# From triangles to a square

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### Abstract

We find and classify triangles which can be made squares by cutting them into four or less pieces.

**AMS subject classification:** 97G40.
**Keywords:** polygon, quadrangle, square, triangle.

## 1. Introduction

It is well known that for any two polygons of the same area we can cut one of them into some finite pieces of polygons and arrange them appropriately to be congruent with the other. One of the most interesting couple is an equilateral triangle and a square (e.g., see the references [1, 2]). In fact, by cutting the equilateral triangle into four pieces of polygons (in fact, into three quadrangles and one triangle) we can arrange them to make a square: Let $\triangle ABC$ be a equilateral triangle with area $S$ and $M$, $N$ be the midpoints of the line segments $AB$, $AC$, respectively. Then we can choose two points $P$ and $Q$ on the line segment $BC$ such that

$$\overline{NP} = \sqrt{S}, \tag{1.1}$$

$$\overline{PQ} = \frac{1}{2}\overline{BC}. \tag{1.2}$$

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Then we can choose $R$ and $T$ on the line segment $\overline{NP}$ such that

$$\overline{QT} \perp \overline{NP}, \tag{1.3}$$

$$\overline{MR} \perp \overline{NP}. \tag{1.4}$$

If we cut the line segments $\overline{NP}$, $\overline{MR}$ and $\overline{QT}$, then the triangle $\triangle ABC$ is divided into three quadrangles and one triangle. Rotating $\square BPRM$ with center $M$ and making the vertex $B$ at one with the vertex $A$, and similarly, rotating $\square CNTQ$ with center $N$ and making the vertex $C$ at one with the vertex $A$, we obtain a pentagon with a corner cut away from a rectangle. Finally, filling its corner with the remaining piece $\triangle PQT$ we obtain a rectangle. It is easy to check that $\overline{NP}$ is the length of a side of the rectangle. Thus, the obtained rectangle is a square.

We can see that the above method works for the triangles such that there exist $P, Q, R, T$ satisfying the conditions $(1.1) \sim (1.4)$. We call such a triangle *a mild triangle* (see Fig 2.1). One of the authors gave a lecture in the Science Education Institute of Gifted in his University dealing with the above subject and actually allowed the students to make cubes from various triangles of their own choice by using above method. Because of bad choice of triangles, some students failed to make squares with the triangles. Considering the problem, the authors realized that an elementary approach characterizes the triangles which can be converted to squares by cutting it into fewer than four pieces of polygons (triangles and quadrangles). In this note we characterize the various mild triangles and how to cut them to make squares.

## 2. Main Results

We locate $\triangle ABC$ in the rectangular coordinate. Without loss of generality we may fix $\overline{BC} = 2$ and let $B = (0, 0)$, $C = (2, 0)$. Then we find the region $D \subset \mathbb{R}^2$ of all points of the remaining vertex $A$ so that $\triangle ABC$ satisfies the condition; there exist $P, Q \in \overline{BC}$, $R, T \in \overline{PN}$ satisfying $(1.1) \sim (1.4)$.
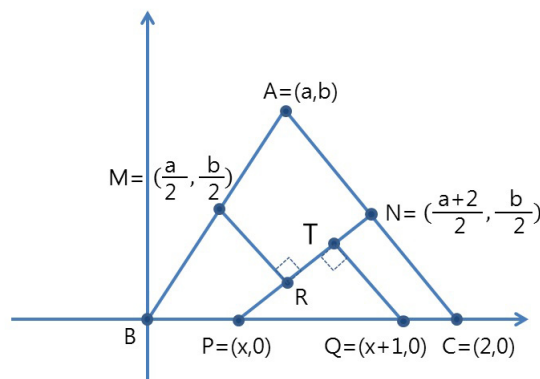


Figure 2.1:

Let $A = (a, b)$, $b > 0$ and $P = (x, 0)$, $Q(x + 1, 0)$. Then from the condition (1.1), the algebraic equation

$$\overline{NP} = \sqrt{\left(x - \frac{a + 2}{2}\right)^2 + \frac{b^2}{4}} = \sqrt{S} = \sqrt{b} \tag{2.1}$$

has a solution $x \in \{x : 0 \le x \le 2\}$. Thus, we have

$$0 < b \le 4. \tag{2.2}$$

Note that if $b = 4$, then $\overline{NP} \perp \overline{BC}$. From conditions (1.2) and (1.3) we have

$$0 \le x \le 1, \ x \le \frac{a + 2}{2}. \tag{2.3}$$

Thus, in view of (2.1) and (2.3) we have $x = \frac{1}{2}(a + 2 - \sqrt{4b - b^2})$ and

$$0 \le \frac{1}{2}(a + 2 - \sqrt{4b - b^2}) \le 1. \tag{2.4}$$

Solving (2.4) we have

$$-2 \le a < 0, \ (a + 2)^2 + (b - 2)^2 \ge 4, \tag{2.5}$$

or

$$a \ge 0, \ a^2 + (b - 2)^2 \le 4. \tag{2.6}$$

Note that if $a + 2)^2 + (b - 2)^2 = 4$ in (2.5), then $P = (0, 0)$, and if $a^2 + (b - 2)^2 = 4$ in (2.6), then $P = (1, 0)$. Let $R = (r_1, r_2)$, $T = (t_1, t_2)$. Then condition (1.3) is equivalent to

$$t_1 \le \frac{a + 2}{2}, \tag{2.7}$$

and condition (1.4) is equivalent to

$$r_1 \ge \frac{1}{2}(a + 2 - \sqrt{4b - b^2}). \tag{2.8}$$

Note that the slope $m$ of $\overline{NP}$ is given by

$$m = \frac{b}{\sqrt{4b - b^2}}.$$

Now, condition (2.7) is equivalent to that the straight line

$$y = -\frac{\sqrt{4b - b^2}}{b}\left(x - \frac{a + 2}{2}\right) + \frac{b}{2}$$

intercepts $x$-axis in the range $x \geq \frac{1}{2}(a + 4 - \sqrt{4b - b^2})$, i.e.,

$$\frac{1}{2}\left(a + 2 + \frac{b^2}{\sqrt{4b - b^2}}\right) \geq \frac{1}{2}(a + 4 - \sqrt{4b - b^2}). \tag{2.9}$$

Solving (2.9) we have

$$b \geq \frac{4}{5}. \tag{2.10}$$

Similarly, condition (2.8) is equivalent to that the straight line

$$y = -\frac{\sqrt{4b - b^2}}{b}\left(x - \frac{a}{2}\right) + \frac{b}{2}$$

intercepts $x$-axis in the range $x \geq \frac{1}{2}(a + 2 - \sqrt{4b - b^2})$, i.e.,

$$\frac{1}{2}\left(a + \frac{b^2}{\sqrt{4b - b^2}}\right) \geq \frac{1}{2}(a + 2 - \sqrt{4b - b^2}). \tag{2.11}$$

Solving (2.11) we have (2.10). Note that $R = P$ and $S = N$ if $b = \frac{4}{5}$.

From (2.2), (2.5), (2.6) and (2.10) the region $D$ is given by

$$D = \{(a, b) : -2 \leq a < 0, \ \frac{4}{5} \leq b \leq 4, \ (a + 2)^2 + (b - 2)^2 \geq 4\}$$

$$\cup \{(a, b) : a \geq 0, \ \frac{4}{5} \leq b \leq 4, \ a^2 + (b - 2)^2 \leq 4\}.$$
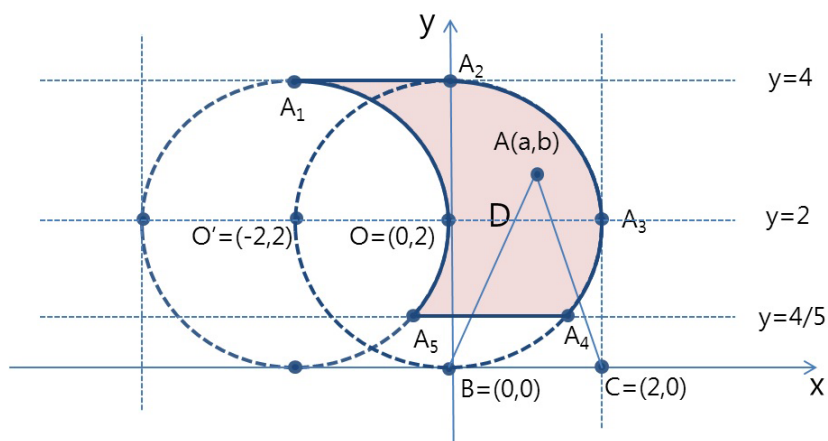


Figure 2.2:

If $A = (a, b)$ is in the interior of $D$, then the points $B, C, M, N, P, Q, R, T$ ($R = T$ when $b = 2$) are distinct. Thus, the triangle $\triangle ABC$ is divided into three quadrangles and one triangle which make a square. If $A$ is in the boundary of $D$, partitions of some different shapes make a square:
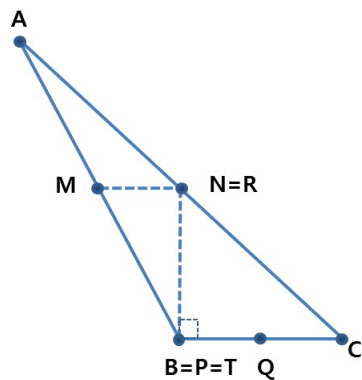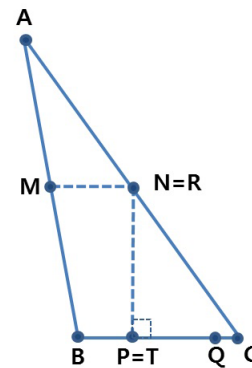
Figure 2.3: $A = A_1$.

Figure 2.4: $A \in \overline{A_1 A_2}$.

1. If $A = A_1$, then $P = T = B$, $R = N$, $Q = (1,0)$ and $\triangle PQT = \emptyset$. Thus, $\triangle ABC$ is divided into three triangles. From now on, we denote by $\triangle ABC = 3Tr$ for short.

2. If $A \in \overline{A_1 A_2}$, then $P = T$, $R = N$ and $\triangle PQT = \emptyset$ ($\triangle ABC = 2Tr + Qu$).
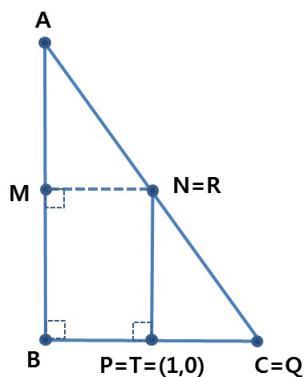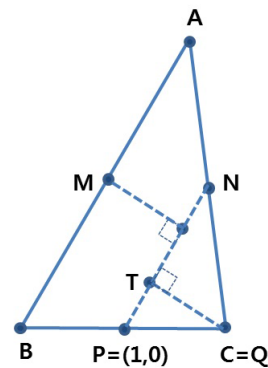
Figure 2.5: $A = A_1$.

Figure 2.6: $A \in \widehat{A_2 A_3}$

3. If $A = A_2$, then $R = N$, $S = P = (1,0)$, $Q = C$ and $\triangle PQT = \emptyset$ ($\triangle ABC = 2Tr + Qu = Tr + Qu$).

4. If $A \in \widehat{A_2 A_3}$, then $P = (1,0)$, $Q = C$ ($\triangle ABC = 2Tr + 2Qu$).
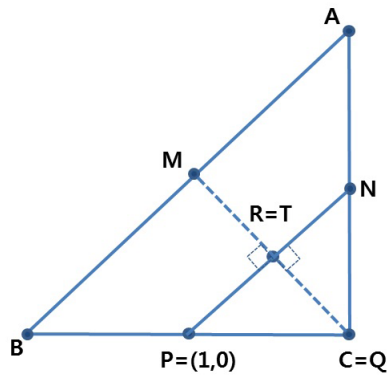
Figure 2.7: $A = A_3$.
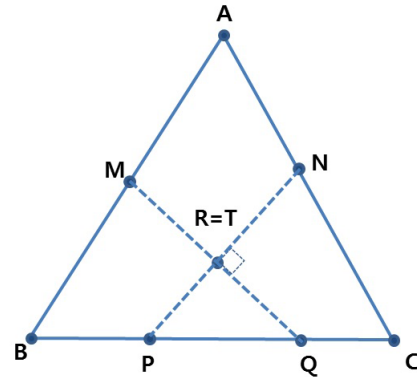


Figure 2.8: $A \in \overline{OA_3}$

5. If $A = A_3$, then $P = (1, 0)$, $Q = C$, $R = T$ ($\triangle ABC = 2Tr + 2Qu = 2Tr$).

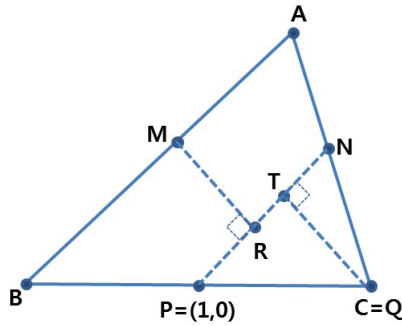6. If $A \in \overline{OA_3}$, then $R = T$ ($\triangle ABC = Tr + 3Qu$).
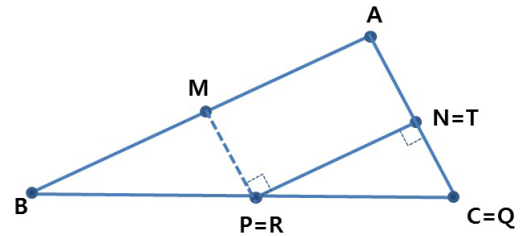


Figure 2.9: $A \in \overset{\frown}{A_3A_4}$.



Figure 2.10: $A = A_4$

7. If $A \in \overset{\frown}{A_3A_4}$, then $P = (1, 0)$, $Q = C$ ($\triangle ABC = 2Tr + 2Qu$).

8. If $A = A_4$, then $R = P$, $C = Q$, $N = T$ and $\square CNTQ = \emptyset$ ($\triangle ABC = 2Tr + Qu$).
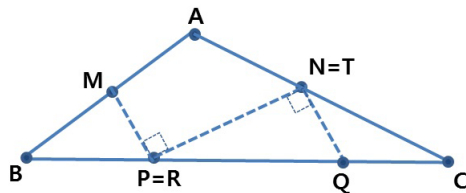


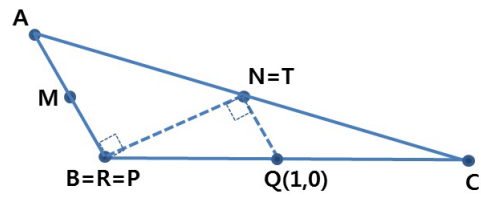Figure 2.11: $A \in \overset{\frown}{A_4A_5}$.



Figure 2.12: $A = A_5$.

9. If $A \in \overline{A_4 A_5}$,, then $R = P$, $T = N$ $(\triangle ABC = 3Tr + Qu)$.

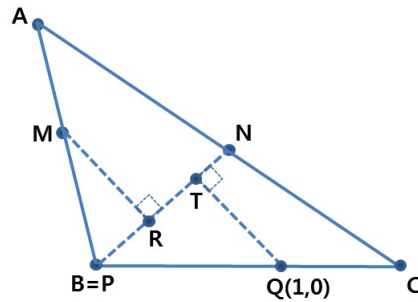10. If $A = A_5$, then $R = P = B$, $Q = (1, 0)$, $T = N$ and $\square BPRM = \emptyset$ $(\triangle ABC = 3Tr)$.
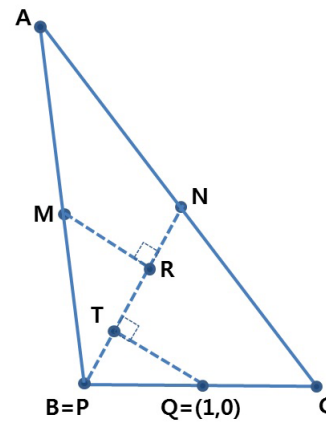


Figure 2.13: $A \in \widehat{O} A_5$.



Figure 2.14: $A \in \widehat{A_1} O$.

11. If $A \in \overline{OA_5}$, then $B = P$, $Q = (1, 0)$ $(\triangle ABC = 2Tr + 2Qu)$.

12. If $A \in \overline{A_1 O}$, then $B = P$, $Q = (1, 0)$ $(\triangle ABC = 2Tr + 2Qu)$.

## Acknowledgment

## References

[1] http://demonstrations.wolfram.com/EquilateralTriangleToSquare/

[2] http://www.instructables.com/id/How-to-make-a-paper-triangle-from-a-square/