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Exact solution of the four velocity Broadwell model

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Abstract

We prove the existence, the boundedness and the uniqueness of the solutions of the initial-boundary value problem for the four velocity Broadwell discrete model of gas, and compute the exact analytic solution for a flow between two moving plates.

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1. Introduction

Discrete models of gas are simplified models of the Boltzmann equation. Since the pioneering works of Carleman and Broadwell [1, 2] and the extension to the general p velocity model by Gatignol ($p \in \mathbb{N}^*$), remarkable progress has been achieved. An important part of the literature devoted to the mathematical aspect of the study of discrete velocity models concern the proof of the global existence of the solution of the initial value problem sometimes with extended conditions [3, 8, 9, 10, 11, 12, 14, 15] and some authors focused on the existence of the solution of the pure boundary value problem [4, 6, 7] in the steady case. The kinetic equations are non linear and difficult to solve analytically. However some authors succeeded in building exact solutions for particular models.

In this work, in order to study the transition of unsteady flows to steady states, we extend a method used in the steady case in [4, 5, 6, 7] to study the existence of solution of the boundary value problem for the general p velocity discrete model to prove the existence, the uniqueness and the boundedness of the solution of the initial-boundary value problem for the four velocity discrete Broadwell model. In the course of the proof we find that, in the unsteady one dimensional case in contrast to the steady case, the method turns to be a genuine method of construction of exact solution.

The paper is organized as follows. In section 2 we briefly describe the Broadwell model, state the initial-boundary value problem and present the main result of the paper which is proved in section 3. In section 4 we apply the results to the resolution of the plane Couette flow problem.

2. Statement of the problem

We consider the unsteady flow of a gas between two parallel infinite and moving plates and the mass and heat transfers with the boundaries. The modelling of this problem of gas dynamics by means of discrete velocity models in the spatially one dimension case leads to a two points initial-boundary value problem.

We choose the origin O of the orthonormal reference $(O, \vec{e}_1, \vec{e}_2)$ of $\mathbb{R}^2$ so that the plates are located in the planes $y = 0$ and $y = l$, $l > 0$. The velocities of the Broadwell model in the basis $(\vec{e}_1, \vec{e}_2)$ are: $\vec{u}_1 = c(-1, 1)$, $\vec{u}_2 = c(1, 1)$, $\vec{u}_3 = c(1, -1)$, $\vec{u}_4 = c(-1, -1)$, $c > 0$. We denote by $N_i(t', x', y')$ the number density of particles of velocity $\vec{u}_i$ in point $M(x', y')$ at time t'. The N_i are continuous functions of t', x' and y'. When we assume that the flow is one dimensional depending only upon the spatial variable y' and the time t' the initial-boundary value problem has the form:

$$\left\{\begin{array}{lcl} \dfrac{\partial N_1}{\partial t'} + c\dfrac{\partial N_1}{\partial y'} &=& 2\sqrt{2}cS\,(N_2N_4 - N_1N_3) = Q(N) \\ \dfrac{\partial N_2}{\partial t'} + c\dfrac{\partial N_2}{\partial y'} &=& -Q(N) \\ \dfrac{\partial N_3}{\partial t'} - c\dfrac{\partial N_3}{\partial y'} &=& Q(N) \\ \dfrac{\partial N_4}{\partial t'} - c\dfrac{\partial N_4}{\partial y'} &=& -Q(N) \\ N_i(0, y') &=& f_i(y'), \quad i = 1,\ 2,\ 3,\ 4 \\ N_i(t', 0) &=& a_i(t'), \quad i = 1,\ 2 \\ N_i(t', l) &=& b_i(t'), \quad i = 3,\ 4 \end{array}\right. \tag{2.1}$$

The functions a_i, b_j and f_k, $i \in \{1,\ 2\}$, $j \in \{3,\ 4\}$, $k \in \{1,\ 2,\ 3,\ 4\}$ are non negative. The main result of the paper is the following:

Theorem 2.1. The problem (2.1) has unique and bounded solution $N = (N_1,\ N_2,\ N_3,\ N_4)$ for bounded initial and boundary data a_i, b_j and f_k, $i \in \{1,\ 2\}$, $j \in \{3,\ 4\}$, $k \in \{1,\ 2,\ 3,\ 4\}$.

3. Existence and boundedness of the solution

In order to prove the existence we put the problem (2.1) in a suitable equivalent form. We first perform the linear bijective change of variables $\mathcal{T} : (t', y') \mapsto (\tau, \eta)$ so that

$\tau = y' + ct'$ and $\eta = y' - ct'$. The system (2.1) becomes:

$$\begin{cases} 2c\dfrac{\partial \widetilde{N}_1}{\partial \underset{\sim}{\tau}} = \widetilde{Q}\left(\widetilde{N}\right) \\ 2c\dfrac{\partial \widetilde{N}_2}{\partial \underset{\sim}{\tau}} = -\widetilde{Q}\left(\widetilde{N}\right) \\ 2c\dfrac{\partial \widetilde{N}_3}{\partial \underset{\sim}{\eta}} = -\widetilde{Q}\left(\widetilde{N}\right) \\ 2c\dfrac{\partial \widetilde{N}_4}{\partial \eta} = \widetilde{Q}\left(\widetilde{N}\right) \\ \widetilde{N}_i\left(\dfrac{\tau+\eta}{2}, \dfrac{\tau+\eta}{2}\right) = \widetilde{f}_i\left(\dfrac{\tau+\eta}{2}\right), \quad i = 1, 2, 3, 4 \\ \widetilde{N}_i\left(\dfrac{\tau-\eta}{2}, \dfrac{\eta-\tau}{2}\right) = \widetilde{a}_i\left(\dfrac{\tau-\eta}{2}\right), \quad i = 1, 2 \\ \widetilde{N}_i\left(l + \dfrac{\tau-\eta}{2}, l - \dfrac{\tau-\eta}{2}\right) = \widetilde{b}_i\left(\dfrac{\tau-\eta}{2}\right), \quad i = 3, 4 \end{cases} \tag{3.2}$$

The quantity $\widetilde{\Theta}$ is the expression of the quantity Θ after the change of variables.

We then put: $\widetilde{N} = (\widetilde{N}_1, \widetilde{N}_2, \widetilde{N}_3, \widetilde{N}_4)$, $\widetilde{\rho}^+\left(\widetilde{N}\right) = \widetilde{N}_1 + \widetilde{N}_2$ and $\widetilde{\rho}^-\left(\widetilde{N}\right) = \widetilde{N}_3 + \widetilde{N}_4$ and consider for σ the following problem:

$$\begin{cases} 2c\dfrac{\partial \widetilde{N}_1}{\partial \underset{\sim}{\tau}} + \sigma\widetilde{N}_1\widetilde{\rho}^+\left(\widetilde{N}\right) = \widetilde{Q}\left(\widetilde{N}\right) + \sigma\widetilde{N}_1\widetilde{\rho}^+\left(\widetilde{N}\right) = \widetilde{Q}_1^\sigma\left(\widetilde{N}\right) \\ 2c\dfrac{\partial \widetilde{N}_2}{\partial \underset{\sim}{\tau}} + \sigma\widetilde{N}_2\widetilde{\rho}^+\left(\widetilde{N}\right) = -\widetilde{Q}\left(\widetilde{N}\right) + \sigma\widetilde{N}_2\widetilde{\rho}^+\left(\widetilde{N}\right) = \widetilde{Q}_2^\sigma\left(\widetilde{N}\right) \\ 2c\dfrac{\partial \widetilde{N}_3}{\partial \underset{\sim}{\eta}} + \sigma\widetilde{N}_3\widetilde{\rho}^-\left(\widetilde{N}\right) = -\widetilde{Q}\left(\widetilde{N}\right) + \sigma\widetilde{N}_3\widetilde{\rho}^-\left(\widetilde{N}\right) = \widetilde{Q}_3^\sigma\left(\widetilde{N}\right) \\ 2c\dfrac{\partial \widetilde{N}_4}{\partial \eta} + \sigma\widetilde{N}_4\widetilde{\rho}^-\left(\widetilde{N}\right) = \widetilde{Q}\left(\widetilde{N}\right) + \sigma\widetilde{N}_4\widetilde{\rho}^-\left(\widetilde{N}\right) = \widetilde{Q}_4^\sigma\left(\widetilde{N}\right) \\ \widetilde{N}_i\left(\dfrac{\tau+\eta}{2}, \dfrac{\tau+\eta}{2}\right) = \widetilde{f}_i\left(\dfrac{\tau+\eta}{2}\right), \quad i = 1, 2, 3, 4 \\ \widetilde{N}_i\left(\dfrac{\tau-\eta}{2}, \dfrac{\eta-\tau}{2}\right) = \widetilde{a}_i\left(\dfrac{\tau-\eta}{2}\right), \quad i = 1, 2 \\ \widetilde{N}_i\left(l + \dfrac{\tau-\eta}{2}, l - \dfrac{\tau-\eta}{2}\right) = \widetilde{b}_i\left(\dfrac{\tau-\eta}{2}\right), \quad i = 3, 4 \end{cases} \tag{3.3}$$

Proposition 3.1. The problem (3.3) is equivalent to problem (3.2).

Proof. The system (3.3) is obtained from system (3.2) by adding $\sigma\widetilde{N}_i\widetilde{\rho}^{\pm}\left(\widetilde{N}\right)$ to the two members of the kinetic equation for $\widetilde{N}_i$. Moreover the transformation $\mathcal{T}$ is a diffeomorphism and the proof is complete. ■

3.1. Existence of solutions of (3.3)

For $T > 0$ and $l > 0$, if $(t, y) \in [0, T] \times [0, l]$ then $(\tau, \eta) \in [0, l+cT] \times [-cT, l-cT]$. Let $J = [0, l + cT] \times [-cT, l - cT]$.

We denote by $\mathcal{C}$ the set of continuous functions defined on J, and by $\mathcal{C}_+$ its subset of non negative functions. $\mathcal{C}^4$ and $\mathcal{C}^4_+$ respectively denote their cartesian products.

We introduce the following norms:

If $x = (\tau, \eta) \in J$ and $M = (M_1, \cdots, M_4) \in \mathcal{C}^4$ then $\|x\| = |\tau| + |\eta|$, $\|M_i\| = \sup\limits_{\|x\|<l+|V_i|T} |M_i(x)|$ and $\|M\| = \sup\limits_{i\in\wedge} \|M_i\|$, with $\wedge = \{1, 2, 3, 4\}$.

Theorem 3.2. The problem (3.3) has a solution which belongs to $\mathcal{C}^4_+$ for sufficiently large σ.

For the proof, consider for $M \in \mathcal{C}^4_+$, the following initial and boundary value problem:

$$\begin{cases} 2c\dfrac{\partial \widetilde{N}_1}{\partial \tau} + \sigma \widetilde{N}_1 \widetilde{\rho}^+(M) & = & \widetilde{Q}(M) + \sigma M_1 \widetilde{\rho}^+(M) = \widetilde{Q}^\sigma_1(M) \\ 2c\dfrac{\partial \widetilde{N}_2}{\partial \tau} + \sigma \widetilde{N}_2 \widetilde{\rho}^+(M) & = & -\widetilde{Q}(M) + \sigma M_2 \widetilde{\rho}^+(M) = \widetilde{Q}^\sigma_2(M) \\ 2c\dfrac{\partial \widetilde{N}_3}{\partial \eta} + \sigma \widetilde{N}_3 \widetilde{\rho}^-(M) & = & -\widetilde{Q}(M) + \sigma M_3 \widetilde{\rho}^-(M) = \widetilde{Q}^\sigma_3(M) \\ 2c\dfrac{\partial \widetilde{N}_4}{\partial \eta} + \sigma \widetilde{N}_4 \widetilde{\rho}^-(M) & = & \widetilde{Q}(M) + \sigma M_4 \widetilde{\rho}^-(M) = \widetilde{Q}^\sigma_4(M) \\ \widetilde{N}_i\left(\dfrac{\tau+\eta}{2}, \dfrac{\tau+\eta}{2}\right) & = & \widetilde{f}_i\left(\dfrac{\tau+\eta}{2}\right), \quad i = 1, 2, 3, 4 \\ \widetilde{N}_i\left(\dfrac{\tau-\eta}{2}, \dfrac{\eta-\tau}{2}\right) & = & \widetilde{a}_i\left(\dfrac{\tau-\eta}{2}\right), \quad i = 1, 2 \\ \widetilde{N}_i\left(l + \dfrac{\tau-\eta}{2}, l - \dfrac{\tau-\eta}{2}\right) & = & \widetilde{b}_i\left(\dfrac{\tau-\eta}{2}\right), \quad i = 3, 4 \end{cases} \tag{3.4}$$

Lemma 3.3. The problem (3.4) has for given $M \in \mathcal{C}^4_+$ an unique solution which belongs to $\mathcal{C}^4_+$ for sufficiently large σ.

Proof. The problem (3.4) is a linear problem associated with (3.3) and it is solved by splitting it into the two following mild problems:

$$\begin{cases} 2c\dfrac{\partial \widetilde{N}_1}{\partial \tau} + \sigma \widetilde{N}_1 \widetilde{\rho}^+(M) & = & \widetilde{Q}^\sigma_1(M) \\ 2c\dfrac{\partial \widetilde{N}_2}{\partial \tau} + \sigma \widetilde{N}_2 \widetilde{\rho}^+(M) & = & \widetilde{Q}^\sigma_2(M) \\ \widetilde{N}_i\left(\dfrac{\tau+\eta}{2}, \dfrac{\tau+\eta}{2}\right) & = & \widetilde{f}_i\left(\dfrac{\tau+\eta}{2}\right), \quad i = 1, 2 \\ \widetilde{N}_i\left(\dfrac{\tau-\eta}{2}, \dfrac{\eta-\tau}{2}\right) & = & \widetilde{a}_i\left(\dfrac{\tau-\eta}{2}\right), \quad i = 1, 2 \end{cases}$$

and

$$\begin{cases} 2c\dfrac{\partial \widetilde{N}_3}{\partial \eta} + \sigma \widetilde{N}_3 \widetilde{\rho}^-(M) & = & \widetilde{Q}_3^\sigma(M) \\ 2c\dfrac{\partial \widetilde{N}_4}{\partial \eta} + \sigma \widetilde{N}_4 \widetilde{\rho}^-(M) & = & \widetilde{Q}_4^\sigma(M) \\ \widetilde{N}_i\left(\dfrac{\tau+\eta}{2}, \dfrac{\tau+\eta}{2}\right) & = & \widetilde{f}_i\left(\dfrac{\tau+\eta}{2}\right), \quad i = 3,\ 4 \\ \widetilde{N}_i\left(l + \dfrac{\tau-\eta}{2}, l - \dfrac{\tau-\eta}{2}\right) & = & \widetilde{b}_i\left(\dfrac{\tau-\eta}{2}\right), \quad i = 3,\ 4 \end{cases}$$

The unique solution of (3.4) is given by:

$$\begin{cases} \widetilde{N}_i(\tau,\eta) & = & \widetilde{a}_i\left(\dfrac{\tau-\eta}{2}\right) g^+(\tau,\eta) \\ & & +\dfrac{1}{2c}\displaystyle\int_{\frac{\tau-\eta}{2}}^{\tau} \widetilde{Q}_i^\sigma(M)(s,\eta) g^+(\tau-s,\eta)\mathrm{d}s,\ i = 1,\ 2 \\ \widetilde{N}_i(\tau,\eta) & = & \widetilde{b}_i\left(\dfrac{\tau-\eta}{2}\right) g^-(\tau,\eta) \\ & & +\dfrac{1}{2c}\displaystyle\int_{l-\frac{\tau-\eta}{2}}^{\eta} \widetilde{Q}_i^\sigma(M)(\tau,s) g^-(\tau,\eta-s)\mathrm{d}s,\ i = 3,\ 4 \end{cases} \tag{3.5}$$

with

$$g^+(\tau,\eta) = \exp\left(-\frac{\sigma}{2c}\int_{\frac{\tau-\eta}{2}}^{\tau} \widetilde{\rho}^+(M)(a,\eta)\mathrm{d}a\right)$$

and

$$g^-(\tau,\eta) = \exp\left(\frac{\sigma}{2c}\int_{l-\frac{\tau-\eta}{2}}^{\eta} \widetilde{\rho}^-(M)(\tau,a)\mathrm{d}a\right).$$

For sufficiently large σ, $\widetilde{Q}_i^\sigma$ is positive $\forall i \in \{1,\ 2,\ 3,\ 4\}$, hence $\widetilde{N}_i(\tau,\eta) > 0$, $\forall(\tau,\eta) \in J$ and $\widetilde{N} \in \mathcal{C}_+^4$. ■

Thus the operator $\mathcal{T}$ defined by $\mathcal{T}(M) = \widetilde{N}$ where $\widetilde{N}$ is the unique solution of (3.4) is well defined and satisfies:

Lemma 3.4. $\mathcal{T}$ is continuous and compact on J.

Proof. We have $\mathcal{T}(M) = \widetilde{N}$ if and only if $\widetilde{N}$ is given by the relations (3.5) from which we deduce:

$$\begin{cases} \left|\widetilde{N}_i(\tau,\eta)\right| & \leqslant \left|\widetilde{a}_i\left(\dfrac{\tau-\eta}{2}\right)\right|\left|g^+(\tau,\eta)\right| \\ & \quad +\dfrac{1}{2c}\left|\displaystyle\int_{\frac{\tau-\eta}{2}}^{\tau}\widetilde{Q}_i^{\sigma}(M)(s,\eta)g^+(\tau-s,\eta)\mathrm{d}s\right|,\ i=1,\ 2 \\ \left|\widetilde{N}_i(\tau,\eta)\right| & \leqslant \left|\widetilde{b}_i\left(\dfrac{\tau-\eta}{2}\right)\right|\left|g^-(\tau,\eta)\right| \\ & \quad +\dfrac{1}{2c}\left|\displaystyle\int_{l-\frac{\tau-\eta}{2}}^{\tau}\widetilde{Q}_i^{\sigma}(M)(\tau,s)g^-(\tau,\eta-s)\mathrm{d}s\right|,\ i=3,\ 4 \end{cases}$$

Using the Generalized Mean Value Theorem, as g^+ and g^- are strictly positive functions, we can find $(\tau_0,\eta_0)\in\left]\dfrac{\tau-\eta}{2},\tau\right[\times\left]l-\dfrac{\tau-\eta}{2},\eta\right[$ such that

$$\begin{cases} \displaystyle\int_{\frac{\tau-\eta}{2}}^{\tau}\widetilde{Q}_i^{\sigma}(M)(s,\eta)g^+(\tau-s,\eta)\mathrm{d}s \\ \quad =\widetilde{Q}_i^{\sigma}(M)(\tau_0,\eta)\displaystyle\int_{\frac{\tau-\eta}{2}}^{\tau}g^+(\tau-s,\eta)\mathrm{d}s, \quad i=1,\ 2 \\ \displaystyle\int_{l-\frac{\tau-\eta}{2}}^{\eta}\widetilde{Q}_i^{\sigma}(M)(\tau,s)g^-(\tau,\eta-s)\mathrm{d}s \\ \quad =\widetilde{Q}_i^{\sigma}(M)(\tau,\eta_0)\displaystyle\int_{l-\frac{\tau-\eta}{2}}^{\eta}g^-(\tau,\eta-s)\mathrm{d}s, \quad i=3,\ 4 \end{cases}$$

Hence

$$\begin{cases} \left|\widetilde{N}_i(\tau,\eta)\right| & \leqslant \left|\widetilde{a}_i\left(\dfrac{\tau-\eta}{2}\right)\right|\left|g^+(\tau,\eta)\right| \\ & \quad +\dfrac{1}{2c}\left|\dfrac{\tau+\eta}{2}\right|\left|\widetilde{Q}_i^{\sigma}(M)(\tau_0,\eta)\right|\left|g^+(\tau,\eta)\right|,\ i=1,\ 2 \\ \left|\widetilde{N}_i(\tau,\eta)\right| & \leqslant \left|\widetilde{b}_i\left(\dfrac{\tau-\eta}{2}\right)\right|\left|g^-(\tau,\eta)\right| \\ & \quad +\dfrac{1}{2c}\left|\dfrac{\tau+\eta}{2}-l\right|\left|\widetilde{Q}_i^{\sigma}(M)(\tau,\eta_0)\right|\left|g^-(\tau,\eta)\right|,\ i=3,\ 4 \end{cases}$$

and

$$\begin{cases} \left|\widetilde{N}_i(\tau,\eta)\right| \leqslant \left|\widetilde{a}_i\left(\dfrac{\tau-\eta}{2}\right)\right|+\dfrac{1}{2c}\left|\dfrac{\tau+\eta}{2}\right|\left|\widetilde{Q}_i^{\sigma}(M)(\tau_0,\eta)\right|, \quad i=1,\ 2 \\ \left|\widetilde{N}_i(\tau,\eta)\right| \leqslant \left|\widetilde{b}_i\left(\dfrac{\tau-\eta}{2}\right)\right|+\dfrac{1}{2c}\left|\dfrac{\tau+\eta}{2}-l\right|\left|\widetilde{Q}_i^{\sigma}(M)(\tau,\eta_0)\right|, \quad i=3,\ 4 \end{cases}$$

since $\left|g^+(\tau,\eta)\right|<1$ and $\left|g^-(\tau,\eta)\right|<1$.

From which we infer

$$\|\mathcal{T}(M)\| \leq \max\left(\|\widetilde{a}_i\|, \|\widetilde{b}_i\|\right) + \frac{l}{2c}\|\widetilde{Q}_i^\sigma(M)\|$$

Thus $\mathcal{T}$ is continuous and bounded since $\widetilde{a}_i$, $\widetilde{b}_i$ and $\widetilde{Q}_i^\sigma$, $i \in \Lambda$ are bounded. Hence if M is bounded then $\widetilde{N} = \mathcal{T}(M)$ is bounded since $\|\mathcal{T}(M)\| \leq \|\mathcal{T}\| \cdot \|M\|$.

Otherwise if $\widetilde{N}$ is the solution of (3.4) then $\forall\, i \in \{1,\ 2,\ 3,\ 4\}$,

$$\left\{\begin{array}{rcl} 2c\dfrac{\partial \widetilde{N}_i}{\partial \tau} + \sigma \widetilde{N}_i \widetilde{\rho}^+(M) &=& \widetilde{Q}_i^\sigma(M),\ i = 1,\ 2 \\ 2c\dfrac{\partial \widetilde{N}_i}{\partial \eta} + \sigma \widetilde{N}_i \widetilde{\rho}^-(M) &=& \widetilde{Q}_i^\sigma(M),\ i = 3,\ 4 \end{array}\right.$$

Thus

$$\left\{\begin{array}{rcl} 2c\dfrac{\partial \widetilde{N}_i}{\partial \tau} &=& \widetilde{Q}_i^\sigma(M) - \sigma \widetilde{N}_i \widetilde{\rho}^+(M),\ i = 1,\ 2 \\ -2c\dfrac{\partial \widetilde{N}_i}{\partial \eta} &=& \widetilde{Q}_i^\sigma(M) - \sigma \widetilde{N}_i \widetilde{\rho}^+(M),\ i = 3,\ 4 \end{array}\right.$$

and

$$\left\{\begin{array}{rcl} \left|\dfrac{\partial \widetilde{N}_i}{\partial \tau}\right| &\leqslant& \dfrac{1}{2c}\widetilde{Q}_i^\sigma(M) + \dfrac{\sigma}{2c}\widetilde{N}_i \widetilde{\rho}^+(M),\ i = 1,\ 2 \\ \left|\dfrac{\partial \widetilde{N}_i}{\partial \eta}\right| &\leqslant& \dfrac{1}{2c}\widetilde{Q}_i^\sigma(M) + \dfrac{\sigma}{2c}\widetilde{N}_i \widetilde{\rho}^+(M),\ i = 3,\ 4 \end{array}\right.$$

Thus if M is bounded, $\dfrac{\partial \widetilde{N}_i}{\partial \tau}$ and $\dfrac{\partial \widetilde{N}_i}{\partial \eta}$ are uniformly bounded and it exists α and β such that

$$\left|\frac{\partial \widetilde{N}_i}{\partial \tau}\right| < \alpha \text{ in } [0, l + cT] \quad \text{and} \quad \left|\frac{\partial \widetilde{N}_i}{\partial \eta}\right| < \beta \text{ in } [-cT, l - cT].$$

Given $x_1 = (\tau_1, \eta_1) \in J$ and $x_2 = (\tau_2, \eta_2) \in J$. We deduce from the Mean Value Theorem, that it exists $z_0 = (\tau_0, \eta_0) \in [x_1,\ x_2] \subset J$ such that

$$\widetilde{N}_i(x_1) - \widetilde{N}_i(x_2) = \mathrm{d}N_i(z_0)(x_1 - x_2)$$

with

$$[x_1,\ x_2] = \left\{z \in \mathbb{R}^2 / z = t(x_1 - x_2) + x_2, t \in [0, 1]\right\}$$

and

$$\mathrm{d}N_i(z_0)(h) = \frac{\partial \widetilde{N}_i}{\partial \tau}(z_0)h_1 + \frac{\partial \widetilde{N}_i}{\partial \eta}(z_0)h_2 \quad \forall h = (h_1, h_2) \in \mathbb{R}^2$$

.
Hence

$$\begin{aligned}\left|\widetilde{N}_i(x_1)-\widetilde{N}_i(x_2)\right| &= |dN_i(z_0)(x_1-x_2)| \\ &\le \|\mathrm{d}N_i(z_0)\|\,\|x_1-x_2\|\end{aligned}$$

with

$$\begin{aligned}\|\mathrm{d}N_i(z_0)\| &= \sup_{\|h\|\le 1}\frac{|\mathrm{d}N_i(z_0)|}{\|h\|} \\ &= \sup_{\|h\|\le 1}\frac{\left|\dfrac{\partial\widetilde{N}_i}{\partial\tau}(z_0)h_1+\dfrac{\partial\widetilde{N}_i}{\partial\eta}(z_0)h_2\right|}{|h_1|+|h_2|}\end{aligned}$$

But

$$\begin{aligned}\left|\frac{\partial\widetilde{N}_i}{\partial\tau}(z_0)h_1+\frac{\partial\widetilde{N}_i}{\partial\eta}(z_0)h_2\right| &\le \left|\frac{\partial\widetilde{N}_i}{\partial\tau}(z_0)\right||h_1|+\left|\frac{\partial\widetilde{N}_i}{\partial\eta}(z_0)\right||h_2| \\ &\le \max\left(\left|\frac{\partial\widetilde{N}_i}{\partial\tau}(z_0)\right|,\left|\frac{\partial\widetilde{N}_i}{\partial\eta}(z_0)\right|\right)(|h_1|+|h_2|)\end{aligned}$$

Thus

$$\begin{aligned}\frac{\left|\dfrac{\partial\widetilde{N}_i}{\partial\tau}(z_0)h_1+\dfrac{\partial\widetilde{N}_i}{\partial\eta}(z_0)h_2\right|}{|h_1|+|h_2|} &\le \max\left(\left|\frac{\partial\widetilde{N}_i}{\partial\tau}(z_0)\right|,\left|\frac{\partial\widetilde{N}_i}{\partial\eta}(z_0)\right|\right) \\ &\le \max(\alpha,\beta)\end{aligned}$$

That is $\|\mathrm{d}N_i(z_0)\| \le \max(\alpha,\beta)$. Then $\left|\widetilde{N}_i(x_1)-\widetilde{N}_i(x_2)\right| \le \max(\alpha,\beta)\|x_1-x_2\|$. It is sufficient that $\|x_1-x_2\| < \dfrac{\varepsilon}{\max(\alpha,\beta)}$ to have $\left|\widetilde{N}_i(x_1)-\widetilde{N}_i(x_2)\right| < \varepsilon$ for all $i \in \{1,\ 2,\ 3,\ 4\}$.

We prove that for all solution $\widetilde{N}$ of (3.4):

$$\forall\varepsilon>0, \exists\xi>0, \|x_1-x_2\|<\xi \Rightarrow \left|\widetilde{N}_i(x_1)-\widetilde{N}_i(x_2)\right|<\varepsilon, \quad \forall\, x_1,\ x_2\in J.$$

The set of the solutions of (3.4) is thus equicontinuous so $\mathcal{T}$ is compact on every bounded subset of $\mathcal{C}_+^4$. ■

Lemma 3.5. Every solution of the equation $\widetilde{N}=\lambda\mathcal{T}\left(\widetilde{N}\right)$, $0<\lambda<1$, is bounded.

Proof. $\widetilde{N}$ is a solution of $\widetilde{N} = \lambda \mathcal{T}\left(\widetilde{N}\right)$ if and only if

$$\left\{\begin{array}{lcll}
2c\dfrac{\partial \widetilde{N}_1}{\partial \tau} + \sigma \widetilde{N}_1 \widetilde{\rho}^+\left(\widetilde{N}\right) & = & \lambda \widetilde{Q}_1^{\sigma}\left(\widetilde{N}\right) & (3.6.1)\\
2c\dfrac{\partial \widetilde{N}_2}{\partial \tau} + \sigma \widetilde{N}_2 \widetilde{\rho}^+\left(\widetilde{N}\right) & = & \lambda \widetilde{Q}_2^{\sigma}\left(\widetilde{N}\right) & (3.6.2)\\
2c\dfrac{\partial \widetilde{N}_3}{\partial \eta} + \sigma \widetilde{N}_3 \widetilde{\rho}^-\left(\widetilde{N}\right) & = & \lambda \widetilde{Q}_3^{\sigma}\left(\widetilde{N}\right) & (3.6.3)\\
2c\dfrac{\partial \widetilde{N}_4}{\partial \eta} + \sigma \widetilde{N}_4 \widetilde{\rho}^-\left(\widetilde{N}\right) & = & \lambda \widetilde{Q}_4^{\sigma}\left(\widetilde{N}\right) & (3.6.4)\\
\widetilde{N}_i\left(\dfrac{\tau+\eta}{2}, \dfrac{\tau+\eta}{2}\right) & = & \lambda \widetilde{f}_i\left(\dfrac{\tau+\eta}{2}\right), \quad i = 1,\ 2,\ 3,\ 4 & (3.6.7)\\
\widetilde{N}_i\left(\dfrac{\tau-\eta}{2}, \dfrac{\eta-\tau}{2}\right) & = & \lambda \widetilde{a}_i\left(\dfrac{\tau-\eta}{2}\right), \quad i = 1,\ 2 & (3.6.8)\\
\widetilde{N}_i\left(l + \dfrac{\tau-\eta}{2}, l - \dfrac{\tau-\eta}{2}\right) & = & \lambda \widetilde{b}_i\left(\dfrac{\tau-\eta}{2}\right), \quad i = 3,\ 4 & (3.6.9)
\end{array}\right. \tag{3.6}$$

As the partial macroscopic densities $\widetilde{\rho}^+\left(\widetilde{N}\right)$ and $\widetilde{\rho}^-\left(\widetilde{N}\right)$ are conserved for the Broadwell model, making the sums (3.6.1) + (3.6.2) and (3.6.3) + (3.6.4), we obtain for their determination the following initial-boundary value problem:

$$\left\{\begin{array}{lcll}
2c\dfrac{\partial\left[\widetilde{\rho}^+\left(\widetilde{N}\right)\right]}{\partial \tau} + (1-\lambda)\sigma\left[\widetilde{\rho}^+\left(\widetilde{N}\right)\right]^2 & = & 0 & (3.7.1)\\
2c\dfrac{\partial\left[\widetilde{\rho}^-\left(\widetilde{N}\right)\right]}{\partial \eta} + (1-\lambda)\sigma\left[\widetilde{\rho}^-\left(\widetilde{N}\right)\right]^2 & = & 0 & (3.7.2)\\
\widetilde{\rho}^+\left(\widetilde{N}\right)\left(\dfrac{\tau+\eta}{2}, \dfrac{\tau+\eta}{2}\right) & = & \lambda\left(\widetilde{f}_1 + \widetilde{f}_2\right)\left(\dfrac{\tau+\eta}{2}\right) & (3.7.3)\\
\widetilde{\rho}^-\left(\widetilde{N}\right)\left(\dfrac{\tau+\eta}{2}, \dfrac{\tau+\eta}{2}\right) & = & \lambda\left(\widetilde{f}_3 + \widetilde{f}_4\right)\left(\dfrac{\tau+\eta}{2}\right) & (3.7.4)\\
\widetilde{\rho}^+\left(\widetilde{N}\right)\left(\dfrac{\tau-\eta}{2}, \dfrac{\eta-\tau}{2}\right) & = & \lambda\left(\widetilde{a}_1 + \widetilde{a}_2\right)\left(\dfrac{\tau-\eta}{2}\right) & (3.7.5)\\
\widetilde{\rho}^-\left(\widetilde{N}\right)\left(l + \dfrac{\tau-\eta}{2}, l - \dfrac{\tau-\eta}{2}\right) & = & \lambda\left(\widetilde{b}_3 + \widetilde{b}_4\right)\left(\dfrac{\tau-\eta}{2}\right) & (3.7.6)
\end{array}\right. \tag{3.7}$$

The unique solution of the system

$$\begin{cases} 2c\dfrac{\partial\left[\widetilde{\rho}^{+}\left(\widetilde{N}\right)\right]}{\partial\tau} + (1-\lambda)\sigma\left[\widetilde{\rho}^{+}\left(\widetilde{N}\right)\right]^2 & = & 0 \\ 2c\dfrac{\partial\left[\widetilde{\rho}^{-}\left(\widetilde{N}\right)\right]}{\partial\eta} + (1-\lambda)\sigma\left[\widetilde{\rho}^{-}\left(\widetilde{N}\right)\right]^2 & = & 0 \end{cases}$$

is obviously

$$\begin{cases} \widetilde{\rho}^{+}\left(\widetilde{N}\right)(\tau,\eta) & = & \dfrac{2c}{(1-\lambda)\sigma\tau - 2cH_1(\eta)} \\ \widetilde{\rho}^{-}\left(\widetilde{N}\right)(\tau,\eta) & = & \dfrac{2c}{(1-\lambda)\sigma\eta - 2cH_2(\tau)} \end{cases}$$

Taking into account the initial conditions (3.7.3), (3.7.4) and the boundary conditions (3.7.5), (3.7.6) we have:

$$\begin{cases} H_1(\eta) & = & -\dfrac{1}{2\lambda\left(\widetilde{f}_1+\widetilde{f}_2\right)(\eta)} - \dfrac{1}{2\lambda\left(\widetilde{a}_1+\widetilde{a}_2\right)(-\eta)} \\ H_2(\tau) & = & \dfrac{(\lambda-1)\sigma l}{2c} - \dfrac{1}{2\lambda\left(\widetilde{f}_3+\widetilde{f}_4\right)(\tau)} - \dfrac{1}{2\lambda\left(\widetilde{b}_3+\widetilde{b}_4\right)(\tau-l)} \end{cases} \tag{3.8}$$

Then

$$\begin{cases} \widetilde{\rho}^{+}\left(\widetilde{N}\right)(\tau,\eta) & = & \dfrac{2c}{(1-\lambda)\sigma\tau + \dfrac{c}{\lambda\left(\widetilde{f}_1+\widetilde{f}_2\right)(\eta)} + \dfrac{c}{2\lambda\left(\widetilde{a}_1+\widetilde{a}_2\right)(-\eta)}} \\ \widetilde{\rho}^{-}\left(\widetilde{N}\right)(\tau,\eta) & = & \dfrac{2c}{(1-\lambda)(\eta-l)\sigma + \dfrac{c}{\lambda\left(\widetilde{f}_3+\widetilde{f}_4\right)(\tau)} + \dfrac{c}{\lambda\left(\widetilde{b}_3+\widetilde{b}_4\right)(\tau-l)}} \end{cases} \tag{3.9}$$

Thus for $0 < \lambda < 1$ and non zero $\widetilde{a}_i, \widetilde{b}_i, \widetilde{f}_i, \widetilde{\rho}^{+}\left(\widetilde{N}\right)$ and $\widetilde{\rho}^{-}\left(\widetilde{N}\right)$ are bounded. The mean density is thus bounded and so are the number densities $\widetilde{N}_i, \forall i \in \{1,\ 2,\ 3,\ 4\}$. ■

We point out the fact that for $\lambda = 1$ the solutions $\widetilde{\rho}^{+}\left(\widetilde{N}\right)$ and $\widetilde{\rho}^{-}\left(\widetilde{N}\right)$ of (3.9) are not singular and moreover verify the conservation equations of the partial macroscopic densities. Accordingly they depend upon one variable.

Finally we conclude to the existence of solution of problem (3.3) by using the fixed point theorem of Schaefer [13]:

Theorem 3.6. Let T be a continuous and compact mapping of a Banach space X into itself, such that the set $\{x \in X, x = \lambda T(x)\}$ is bounded $\forall\,\lambda,\ 0 < \lambda < 1$. Then T has a fixed point.

3.2. Uniqueness of the solution of (3.3)

For $\lambda = 1$, the densities (3.9) are solutions of the conservation equations of the partial densities of model. Hence $\widetilde{\rho}^+(\widetilde{N})$ and $\widetilde{\rho}^-(\widetilde{N})$ are known and we have:

$$\begin{cases} \widetilde{\rho}^+(\widetilde{N})(\eta) = (\widetilde{N}_1 + \widetilde{N}_2)(\eta), \\ \widetilde{\rho}^-(\widetilde{N})(\tau) = (\widetilde{N}_3 + \widetilde{N}_4)(\tau). \end{cases}$$

Then

$$\begin{cases} \widetilde{N}_2(\tau,\eta) &= \widetilde{\rho}^+(\widetilde{N})(\eta) - \widetilde{N}_1(\tau,\eta) \\ \widetilde{N}_4(\tau,\eta) &= \widetilde{\rho}^-(\widetilde{N})(\tau) - \widetilde{N}_3(\tau,\eta) \end{cases}$$

and the system (3.3) becomes:

$$\begin{cases} 2c\dfrac{\partial \widetilde{N}_1}{\partial \tau} &= 2\sqrt{2}cS\left[\widetilde{\rho}^+(\widetilde{N})\widetilde{\rho}^-(\widetilde{N}) - \widetilde{N}_1\widetilde{\rho}^+(\widetilde{N}) - \widetilde{N}_3\widetilde{\rho}^+(\widetilde{N})\right] \\ &= \widetilde{Q}(\widetilde{N}) \\ 2c\dfrac{\partial \widetilde{N}_3}{\partial \eta} &= -\widetilde{Q}(\widetilde{N}) \\ \widetilde{N}_2(\tau,\eta) &= \widetilde{\rho}^+(\widetilde{N})(\eta) - \widetilde{N}_1(\tau,\eta) \\ \widetilde{N}_4(\tau,\eta) &= \widetilde{\rho}^-(\widetilde{N})(\tau) - \widetilde{N}_3(\tau,\eta) \\ \widetilde{N}_i\left(\dfrac{\tau+\eta}{2}, \dfrac{\tau+\eta}{2}\right) &= \widetilde{f}_i\left(\dfrac{\tau+\eta}{2}\right), \quad i = 1,\ 2,\ 3,\ 4 \\ \widetilde{N}_i\left(\dfrac{\tau-\eta}{2}, \dfrac{\eta-\tau}{2}\right) &= \widetilde{a}_i\left(\dfrac{\tau-\eta}{2}\right), \quad i = 1,\ 2 \\ \widetilde{N}_i\left(l + \dfrac{\tau-\eta}{2}, l - \dfrac{\tau-\eta}{2}\right) &= \widetilde{b}_i\left(\dfrac{\tau-\eta}{2}\right), \quad i = 3,\ 4 \end{cases} \tag{3.10}$$

We get a system of two linear partial differential equations with non characteristic initial and boundary conditions and which therefore has an unique solution. The fact that $\widetilde{\rho}^+(\widetilde{N})$ and $\widetilde{\rho}^-(\widetilde{N})$ depend upon one variable permits the computation of the exact analytic solutions in the form:

$$\begin{cases} \widetilde{N}_1(\tau,\eta) &= c_1\widetilde{\rho}^+(\eta)F(\tau,\eta) \\ \widetilde{N}_3(\tau,\eta) &= \left[1 - c_1\left(1 + \dfrac{c_0}{d}\right)F(\tau,\eta)\right]\widetilde{\rho}^-(\widetilde{N})(\tau) \\ \widetilde{N}_2(\tau,\eta) &= \widetilde{\rho}^+(\widetilde{N})(\eta) - \widetilde{N}_1(\tau,\eta) \\ \widetilde{N}_4(\tau,\eta) &= \widetilde{\rho}^-(\widetilde{N})(\tau) - \widetilde{N}_3(\tau,\eta) \end{cases} \tag{3.11}$$

with $F(\tau,\eta) = \exp\left(\displaystyle\int c_0\widetilde{\rho}^-(\widetilde{N})(\tau)d\tau + \int \frac{c_0 d}{c_0 + d}\widetilde{\rho}^+(\widetilde{N})(\eta)\mathrm{d}\eta\right)$, $d = S\sqrt{2}$, c_0 and c_1 are constants resulting from the integration of the partial differential equations of (3.10) which will be determined by the boundary conditions on N_1 and N_3.

4. Unsteady plane Couette flow

We apply the results of the preceding section to study a gas flow between two parallel infinite walls. We choose the origin O of the orthonormal reference $(O, \vec{e}_1, \vec{e}_2)$ so that the walls are limited by the planes $y = -\frac{h}{2}$ and $y = \frac{h}{2}$, $h > 0$, and assume as it is usual in the treatment of the plane Couette flow or the plane evaporation and condensation flows in gas dynamics [1, 2, 3, 4, 6] that the flow is one dimensional depending upon the spatial variable y' and the time t'. We use the diffuse reflection boundary conditions to properly take into account the interactions between the gas and its boundaries. Accordingly we prescribe that the microscopic densities of the incoming flux of gas near the plates are proportional to the corresponding microscopic densities of the discrete gas in Maxwellian equilibrium with the plates.

The macroscopic variables of the flow are the mean density N, the longitudinal velocity U and the transversal velocity V given by:

$$\begin{aligned} N &= N_1 + N_2 + N_3 + N_4 \\ NU &= -N_1 + N_2 - N_3 + N_4 \\ NV &= N_1 + N_2 - N_3 - N_4 \end{aligned} \tag{4.1}$$

The Maxwellian densities of the model associated with the macroscopic variables N, U and V are:

$$\begin{aligned} N_{1M} &= \frac{N}{4}\left(1+\frac{V}{c}\right)\left(1-\frac{U}{c}\right), \qquad N_{3M} = \frac{N}{4}\left(1-\frac{V}{c}\right)\left(1-\frac{U}{c}\right), \\ N_{2M} &= \frac{N}{4}\left(1+\frac{V}{c}\right)\left(1+\frac{U}{c}\right), \qquad N_{4M} = \frac{N}{4}\left(1-\frac{V}{c}\right)\left(1+\frac{U}{c}\right). \end{aligned} \tag{4.2}$$

The microscopic densities of the discrete gas in Maxwellian equilibrium with a wall are the Maxwellian densities associated with 1 and the longitudinal and transversal velocities in the wall. Denoting by $U_w^{\pm}$ and $V_w^{\pm}$ respectively the longitudinal and the transversal velocities in the walls and $\Lambda^{\pm}$ the respective accommodation coefficients, the boundary conditions of diffuse reflection are:

$$\begin{aligned} N_1\left(t', -\frac{h}{2}\right) &= \frac{\Lambda^-}{4}\left(1+\frac{V_w^-}{c}\right)\left(1-\frac{U_w^-}{c}\right), \\ N_2\left(t', -\frac{h}{2}\right) &= \frac{\Lambda^-}{4}\left(1+\frac{V_w^-}{c}\right)\left(1+\frac{U_w^-}{c}\right), \\ N_3\left(t', \frac{h}{2}\right) &= \frac{\Lambda^+}{4}\left(1-\frac{V_w^+}{c}\right)\left(1-\frac{U_w^+}{c}\right), \\ N_4\left(t', \frac{h}{2}\right) &= \frac{\Lambda^+}{4}\left(1-\frac{V_w^+}{c}\right)\left(1+\frac{U_w^+}{c}\right). \end{aligned} \tag{4.3}$$

We assume in addition that the gas is in Maxwellian equilibrium associated to the macroscopic variables N_0, U_0 and V_0 at the start so the initial conditions are:

$$
\begin{aligned}
N_1(0,y') &= \frac{N_0}{4}\left(1+\frac{V_0}{c}\right)\left(1-\frac{U_0}{c}\right), \quad N_3(0,y') = \frac{N_0}{4}\left(1-\frac{V_0}{c}\right)\left(1-\frac{U_0}{c}\right),\\
N_2(0,y') &= \frac{N_0}{4}\left(1+\frac{V_0}{c}\right)\left(1+\frac{U_0}{c}\right), \quad N_4(0,y') = \frac{N_0}{4}\left(1-\frac{V_0}{c}\right)\left(1+\frac{U_0}{c}\right).
\end{aligned} \tag{4.4}
$$

The quantities $\Lambda^{\pm}$, $U_w^{\pm}$ and $V_w^{\pm}$ are functions of t'. The quantities N_0, U_0 and V_0 are functions of y'. The characteristic values of the flow are chosen as the mean density N_c, the distance between the plates h, the characteristic speed c and the characteristic time t_c. We introduce the following non dimensional variables and parameters:

$$
\begin{cases}
n_0 = \dfrac{N_0}{N_c}, \quad n_i = \dfrac{N_i}{N_c},\ i = 1,\ 2,\ 3,\ 4, \quad t = \dfrac{t'}{t_c}, \quad y = \dfrac{y'}{h}\\
u_0 = \dfrac{U_0}{c}, \quad v_0 = \dfrac{V_0}{c}, \quad u_w^{\pm} = \dfrac{U_w^{\pm}}{c}, \quad v_w^{\pm} = \dfrac{V_w^{\pm}}{c}, \quad \lambda^{\pm} = \dfrac{\Lambda^{\pm}}{N_c}\\
Kn = \dfrac{1}{SN_c h}, \quad \gamma = \dfrac{h}{ct_c}
\end{cases} \tag{4.5}
$$

and the initial boundary value problem arising from the modelling of a flow between two parallel and infinite walls by the discrete four velocity Broadwell model is stated as follows:

$$
\begin{cases}
\dfrac{\partial n_1}{\partial t} + \dfrac{1}{\gamma}\dfrac{\partial n_1}{\partial y} = \dfrac{2\sqrt{2}}{\gamma Kn}(n_2 n_3 - n_1 n_4) = Q(n)\\
\dfrac{\partial n_2}{\partial t} + \dfrac{1}{\gamma}\dfrac{\partial n_2}{\partial y} = -Q(n)\\
\dfrac{\partial n_3}{\partial t} - \dfrac{1}{\gamma}\dfrac{\partial n_3}{\partial y} = Q(n)\\
\dfrac{\partial n_4}{\partial t} - \dfrac{1}{\gamma}\dfrac{\partial n_4}{\partial y} = -Q(n)\\
n_1(0,y) = \dfrac{n_0}{4}(1+v_0)(1-u_0)\\
n_2(0,y) = \dfrac{n_0}{4}(1+v_0)(1+u_0)\\
n_3(0,y) = \dfrac{n_0}{4}(1-v_0)(1-u_0)\\
n_4(0,y) = \dfrac{n_0}{4}(1-v_0)(1+u_0)\\
n_1\left(t,-\dfrac{1}{2}\right) = \dfrac{\lambda^-}{4}\left(1+v_w^-\right)\left(1-u_w^-\right)\\
n_2\left(t,-\dfrac{1}{2}\right) = \dfrac{\lambda^-}{4}\left(1+v_w^-\right)\left(1+u_w^-\right)\\
n_3\left(t,\dfrac{1}{2}\right) = \dfrac{\lambda^+}{4}\left(1-v_w^+\right)\left(1-u_w^+\right)\\
n_4\left(t,\dfrac{1}{2}\right) = \dfrac{\lambda^+}{4}\left(1-v_w^+\right)\left(1+u_w^+\right)
\end{cases} \tag{4.6}
$$

We assume that at the initial state the mean density n_0 and the macroscopic velocity $\vec{u}_0 = (u_0, v_0)$ are constants. Taking advantage of the results of section 3 we find the solution of system (4.6) in the form:

$$\left\{\begin{array}{rcll} n_1(t,y) &=& c_1 G(t,y) & (4.7.1)\\ n_2(t,y) &=& \dfrac{n_0(1+v_0)}{2} - n_1(t,y) & (4.7.3)\\ n_3(t,y) &=& \dfrac{c_1\left[2c_0 + d\lambda^+(1-v_w^+)\right]G(t,y)}{d\lambda^-(1+v_w^-)} & (4.7.2)\\ n_4(t,y) &=& \dfrac{n_0(1-v_0)}{2} - n_3(t,y) & (4.7.4)\\ G(t,y) &=& \exp\left(\dfrac{(y+gt)c_0\left[2c_0 + d\lambda^+(1-v_w^+)\right] + (y-gt)c_0 d\lambda^-(1+v_w^-)}{2c_0 + d\lambda^+(1-v_w^+)}\right) & (4.7.5)\\ c_1 &=& \dfrac{\lambda^-(1+v_w^-)(1-u_w^-)}{4}\exp\left(\dfrac{c_0\left[2c_0 + d\lambda^+(1-v_w^+)\right] + c_0 d\lambda^-(1+v_w^-)}{2\left[2c_0 + d\lambda^+(1-v_w^+)\right]}\right) & (4.7.6)\\ n_0 &=& \dfrac{\lambda^+(1-v_w^+) + \lambda^-(1+v_w^-)}{2} & (4.7.7)\\ v_0 &=& \dfrac{\lambda^-(1+v_w^-) - \lambda^+(1-v_w^+)}{\lambda^+(1-v_w^+) + \lambda^-(1+v_w^-)} & (4.7.8) \end{array}\right. \quad (4.7)$$

Where $d = \dfrac{\sqrt{2}}{Kn}$, $g = \dfrac{1}{\gamma}$ and the constant c_0 is a solution of the equation:

$$\begin{aligned} &\left[2c_0 + d\lambda^+(1-v_w^+)\right](u_w^- - 1)\exp\left(\frac{c_0\left[2c_0 + d\lambda^+(1-v_w^+) + d\lambda^-(1+v_w^-)\right]}{\left[2c_0 + d\lambda^+(1-v_w^+)\right]}\right)\\ &\quad + d\lambda^+(1-v_w^+)(1-u_w^+) = 0 \end{aligned} \tag{4.8}$$

We point out the fact that the equation (4.8) does not depend of γ. Moreover the initial conditions of the flow and the macroscopic variables of the boundaries are given so the relations (4.7.7) and (4.7.8) are in fact the equations for the determination of the accommodation coefficients $\lambda^\pm$.

For the plane Couette flow the walls are impermeable and $v_w^\pm = v_0 = 0$ so we have for this solution $\lambda^\pm = n_0$. Hence taking the characteristic density $N_c = N_0$ we have $n = n_0 = 1$ and $v = v_0 = 0$ in the flow. Only the longitudinal velocity u varies in the flow.

The results obtained with the Broadwell model show the dependence of the velocity slip upon the Knudsen number Kn. At the steady state as it shown in Figure 1, the velocity slip increases with Kn. However its rate of variation is not uniform. The variation is rapid for low and transitional Knudsen nombers and weak for large Knudsen numbers Figure 2.

5. Conclusion

We show that the initial-boundary value problem in one spatial dimension has an unique bounded solution for the Broadwell four velocity discrete model. Only positivity and

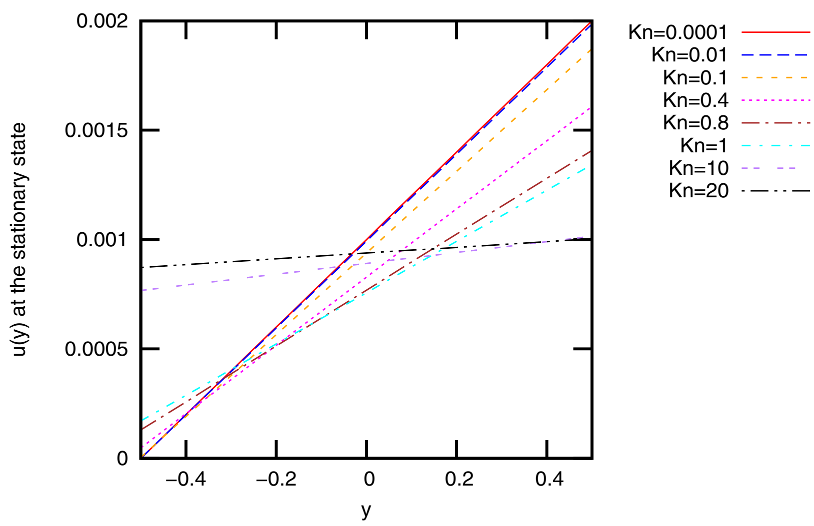

Figure 1: Longitudinal velocity u at the stationary state for $u_w^- = 0$ and $u_w^+ = 0,002$

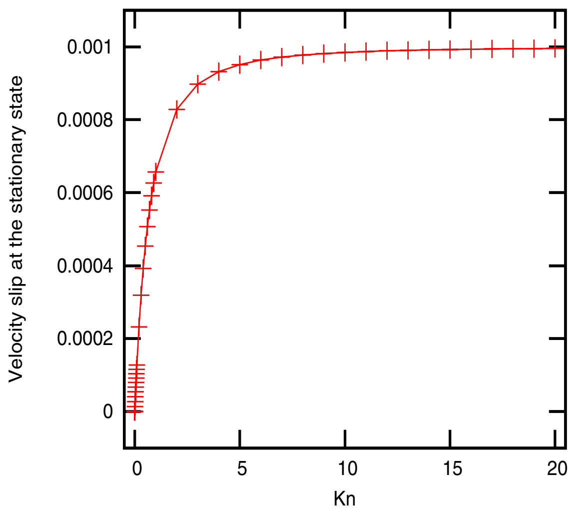

Figure 2: Velocity slip at the stationary state for $u_w^- = 0$ and $u_w^+ = 0,002$

boundedness are assumed for the data so the results apply for a wide range of mass and heat transfer phenomena such as Couette flows and evaporation and condensation flows. The method is simple and can be extended to other discrete velocity models at least for the proof of the existence and the boundedness of the solutions. It will be interesting to check its applicability to discrete models with multiple collisions. We build exact analytic solutions for the initial boundary values problem and use them to study the plane Couette flow. We find that the velocity slip increases rapidly with the Knudsen number Kn for low and transitional Kn and weakly for large Kn. It tends towards the half of the difference of the walls longitudinal velocities when Kn tends towards infinity.

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