Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 2 (2016), pp. 1691-1697

http://www.ripublication.com/gjpam.htm

Concomitants of Order Statistics of a New Bivariate Finite Range Distribution (NBFRD)

Sabir Ali Siddiqui

Department of Mathematics and Sciences, CAAS, Dhofar University, Salalah, Sultanate of Oman.

Syed Nasser Andrabi and Sanjay Jain

Department of Statistics, St. John's College, Agra, India.

Masood Alam

Department of Mathematics, FP, Sultan Qaboos University, Oman.

Raed Abdelkarim

Department of Mathematics and Sciences, CAAS, Dhofar University, Salalah, Sultanate of Oman.

Abstract

In the present paper the probability density function of concomitants of order statistics has been obatained. The basic distribution for both random variables is assumed to be Mukherji-Islam distribution. Then the moments of the concomitants have been evaluated.

AMS subject classification:
Keywords: Order statistics (O.S.), Concomitants.

1. Introduction

The study of concomitants has attracted many research workers, Zippin and Armitage (1966) used theory of concomitants in survival analysis. Yang (1977) gave a general distribution theory on concomitants. Some other reaserch workers can be named here as, Nagaraja and David (1994), Joshi and Nagaraja (1995). Vianna and Lee (2006), Siddiqui et al. (2011, 2012), Das et al. (2012) and Ziaei et al. (2014). Frechet (1951) noted that in case of bivariate distribution,
Since we have,

$$P[T_1 \leq t_1) \cap (T_2 \leq t_2)] \leq Min[P(T_1 \leq t_1), P(T_2 \leq t_2)]$$

hence the relationship will exists,

$$F_{t_1}, t_2(t_1, t_2) \leq Min[F_{t_1}, Ft_2] \tag{1}$$

and since

$$P[(X_1 \geq x_1) \cup (X_2 \geq x_2)] \leq P(X_1 \geq x_2) + P(X_2 \geq x_2)$$

it follows that

$$1 - F_{x_1,x_2}(x_1, x_2) \leq [1 - F_{x_1}(x_1)] + [1 - F_{x_2}(x_2)]$$

so the relation (2) is,

$$F_{x_1,x_2}(x_1, x_2) \geq F_{x_1}(x_1) + F_{x_2}(x_2) - 1 \tag{2}$$

Frechet (1951) suggested that$F_{x_1}(x_1) and F_{x_2}(x_2)$ should include the limits in (1) and (2) as limiting cases. In particular, he suggested the system

$$F_{x_1,x_2}(x_1, x_2) = \theta max[F_{x_1}(x_1) + F_{x_2}(x_2) - 1, 0] + (1 - \theta)mix[F_{x_1}(x_1), F_{x_2}(x_2)] \tag{3}$$

This system is for dependent variables.

A system that does include this case [but not the limits in (1) and (2)] is given by Morgenstern (1956) as,

$$F_{y_1,y_2}(y_1, y_2) = F_{y_1}(y_1).F_{y_2}(y_2)[1 + \delta(1 - F_{y_1}(y_1))(1 - F_{y_2}(y_2)) \tag{4}$$

In the present paper a new bivariate finite range model (NBFRM) has been developed under the system given by Morgenstern (1956) by considering Mukherjee-Islam Model as the basic model.

1.1. Mukherjii-Islam Distribution at a Glance

The cdf of Mukherjii-Islam Distribution (1983) is

$$\begin{aligned} G(y) &= \frac{y^p}{\theta^p}, \quad where, \quad p, \theta > 0, \theta \geq \ y > 0 \\ Or \quad &= 0, \quad otherwise \end{aligned}$$

Here θ and p are scale and shape parameters, the pdf of above cdf (5) will be,

$$g(y) = \frac{py^p}{\theta^p} \tag{5}$$

where $p, \theta > 0, \theta \geq y > 0$ The reliability function $R(x)$ and hazard rate $h(x)$ function of the distribution are;

$$R(t) = 1 - \left(\frac{t}{\theta}\right)^p, \quad \text{where } \ p, \theta > 0, \ \theta \geq t > 0 \tag{6}$$

$$h(t) = \frac{pt^{p-1}}{1 - \left(\frac{t}{\theta}\right)^p}, \quad \text{where } \ p, \theta > 0 \tag{7}$$

2. Marginal, Conditional and Density Functions using Morgenstern System

2.1. Marginal Probability Density Functions

Using Mukherjii-Islam (1983) and exponential distributions with parameters(p, θ) and θ respectively and following the Morgenstern (1956) for $\delta = -1$

$$F(x, y) = \frac{x^p}{\theta^p}\left[1 - e^{-\theta y}\left(1 - \frac{x^p}{\theta^p}\right)\right], \quad 0 < x < \theta, \ y \geq 0 \tag{8}$$

Will be the joint distribution.

This is termed here as a new finite range bivariate distribution (NBFRD). Its probability density function will be as follows

$$f(x, y) = 2\theta e^{-\theta y}\frac{px^{p-1}}{\theta^p}\left[1 - \left(\frac{x^p}{\theta^p} + e^{-\theta y}\right) + 2e^{-\theta y}\frac{x^p}{\theta^p}\right], \quad 0 < x < \theta, \ y \geq 0 \tag{9}$$

The marginal probability density function of X can be obtained as;

$$g(x) = \frac{px^{p-1}}{\theta^p}; \quad 0 < x < \theta \tag{10}$$

and the marginal distribution function of X is;

$$G(x) = \frac{x^p}{\theta^p}; \quad 0 < x < \theta \tag{11}$$

Similarly,

$$h(y) = \theta e^{-\theta y}; \ y \ \geq \ 0 \tag{12}$$

$$H(y) = 1 - e^{-\theta y}; \quad y \ \geq \ 0 \tag{13}$$

are the marginal pdf and cdf of hazzard rate function. Also

$$E(Y) = \frac{1}{\theta} \tag{14}$$

and,

$$E(Y^2) = \frac{2}{\theta^2} \tag{15}$$

2.2. Conditional Probability Density Functions

The conditional pdf of Y comes out as follows;

$$h(y/x) = 2\theta e^{-\theta y}\left[1 - \left(\frac{x^p}{\theta^p} + e^{-\theta y}\right) + 2e^{-\theta y}\frac{x^p}{\theta^p}\right] \tag{16}$$

Similarly, the conditional pdf of X is;

$$g(x/y) = 2\theta e^{-\theta y}\frac{px^{p-1}}{\theta^p}\left[1 - \left(\frac{x^p}{\theta^p} + e^{-\theta y}\right) + 2e^{-\theta y}\frac{x^p}{\theta^p}\right] \tag{17}$$

2.3. Probaility Density Function of Order Statistics

$$f_{r:n}(x) = C_{r:n}\frac{px^{rp-1}}{\theta^p r}\left[1 - \left(\frac{x}{\theta}\right)^p\right]^{n-r} \tag{18}$$

where $C_{r:n} = \dfrac{n!}{(r-1)!(n-r)!}$, is the pdf of rth order statistics.

$$f_{1:n} = n\frac{px^{p-1}}{\theta^p}\left[1 - \left(\frac{x}{\theta}\right)^p\right]^{n-1} \tag{19}$$

pdf for $r = 1$, and

$$f_{r,s:n}(x_1, x_2) = C_{r,s:n}\frac{p^2 x_1^{pr-1} x_2^{p-1}}{\theta^{p(r+1)}}\left[\left(\frac{x_2}{\theta}\right)^p - \left(\frac{x_1}{\theta}\right)^p\right]^{s-r-1}\left[1 - \left(\frac{x_2}{\theta}\right)^p\right]^{n-s} \tag{20}$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!},$$

is the joint pdf of rth and sth O.S.
For random variable Y;

$$f_{1:n}(y) = n.\theta e^{-n\theta y} \tag{21}$$

Eq. (21) is the pdf of first O.S.

2.4. Probability Density Function of Concomitants

Using David (1981), We have

$$g_{[1:n]}(y) = n\Sigma_{k=0}^{n-1} n-1_{C_k}(-1)^{n-k-1}\left[\frac{2}{n-k}\theta e^{-\theta y} - \frac{2}{n-k+1}\theta e^{-\theta y} - \frac{1}{n-k}2\theta e^{-2\theta y} + \frac{4}{n-k+1}\theta e^{-2\theta y}\right]$$

Similarly, $g_{[1:n]}(x)$ can be obtained as;

$$g_{[1:n]}(x) = \frac{1}{n+1}[2nf_1(p) - (n-1)f_2(p)] \tag{22}$$

where

$$f_1(p) = \frac{px^{p-1}}{\theta^p} \text{ and } f_2(p) = \frac{2px^{2p-1}}{\theta^{2p}}$$

Now, following rule,

$$g_{[r:n]}(y) = \Sigma_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}g_{[1:i]}(y) \tag{23}$$

$$= \Sigma_{i=n-r+1}^{n}\alpha\binom{i-1}{n-r}\binom{n}{i}(i\Sigma_{k=0}^{i-1}i-1_{C_k}(-1)^{i-k-1})\left[\frac{2}{n-k}\theta e^{-\theta y} - \frac{2}{n-k+1}\theta e^{-\theta y} - \frac{1}{n-k}2\theta e^{-2\theta y} + \frac{4}{n-k+1}\theta e^{-2\theta y}\right]$$

Where, $\alpha = (-1)^{i-n+r-1}$. We get the pdf of rth concomitant. Also, for x, we can get through

$$g_{[r:n]}(x) = \Sigma_{i=n-r+1}^{n}\binom{i-1}{n-r}\binom{n}{i}\left(i\left[\frac{1}{i+1}[2if_1(p) - (i-1)f_2(p)]\right]\right) \tag{24}$$

3. Moments of Concomitants

The kth moment about origin of the first concomitant i.e. of $Y_{[1:n]}$ is given by,

$$\mu_y^k[1:n] = \int_0^\infty y^k g_{[1:n]}(y)dy$$
$$= n\Sigma_{j=0}^{n-1}n-1_{C_j}(-1)^{n-j-1}\frac{k!}{\theta^k}\left[\frac{2}{n+1-j}\left(\frac{1}{2^k}-1\right)\right]$$

Now,

$$\mu_y^k[r:n] = \Sigma_{i=n-r+1}^{n}\alpha\binom{i-1}{n-r}\binom{n}{i}\mu_y^k[r:n] \tag{25}$$

$$= \Sigma_{i=n-r+1}^{n} \alpha \binom{i-1}{n-r} \binom{n}{i} \Big(i \Sigma_{j=0}^{i-1} i - 1_{C_j} \alpha$$
$$(-1)^{i-j-1} \frac{k!}{\theta^k} \left[\frac{2}{i+1-j} \left(\frac{1}{2^k} - 1 \right) + \frac{1}{i-j} \left(2 - \frac{1}{2^k} \right) \right] \Big)$$

Eq. (29) gives the kth moment of $Y_{[1:n]}$. Now,

$$\mu^1_{y[r:n]} = \Sigma_{i=n-r+1}^{n} \alpha \binom{i-1}{n-r} \binom{n}{i} \Big(i \Sigma_{j=0}^{i-1} i - 1_{C_j}$$
$$(-1)^{i-j-1} \frac{1}{\theta} \left[\frac{3}{2(i-j)} - \frac{1}{i-j+1} \right]$$

In Eq. (30) we get it for $k = 1$. The mean of $X_{[r:n]}$ will be;
$E(X_{[r:n]}) = \mu^1 x[r : n]$ and in detail:

$$E(X_[r : n]) = \mu^1 x[r : n] = \Sigma_{i=n-r+1}^{n} \alpha \binom{i-1}{n-r} \binom{n}{i}$$
$$\Big(i \Sigma_{j=0}^{i-1} i - 1_{C_j} \ (-1)^{i-j-1} \theta 2\alpha$$
$$\times \left[\frac{1}{(i-j+1)(\alpha+1)} \left(\frac{1}{\alpha+1} - \frac{2}{2\alpha+1} \right) + \frac{1}{(i-j)(2\alpha+1)} \right]$$

Similarly the variance of can be obtained as

$$Var(y_{[r:n]}) = \mu^2_{y[r:n]} - (\mu^1_{y[r:n]})^2$$

where

$$\mu^2_{y[r:n]} = \Sigma_{i=n-r+1}^{n} \alpha \binom{i-1}{n-r} \binom{n}{i} \Big(i \Sigma_{j=0}^{i-1} i - 1_{C_j}$$
$$(-1)^{i-j-1} \theta^2 2\beta \left[\frac{1}{(i-j+1)(\beta+2)} \left(\frac{1}{\beta+2} - \frac{1}{\beta+1} \right) + \frac{1}{(i-j)(\beta+1)} \right]$$

References

[1] Das, K.K., Das, B. and Baruah, B.K., A Comparative Study on Concomitants of order statistics and Record Statistics for Weighted Inverse Gaussian Distribution, International Journal of Scientific and Research Publications, Vol. 2, Issue 6, (2012).

[2] David, H.A., Order Statistics, 2nd Ed., John Wiley, New York, (1981).

[3] Frechet, M., Sur le tableau de correlation doesn't les marges sont donnesž, Annales e Universite de Lyon, Ser. III, 14, 53–77, (1951)

[4] Joshi, S.N. and Nagarajan, H.N., Joint distribution of maxima of concomitants of subsets of order statistics, Bernaulli, 1(3), 245–255, (1995).

[5] Morgenstern, D., Einfache beispiele zweidimensionaler Verteinungenž, Metteilungsblatt fiir Mathematische Statstik, 8, 234–235, (1956).

[6] Mukheerji, S.P. and Islam, A., A finite range distribution of failures times, Naval Research Logistics Quarterly, Vol. 30, pp. 487–491, (1983).

[7] Nagaraja, H.N. and David, H.A., Distribution of the maximum of concomitants of selected order statistics, The Annals of Statistics, 22(1), 478–494, (1994).

[8] Siddiqui, SA et al., Moments and joint distribution of concomitants of order statistics, Journal of Reliability and Statistical Studies, 4(2), p. 25–33. (2011).

[9] Siddiqui, S.A., Jain, S., Andrabi, S.N., Siddiqui, I., Alam, M. and Chalkoo, P.A., Moments of concomitants of order statistics for a new finite range bivariate distribution (NFRBD), Journal of Reliability and Statistical Studies, 5(2), 77–84, (2012).

[10] Viana, M. A. G. and Lee, H., Correlation analysis of ordered symmetrically dependent observations and their concomitants of order statistics. Scandinavian Journal of Statistics, 34(2), 327–340, (2006).

[11] Yang, S.S., General distribution theory of the concomitants of order statistics, The Annals of Statistics, 5, 996–1002, (1977).

[12] Ziaei, A.R., Sheikhi, A. and Amirzadeh, V., Regression analysis using order statistics and their concomitants, SORT, 38(1), (2014).

[13] Zippin, C. and Armitage, P., Use of concomitant variables and incomplete survival information in the estimation of an exponential survival parameter. Biometrics, 22, 665–672, (1966).