

On multiple twisted λ -Daehee polynomials

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Abstract

In this paper, we consider multiple λ -Daehee polynomials (or called the twisted multiple Daehee polynomials) which are derived from the bosonic p -adic integral on \mathbb{Z}_p . In addition, we give some new identities and formulae of those polynomials.

AMS subject classification:

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1. Introduction

Let p be a fixed prime number. Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{C}_p will denote the ring of p -adic integers, the field of p -adic rational numbers and the completion of the algebraic closure of \mathbb{Q}_p . The p -adic norm is normalized as $|p|_p = \frac{1}{p}$. Let $f(x)$ be

uniformly differentiable function on \mathbb{Z}_p . Then the p -adic invariant integral on \mathbb{Z}_p (or called the bosonic p -adic integral on \mathbb{Z}_p) is defined as

$$\int_{\mathbb{Z}_p} f(x)d\mu_0(x) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \text{ (see [12])}. \tag{1.1}$$

From (1.1), we note that

$$\int_{\mathbb{Z}_p} f(x + 1)d\mu_0(x) - \int_{\mathbb{Z}_p} f(x)d\mu_0(x) = f'(0), \tag{1.2}$$

and

$$\int_{\mathbb{Z}_p} f(x + n)d\mu_0(x) - \int_{\mathbb{Z}_p} f(x)d\mu_0(x) = \sum_{l=0}^{n-1} f'(l), \text{ (} n \in \mathbb{N}\text{)}. \tag{1.3}$$

From (1.1), we can derive the following equation:

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_0(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.4}$$

where $B_n(x)$ are called ordinary Bernoulli polynomials. When $x = 0$, $B_n = B_n(0)$ are called Bernoulli numbers. Let $C_{p^n} = \{\xi \mid \xi^{p^n} = 1\}$ be cyclic group of order p^n . Then $T_p = \lim_{n \rightarrow \infty} C_{p^n}$, ($n \geq 0$). In [12], the twisted Bernoulli polynomials are defined by Kim to be

$$\int_{\mathbb{Z}_p} \xi^y e^{(x+y)t} d\mu_0(y) = \frac{t}{\xi e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x \mid \lambda) \frac{t^n}{n!}, \tag{1.5}$$

where $\xi \in T_p$.

When $x = 0$, $B_n(\xi) = B_n(0 \mid \xi)$ are called the twisted Bernoulli numbers. It is well known that the Daehee polynomials are defined by the generating function to be

$$\int_{\mathbb{Z}_p} (1 + t)^{x+y} d\mu_0(y) = \frac{\log(1 + t)}{t} (1 + t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}, \tag{1.6}$$

where $t \in \mathbb{C}_p$ with $|t|_p < p^{\frac{-1}{p-1}}$, (see [3 – 10]). For $\xi \in T_p$, the twisted Daehee polynomials are also given by be generation function to be

$$\int_{\mathbb{Z}_p} \xi^y (1 + t)^{x+y} d\mu_0(y) = \frac{\log(1 + t)}{\xi(1 + t) - 1} (1 + t)^x = \sum_{n=0}^{\infty} D_n(x \mid \xi) \frac{t^n}{n!}, \tag{1.7}$$

where $t \in C_p$ with $|t|_p < p^{\frac{-1}{p-1}}$ and $\xi \in T_p$ (see [4]). When $x = 0$, $D_n(\xi) = D_n(0 \mid \xi)$ are called the twisted Daehee numbers.

The higher-order twisted Bernoulli polynomials are given by the generating function to be

$$\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \xi^{x_1+x_2+\dots+x_r} e^{(x_1+x_2+\dots+x_r+x)t} d\mu_0(x_1) \dots d\mu_0(x_r) = \left(\frac{t}{\xi e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x | \lambda) \frac{t^n}{n!}, \quad (\text{see [6, 9]}). \tag{1.8}$$

and the twisted higher-order Daehee polynomials are also given by the multivariate integral on \mathbb{Z}_p as follows:

$$\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \xi^{x_1+x_2+\dots+x_r} (1+t)^{(x_1+x_2+\dots+x_r+x)t} d\mu_0(x_1) \dots d\mu_0(x_r) = \left(\frac{\log(1+t)}{\xi(1+t) - 1}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x | \lambda) \frac{t^n}{n!}, \quad (\text{see [6, 9]}). \tag{1.9}$$

In this paper, we consider the multiple twisted Daehee polynomials and multiple twisted Bernoulli polynomials and we give some relations between multiple twisted Daehee polynomials and multiple twisted Bernoulli polynomials. Recently, several authors have studied Daehee polynomials (see [1–10]).

2. Multiple twisted Daehee and Bernoulli polynomials

In this section, we assume that $\lambda_1, \lambda_2, \dots, \lambda_r \in T_p$ and $t \in \mathbb{C}_p$ with $|t| < p^{\frac{-1}{p-1}}$.

Now, we define the multiple twisted λ -Bernoulli polynomials which are given by the generation function to be

$$\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \left(\prod_{l=1}^r \lambda_l^{x_l}\right) e^{(x_1+x_2+\dots+x_r+x)t} d\mu_0(x_1) \dots d\mu_0(x_r) = \left(\frac{t}{\lambda_1 e^t - 1}\right) \times \left(\frac{t}{\lambda_2 e^t - 1}\right) \times \dots \times \left(\frac{t}{\lambda_r e^t - 1}\right) e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x | \lambda_1, \lambda_2, \dots, \lambda_r) \frac{t^n}{n!}. \tag{2.1}$$

From (2.1), we note that

$$B_n^{(r)}(x | \lambda, \lambda, \dots, \lambda) = B_n^{(r)}(x | \lambda), \quad B_n^{(r)}(x | 1) = B_n^{(r)}(x), \tag{2.2}$$

where $B_n^{(r)}$ are called higher-order Bernoulli polynomials which are defined by the generation function to be

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad (r \in \mathbb{N}). \tag{2.3}$$

The multiple twisted λ -Daehee polynomials are defined by the generating function to be

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\prod_{l=1}^r \lambda_l^{x_l} \right) (1+t)^{(x_1+x_2+\cdots+x_r+x)t} d\mu_0(x_1) \cdots d\mu_0(x_r) \\ &= \left(\frac{t}{\lambda_1(1+t) - 1} \right) \times \left(\frac{t}{\lambda_2(1+t) - 1} \right) \times \cdots \times \left(\frac{t}{\lambda_r(1+t) - 1} \right) (1+t)^x \quad (2.4) \\ &= \sum_{n=0}^{\infty} D_n^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) \frac{t^n}{n!}. \end{aligned}$$

From (1.9) and (2.4), we note that

$$D_n^{(r)}(x \mid \lambda, \lambda, \dots, \lambda) = D_n^{(r)}(x \mid \lambda), \quad (n \geq 0). \quad (2.5)$$

From (2.4), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} D_n^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\prod_{l=1}^r \lambda_l^{x_l} \right) e^{(x_1+x_2+\cdots+x_r+x)\log(1+t)} d\mu_0(x_1) \cdots d\mu_0(x_r) \\ &= \left(\frac{\log(1+t)}{\lambda_1 e^{\log(1+t)} - 1} \right) \times \left(\frac{\log(1+t)}{\lambda_2 e^{\log(1+t)} - 1} \right) \times \cdots \times \left(\frac{\log(1+t)}{\lambda_r e^{\log(1+t)} - 1} \right) e^{x\log(1+t)} \\ &= \sum_{m=0}^{\infty} B_m^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) \frac{1}{m!} (\log(1+t))^m \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n S_1(n, m) B_m^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) \right) \frac{n}{n!}. \quad (2.6) \end{aligned}$$

Thus, by comparing the coefficients on the both sides of (2.6), we get

$$D_n^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) = \sum_{m=0}^n S_1(n, m) B_m^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r), \quad (n \geq 0), \quad (2.7)$$

where $S_1(n, m)$ is the stirling number of the first kind. By replacing t by e^{t-1} in (2.4),

we get

$$\begin{aligned}
 & \sum_{n=0}^{\infty} B_n^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) \frac{t^n}{n!} \\
 &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\prod_{l=1}^r \lambda_l^{x_l} \right) e^{(x_1+x_2+\cdots+x_r+x)t} d\mu_0(x_1) \cdots d\mu_0(x_r) \\
 &= \sum_{m=0}^{\infty} D_m^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) \frac{1}{m!} (e^t - 1)^m \tag{2.8} \\
 &= \sum_{m=0}^{\infty} D_m^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) \sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n S_2(n, m) D_m^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) \right) \frac{t^n}{n!},
 \end{aligned}$$

where $S_2(n, m)$ is the stirling number of the second kind. Thus, by comparing the coefficients on the both sides of (2.8), we get

$$B_n^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) = \sum_{m=0}^n S_2(n, m) D_m^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r), \quad (n \geq 0). \tag{2.9}$$

Therefore, by (2.7) and (2.9), we obtain the following theorem.

Theorem 2.1. For $n \geq 0, r \in \mathbb{N}$, we have

$$B_n^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) = \sum_{m=0}^n S_2(n, m) D_m^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r),$$

and

$$D_n^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) = \sum_{m=0}^n S_1(n, m) B_m^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r),$$

where $\lambda_i \neq \lambda_j$ if $i \neq j$, and $\lambda_i \neq 1 (i = 1, 2, \dots, r)$.

By (2.1), we easily get

$$\begin{aligned}
 & \sum_{n=0}^{\infty} B_n^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) \frac{t^n}{n!} \\
 &= \left(\frac{t}{\lambda_1 e^t - 1} \right) \times \left(\frac{t}{\lambda_2 e^t - 1} \right) \times \dots \times \left(\frac{t}{\lambda_r e^t - 1} \right) e^{xt} \\
 &= \sum_{k=1}^{\infty} \frac{t^r (-1)^r}{\lambda_k e^t - 1} \prod_{j=1, j \neq k}^r \left(1 - \frac{\lambda_j}{\lambda_k} \right)^{-1} e^{xt} \\
 &= \sum_{n=0}^{\infty} \left(t^{r-1} (-1)^r \sum_{k=1}^r B_n(x \mid \lambda_k) \prod_{j=1, j \neq k}^r \left(1 - \frac{\lambda_j}{\lambda_k} \right)^{-1} \right) \frac{t^n}{n!},
 \end{aligned} \tag{2.10}$$

where $\lambda_i \neq 1, (i = 1, 2, \dots, r)$. From (2.10), we note that

$$\begin{aligned}
 B_0^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) &= B_1^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) \\
 &= \dots = B_{r-1}^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) = 0
 \end{aligned} \tag{2.11}$$

From (2.10) and (2.11), we can derive the following equation:

$$\begin{aligned}
 & (-1)^r \sum_{k=1}^r \frac{B_{n+1}(x \mid \lambda_k)}{n+1} \prod_{j=1, j \neq k}^r \left(1 - \frac{\lambda_j}{\lambda_k} \right)^{-1} \\
 &= \frac{1}{\binom{n+r}{r} r!} B_{n+r}^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r), \quad (n \in \mathbb{N} \cup \{0\}, r \in \mathbb{N}).
 \end{aligned} \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 2.2. For $n \in \mathbb{N} \cup \{0\}$, and $r \in \mathbb{N}$ we have

$$\begin{aligned}
 & (-1)^r \sum_{k=1}^r \frac{B_{n+1}(x \mid \lambda_k)}{n+1} \prod_{j=1, j \neq k}^r \left(1 - \frac{\lambda_j}{\lambda_k} \right)^{-1} \\
 &= \frac{1}{\binom{n+r}{r} r!} B_{n+r}^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r),
 \end{aligned}$$

where $\lambda_i \neq \lambda_j$ if $i \neq j$, and $\lambda_i \neq 1 (i = 1, 2, \dots, r)$.

By (2.4), we get

$$\begin{aligned}
 & \sum_{n=0}^{\infty} D_n^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) \frac{t^n}{n!} \\
 &= \left(\frac{t}{\lambda_1(1+t) - 1} \right) \times \left(\frac{t}{\lambda_2(1+t) - 1} \right) \times \dots \times \left(\frac{t}{\lambda_r(1+t) - 1} \right) (1+t)^x \\
 &= \sum_{k=1}^r \frac{(-1)^r t^r}{\lambda_k(1+t) - 1} \prod_{j=1, j \neq k}^r \left(1 - \frac{\lambda_j}{\lambda_k} \right)^{-1} (1+t)^x \\
 &= \sum_{n=0}^r \left((-1)^r t^{r-1} \sum_{k=1}^r D_n(x \mid \lambda_k) \prod_{j=1, j \neq k}^r \left(1 - \frac{\lambda_j}{\lambda_k} \right)^{-1} \right) \frac{t^n}{n!}
 \end{aligned} \tag{2.13}$$

From, (1.7) and (2.4), we have

$$\lambda D_n(\lambda) + n\lambda D_{n-1}(\lambda) - D_n(\lambda) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n \neq 1, \end{cases} \quad D_0(\lambda) = 0,$$

and

$$\begin{aligned}
 D_0^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) &= D_1^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) \\
 &= \dots \\
 &= D_{r-1}^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) = 0
 \end{aligned}$$

By (2.13), we get

$$\begin{aligned}
 & (-1)^r \sum_{k=1}^r \frac{D_{n+1}(x \mid \lambda_k)}{n+1} \prod_{j=1, j \neq k}^r \left(1 - \frac{\lambda_j}{\lambda_k} \right)^{-1} \\
 &= D_{n+r}^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) \frac{1}{r! \binom{n+r}{r}}, \quad (n \in \mathbb{N} \cup \{0\}, r \in \mathbb{N})
 \end{aligned} \tag{2.14}$$

Therefore, by (2.13) and (2.14), we obtain the following theorem.

Theorem 2.3. For $n \geq 0$ and $r \in \mathbb{N}$, we have

$$\begin{aligned}
 & \frac{1}{r! \binom{n+r}{r}} D_{n+r}^{(r)}(x \mid \lambda_1, \lambda_2, \dots, \lambda_r) \\
 &= (-1)^r \sum_{k=1}^r \frac{D_{n+1}(x \mid \lambda_k)}{n+1} \prod_{j=1, j \neq k}^r \left(1 - \frac{\lambda_j}{\lambda_k} \right)^{-1},
 \end{aligned}$$

where $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_r$ and $\lambda_i \neq 1 (i = 1, 2, \dots, r)$.

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