

On determinants of tridiagonal matrices with $(-1, 1)$ -diagonal or superdiagonal in relation to Fibonacci numbers

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Abstract

The aim of the paper is to find some new determinants connected with Fibonacci numbers. We generalize the result provided in Strang's book because we derive that two sequences of similar tridiagonal matrices are connected with Fibonacci numbers.

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1. Introduction

The Fibonacci sequence (or the sequence of Fibonacci numbers) $(F_n)_{n \geq 0}$ is the sequence of positive integers satisfying the recurrence

$$F_{n+2} = F_{n+1} + F_n \quad (1)$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. Let α and β be the roots of the characteristic equation $x^2 - x - 1 = 0$ (thus $\alpha = \frac{1 + \sqrt{5}}{2}$ and $\beta = \frac{1 - \sqrt{5}}{2}$). Then Binet's formula for the Fibonacci numbers has the form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

The Fibonacci numbers have many remarkable properties and many of their features appear throughout nature (cf. [6, 15, 14]).

Strang [12] included probably the first example of determinant of $n \times n$ matrix, which is equal to the Fibonacci number, as he showed that the determinant of the $n \times n$ matrix

$$\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & 1 & \ddots & \ddots & \vdots \\ 0 & \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix} \quad (2)$$

is equal to F_{n+1} for $n \geq 1$. Matrix (2) is a special case of a tridiagonal matrix, that is a square matrix $\mathbb{A} = (a_{jk})$ of the order n , where $a_{jk} = 0$ for $|k - j| > 1$ and $1 \leq j, k \leq n$, i. e.,

$$\mathbb{A}(n) = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & \ddots & \vdots \\ 0 & a_{3,2} & a_{3,3} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{n-1,n} \\ 0 & \cdots & 0 & a_{n,n-1} & a_{n,n} \end{pmatrix}.$$

Many authors derived the similar types of matrices whose determinants are related to Fibonacci numbers or different kinds of their generalizations, e. g. k -generalized Fibonacci numbers, cf. [5, 8, 7, 4, 9, 10, 16].

At this point we turn our attention to the relation of determinants of special tridiagonal matrices with Fibonacci numbers. Trojovský [13] dealt with the sequence of generalized matrices to matrix (2), which has also determinant equal to F_{n+1} . We show that matrix (2) can be easily changed into two different sequences of matrices, whose determinants are connected with Fibonacci numbers.

2. Main results

We formulate the following theorem on determinants of sequences of tridiagonal matrices with alternating 1 's and -1 's on the superdiagonal.

Theorem 2.1. Let $(\mathbb{B}^\delta(n) = (b_{jk}^\delta)_{1 \leq j, k \leq n})_{n \geq 1}$, where $\delta \in \{0, 1\}$, be a sequence of tridiagonal matrices in the form

$$b_{jk}^\delta = \begin{cases} 1, & k = j \text{ or } k = j - 1; \\ (-1)^{j+\delta}, & k = j + 1; \\ 0, & \text{otherwise} \end{cases}$$

i. e.,

$$\mathbb{B}^\delta(n) = \begin{pmatrix} 1 & (-1)^{1-\delta} & 0 & \dots & \dots & 0 \\ 1 & 1 & (-1)^{2-\delta} & \ddots & \ddots & \vdots \\ 0 & 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & (-1)^{n-2-\delta} & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 & (-1)^{n-1-\delta} \\ 0 & \dots & \dots & 0 & 1 & 1 \end{pmatrix}.$$

Then

$$\det \mathbb{B}^\delta(n) = \begin{cases} F_{\frac{n+4-6\delta}{2}} & n \equiv 0 \pmod{2}; \\ F_{\frac{n+1}{2}}, & n \equiv 1 \pmod{2}. \end{cases} \tag{3}$$

Now we show that matrix (2) can be changed into the sequence of matrices with alternating 1's and -1's on the diagonal, whose determinants are equal to a term of the Fibonacci sequence with the plus or minus sign.

Theorem 2.2. Let $(\mathbb{C}^\delta(n) = (c_{jk}^\delta)_{1 \leq j, k \leq n})_{n \geq 1}$, where $\delta \in \{0, 1\}$, be a sequence of tridiagonal matrices in the form

$$c_{jk}^\delta = \begin{cases} 1, & k = j \pm 1; \\ (-1)^{j+\delta}, & k = j; \\ 0, & \text{otherwise} \end{cases}$$

i. e.,

$$\mathbb{C}^\delta(n) = \begin{pmatrix} (-1)^{1+\delta} & 1 & 0 & \dots & \dots & 0 \\ 1 & (-1)^{2+\delta} & 1 & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & (-1)^{n-2+\delta} & 1 & 0 \\ \vdots & \ddots & \ddots & 1 & (-1)^{n-1+\delta} & 1 \\ 0 & \dots & \dots & 0 & 1 & (-1)^{n+\delta} \end{pmatrix}.$$

Then

$$\det \mathbb{C}^\delta(n) = (-1)^{\frac{n}{2}(n+1-2\delta)} F_{n+1}. \tag{4}$$

3. An auxiliary result

We recall the following lemma, which can be proved using cofactor expansion on the last row and subsequently on the last column of matrix $\mathbb{H}(n)$.

Lemma 3.1. (Lemma 3.1 of [3]) Let $\{\mathbb{H}(n), n = 1, 2, \dots\}$ be a sequence of tridiagonal matrices of the form

$$\mathbb{H}(n) = \begin{pmatrix} h_{1,1} & h_{1,2} & 0 & \cdots & 0 \\ h_{2,1} & h_{2,2} & h_{2,3} & \ddots & \vdots \\ 0 & h_{3,2} & h_{3,3} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & h_{n-1,n} \\ 0 & \cdots & 0 & h_{n,n-1} & h_{n,n} \end{pmatrix}.$$

Then the successive determinants of $\mathbb{H}(n)$ are given by recursive formula

$$\begin{aligned} \det \mathbb{H}(1) &= h_{1,1}; \\ \det \mathbb{H}(2) &= h_{1,1}h_{2,2} + h_{1,2}h_{2,1}; \\ \det \mathbb{H}(n) &= h_{n,n} \det \mathbb{H}(n-1) + h_{n-1,n}h_{n,n-1} \det \mathbb{H}(n-2). \end{aligned} \tag{5}$$

4. Proofs of the main results

4.1. Proof of Theorem 1

Case $\delta = 0$.

Using Lemma 3.1 we have $\det \mathbb{B}^0(1) = 1$, $\det \mathbb{B}^0(2) = 2$, and for $n \geq 3$ we obtain the following recurrence relation

$$\det \mathbb{B}^0(n) = \det \mathbb{B}^0(n-1) + (-1)^n \det \mathbb{B}^0(n-2). \tag{6}$$

Alladi and Hoggatt [1] mentioned recurrence relation (6). They showed that the number of palindromic compositions of positive integer n using only numbers 1 and 2 is described by (6) and they proved (in a different notation) that

$$\det \mathbb{B}^0(n) = \begin{cases} F_{\frac{n+4}{2}}, & n \equiv 0 \pmod{2}; \\ F_{\frac{n+1}{2}}, & n \equiv 1 \pmod{2} \end{cases} \tag{7}$$

holds for any positive integer n . Therefore identity (3) holds.

Case $\delta = 1$.

We easily obtain $\det \mathbb{B}^1(1) = 1$ and $\det \mathbb{B}^1(2) = 0$. Using cofactor expansion on the

first row and then on the first column of $\det \mathbb{B}^0(n + 1)$ we get for $n \geq 3$ the following

$$\begin{aligned} & \det \begin{pmatrix} 1 & (-1)^1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 1 & 1 & (-1)^2 & 0 & & & & \vdots \\ 0 & 1 & 1 & (-1)^3 & 0 & & & 0 \\ \vdots & 0 & 1 & 1 & \ddots & \ddots & & 0 \\ 0 & & \ddots & \ddots & \ddots & (-1)^{n-2} & 0 & \vdots \\ 0 & & & 0 & 1 & 1 & (-1)^{n-1} & 0 \\ \vdots & & & & 0 & 1 & 1 & (-1)^n \\ 0 & \dots & \dots & 0 & \dots & 0 & 1 & 1 \end{pmatrix} \\ &= \det \mathbb{B}^1(n) + \\ & \det \begin{pmatrix} 1 & (-1)^2 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & (-1)^3 & 0 & & & 0 \\ 0 & 1 & 1 & \ddots & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & (-1)^{n-2} & 0 & \vdots \\ 0 & & 0 & 1 & 1 & (-1)^{n-1} & 0 \\ \vdots & & & 0 & 1 & 1 & (-1)^n \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix} \cdot \\ &= \det \mathbb{B}^1(n) + \det \begin{pmatrix} 1 & (-1)^3 & 0 & \dots & \dots & 0 \\ 1 & 1 & (-1)^4 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & 1 & 1 & (-1)^{n-1} & 0 \\ \vdots & \ddots & 0 & 1 & 1 & (-1)^n \\ 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix} \end{aligned}$$

Thus we get the following recurrence

$$\det \mathbb{B}^0(n + 1) = \det \mathbb{B}^1(n) + \det \mathbb{B}^0(n - 1)$$

for any $n \geq 3$. This recurrence can be rewritten as follows

$$\det \mathbb{B}^1(n) = \det \mathbb{B}^0(n + 1) - \det \mathbb{B}^0(n - 1)$$

and using identity (7) we obtain

$$\begin{aligned} \det \mathbb{B}^1(n) &= \det \mathbb{B}^0(n + 1) - \det \mathbb{B}^0(n - 1) \\ &= \begin{cases} F_{\frac{n+2}{2}} - F_{\frac{n}{2}} = F_{\frac{n-2}{2}}, & n \equiv 0 \pmod{2}; \\ F_{\frac{n+5}{2}} - F_{\frac{n+3}{2}} = F_{\frac{n+1}{2}}, & n \equiv 1 \pmod{2}. \end{cases} \end{aligned}$$



4.2. Proof of Theorem 2

Case $\delta = 1$.

For simplicity of notation, we let D_n stand for $\det \mathbb{C}^1(n)$. Using Lemma 3.1 we have $D_1 = 1, D_2 = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$, and for $n \geq 3$ we obtain the following recurrence relation

$$D_n = (-1)^{n+1} D_{n-1} - D_{n-2},$$

for $n \geq 3$. By substitution $D_n = (-1)^{\frac{n}{2}(n-1)} d_n$ we get

$$\begin{aligned} (-1)^{\frac{n+2}{2}(n+1)} d_{n+2} &= (-1)^{n+1} (-1)^{\frac{n+1}{2}n} d_{n+1} - (-1)^{\frac{n}{2}(n-1)} d_n, \\ d_{n+2} &= (-1)^{n+1+(n+1)\frac{n}{2}+(n+2)\frac{n+1}{2}} d_{n+1}, \\ &\quad - (-1)^{\frac{n}{2}(n-1)+(n+2)\frac{n+1}{2}} d_n, \\ d_{n+2} &= (-1)^{(n+1)(n+2)} d_{n+1} + (-1)^{(n-1)(n+2)} d_n. \end{aligned}$$

As $(n + 1)(n + 2)$ and $(n - 1)(n + 2)$ are even for any positive integer n we obtain the following recurrence for d_n

$$d_{n+2} = d_{n+1} + d_n, \quad d_0 = 1, \quad d_1 = 1.$$

This is recurrence (1) for Fibonacci numbers with shifted initial conditions one place to the right, hence $d_n = F_{n+1}$. Thus, we finally obtain

$$\det \mathbb{C}^1(n) = (-1)^{\frac{n}{2}(n-1)} F_{n+1}. \tag{8}$$

Therefore identity (4) holds.

Case $\delta = 0$.

We easily obtain $\det \mathbb{C}^0(1) = 1 = F_2, \det \mathbb{C}^0(2) = -2 = -F_3$, and for $n \geq 3$ using cofactor expansion on the first row and subsequently on the first column of $\det \mathbb{C}^1(n + 1)$ we get the following

$$\det \begin{pmatrix} (-1)^2 & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & (-1)^3 & 1 & 0 & 0 & \dots & \vdots \\ 0 & 1 & (-1)^4 & 1 & 0 & \ddots & 0 \\ 0 & 0 & 1 & (-1)^5 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 & (-1)^{n+1} & 1 \\ 0 & 0 & \dots & 0 & 0 & 1 & (-1)^{n+2} \end{pmatrix}$$

$$\begin{aligned}
 &= \det \mathbb{C}^0(n) \\
 &\quad + (-1)^3 \det \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & (-1)^4 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & (-1)^5 & 0 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ 0 & & 0 & 1 & (-1)^n & 1 & 0 \\ \vdots & & & 0 & 1 & (-1)^{n+1} & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & (-1)^{n+2} \end{pmatrix} \\
 &= \det \mathbb{C}^0(n) - \det \begin{pmatrix} (-1)^4 & 1 & 0 & 0 & \cdots & 0 \\ 1 & (-1)^5 & 1 & 0 & \ddots & \vdots \\ 0 & 1 & (-1)^6 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & 1 & (-1)^{n+1} & 1 \\ 0 & 0 & \cdots & 0 & 1 & (-1)^{n+2} \end{pmatrix}.
 \end{aligned}$$

Thus we get the following recurrence

$$\det \mathbb{C}^1(n + 1) = \det \mathbb{C}^0(n) - \det \mathbb{C}^1(n - 1),$$

which implies

$$\det \mathbb{C}^0(n) = \det \mathbb{C}^1(n + 1) + \det \mathbb{C}^1(n - 1)$$

for $n \geq 3$. Using identity (8) we obtain

$$\begin{aligned}
 \det \mathbb{C}^0(n) &= \det \mathbb{C}^1(n + 1) + \det \mathbb{C}^1(n - 1) \\
 &= (-1)^{\frac{n-1}{2}(n-2)} F_n + (-1)^{\frac{n+1}{2}n} F_{n+2} \\
 &= (-1)^{\frac{n}{2}(n+1)} (F_{n+2} + (-1)^{1-2n} F_n) \\
 &= (-1)^{\frac{n}{2}(n+1)} (F_{n+2} - F_n) \\
 &= (-1)^{\frac{n}{2}(n+1)} F_{n+1}.
 \end{aligned}$$

■

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