10}, 10^{-12} and maximum error results. This displays that the implicit block multistep method of order six which is particularly designed for nonstiff ODEs has shown to be more efficient and consistent applying variable step size approach in solving real life mathematical model.

Remark

The implementation of an order six implicit block multistep method is performed on windows operating system and coded in Mathematica language.

Acknowledgement

The authors would like to thank the Covenant University for providing financial support through grants during the study period.

A Conflict of Interest

The Author(s) declares that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

The authors' contribution includes design and analysis of the study, preparation and review of the write up, implementation of the method as well as final revision of the write up.

References

[1] Akinfenwa, AO, Jator, SN, Yao, NM. Continuous block backward differentiation formula for solving stiff ordinary differential equations. Computers and Mathematics with Applications 2013; **65**: 996-1005.

[2] Anake, TA, Adesanya, AO, Oghonyon, GJ, Agarana, MC. Block algorithm for general third order ODEs. Icastor Journal of Mathematical Sciences 2013; **7**: 127-136.

[3] Awoyemi, DO. A P-stable linear multistep method for solving general third order ODEs. International Journal of Computer Mathematics 2003; **80**: 987-993.

[4] Awoyemi, DO, Kayode, SJ, Adoghe, LO. A five-step p-stable method for the numerical integration of third-order ODEs. American Journal of Computational Mathematics 2014; **4**: 119-126.

[5] Cash, JR., Semnani, S. A new approach to solving nonstiff initial-value problems. Journal of Computational and Applied Mathematics 1993;**45**: 41-46.

[6] Ehigie, JO, Okunuga, SA, Sofoluwe, AB. Three-point block methods for direct integration of general second order ODEs. Advances in Numerical Analysis 2011; **2011**: 1-14.

[7] Faires, JD, Burden, RL. Initial-value problems for odes: Variable step-size multistepmethods.Brooks/Cole: Dublin City University, 2012.

[8] Fatunla, S.O. Block methods for second order IVPs. Int. J. Comput. Math. 1990; **41**: 55–63.

[9] http://www.math.utah.edu/~gustafso/2250systems-de.pdf

[10] Ken, YL, Ismail, IF, Suleiman, M. Block methods for special second order ODEs. Lambert Academic Publishing: Universiti Putra Malaysia, 2011.

[11] Lambert, JD. Computational Methods in Ordinary Differential Equations. John Wiley &Sons: New York, 1993.

[12] Majid, ZA, Mohammed, BS. Implementation of four-point fully implicit block method for solving ordinary differential equations. Applied Mathematics and Computation 2007; **184**: 514-522.

[13] Majid, ZA, Azmi, NA, Suleiman,M, Ibrahim, ZB. Solving directly general third order odes using two-point four step block method. Sains Malaysiana 41, 623-632 (2012)

[14] Mehrkanoon, S, Majid, ZA, Suleiman, M. A variable step implicit block multistep method for solving first-order ODEs. Journal of Computational and Applied Mathematics 2010; **233**: 2387-2394.

[15] Mehrkanoon, S. A direct variable step block multistep method for solving general third-order ODEs. Numer Algor 2010; **57**: 53-66.

[16] Mohammed, AA, Hamza, A, Mohammed, U. A self-starting hybrid linear multistep method for a direct solution of the general second-order initial value problem. IOSR Journal of Mathematics (IOSR) 2013; **4**:7-13.

[17] Olabode, BT, Yusuph, Y. A new block method for special third order ODEs. Journal of Mathematics and Statistics 2009; **5**: 67-170.

[18] Omar, Z, Kuboye, JO. Computation of an accurate implicit block method for solving third order ODEs Directly. Global Journal of Pure and Applied Mathematics 2015; **11**: 177-186.

[19] Uri, MA, Linda, RP. Computer Methods for ODEs-Differential Algebraic Equations. SIAM: USA, 1998.

[20] Xie, L., Tian, H. Continuous parallel block methods and their applications. Applied Mathematics and Computation 2014; **241**: 356-370.

[21] You, X, Chen, Z. Direct integrators of Runge-Kutta type for special third-order ODEs. Applied Numerical Mathematics 2013; **74**:128-150.