

SINC BASED METHOD FOR SOLVING TWO-DIMENSIONAL FREDHOLM INTEGRAL EQUATIONS

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Abstract

Throughout this paper, we study the Sinc collocation method applied to obtain a numerical solution of two-dimensional Fredholm integral equation of first and second kinds. This method has been shown a powerful numerical tool for finding an accurate numerical solution. The Sinc function properties are provided and the global convergence analysis is given to guarantee the efficiency of our method. Moreover, we apply the method for some numerical examples to ensure the validity of the Sinc technique.

Keywords: *Fredholm, two-dimensional, Sinc Approximation.*

1. INTRODUCTION

Numerous problems in engineering, physics and mechanics are modeled as a two-dimensional Fredholm integral equations [1], [2]. In particular, the two-dimensional integral equations appear in electromagnetic, electrodynamic, heat and mass transfer, population and image processing [3], [4], [5] and [6]. Where the out-of-focus images are modeled as two-dimensional linear Fredholm integral equation of the first kind, and by solving the integral equation the given noising images are constructed. So, the numerical solution of the two-dimensional integral equation took the attention of many researches, where many different methods are used to solve the integral equation. For example, Haar wavelet [7], Coiflet wavelet [8] are a new methods based on the wavelet basis, in [9], Han and Wang approximated the two-dimensional Fredholm integral equations by the Galerkin iterative method. In addition, Nystrom

method [9], collocation method [10], and Gaussian radial basis function [11] are the commonly methods used.

In this paper, we consider the two-dimensional linear Fredholm integral equation of the first and second kind respectively of the form

$$g(x, y) = \int_a^b \int_a^b k(x, y, s, t) f(s, t) ds dt \quad (1.1)$$

$$f(x, y) = g(x, y) + \int_a^b \int_a^b k(x, y, s, t) f(s, t) ds dt \quad (1.2)$$

Here, $f(x, y)$ is the unknown function, we assume that the functions f, g are sufficiently smooth for $(x, y) \in [a, b]^2 \times [a, b]^2$, and $k(x, y, s, t)$ is continuous on $(x, s) \times (y, t) \in [a, b]^2 \times [a, b]^2$.

2. SINC FUNCTION

In this section, we will give a brief introduction of the Sinc function and its properties, in addition to some definitions and theorems that are required for function approximation, where the Sinc function is defined in details in [12]. The Sinc function is defined in the real line as follows

$$\text{sinc}(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases},$$

and the normalized Sinc function has the form

$$\text{Sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases} \quad (2.1)$$

This function is defined by Borel and Whittaker. For any $h > 0$ the Sinc function (2.1) is translated with spaced nodes jh and scaled by h as follows;

$$S(j, h)(x) = \text{Sinc}\left(\frac{x - jh}{h}\right), \quad j = 0, \pm 1, \pm 2, \dots \quad (2.2)$$

Definition 1. Let $H^1(D_E)$ denote the family of all analytic functions in the infinite strip

$$D_E = \{z = u + iv; |\text{Im}(z)| = |v| < d, d \leq \pi/2\}.$$

For a given function $f(x)$, $-\infty < x < \infty$, the Sinc function interpolation is defined by the Sinc basis functions $S(j, h)(x)$

$$D^2_d = \left\{ z \in C; \left| \arg \left(\frac{z-a}{b-z} \right) \right| < d \right\}, \quad -\infty < a < b < \infty$$

$$P^N(f)(x) = \sum_{k=-\infty}^{\infty} f(kh)S(k, h)(x) \tag{2.3}$$

The function approximation (2.3) is known as Whittaker expansion. In fact (2.3) gives the function approximation on the whole real line, and to have an approximation for function which is defined on closed interval $[a, b]$, we consider a conformal map $\varphi(x)$ defines the Sinc function over a closed interval $[a, b]$ as follows.

$$\varphi(x) = \ln \frac{x-a}{b-x}, \tag{2.4}$$

which maps the eye-shaped region $\left\{ z = x + iy; \left| \arg \left(\frac{z-a}{b-z} \right) \right| < d < \frac{\pi}{2} \right\}$ onto D_E , and the $\varphi(x)$ is a one-to-one function on the interval (a, b) onto the real line. Therefore, the basis functions on the interval $[a, b]$ are given by the composition

$$S(j, h)(x) \circ \varphi(x) = \text{Sin} \left(\frac{\varphi(x) - jh}{h} \right).$$

The Sinc collocation points x_k are defined for $h > 0$ by

$$x_k = \varphi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}}, \quad k = 0, \pm 1, \pm 2, \dots \tag{2.5}$$

Theorem 1. Let $f(x) \in L_\alpha(D_E)$. Then there exist a positive constants c_0 and r_0 independent of N , such that

$$\sup_{x \in \Gamma} \left| f(z) - \sum_{k=-N}^N f(z_k)S(k, h) \circ \varphi(z) \right| < c_0 \exp(-(\pi d \alpha N))^{\frac{1}{2}} \tag{2.6}$$

where N is appositve integer and $h = \sqrt{\frac{\pi d}{\alpha N}}$.

For a continuous function $f(x, y)$ is on the rectangle $[a, b]^2$, then the Sinc interpolation is defined as

$$P_N(f)(x, y) = \sum_{k=-N}^N \sum_{k'=-N}^N f(x_k, y_{k'}) S(k, h) \circ \varphi(x) S(k', h) \circ \varphi(y) \quad (2.7)$$

Where x_k and $y_{k'}$, $k, k' = -N, \dots, N$ are the Sinc collocation points defined in (2.5) and

$h = \left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}}$. Theorem 2: For a given constants α, d and integer N , $h = \left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}}$, and

$P_N(f)(x, y)$ is the Sinc interpolation for the function $f(x, y)$ (2.7). Then

$$\sup_{(x,y) \in [a,b]^2} |f(x, y) - P_N(f)(x, y)| \leq (c_1 + c_2 \log N) N^{\frac{1}{2}} \exp\left(-(\alpha d \pi N)^{\frac{1}{2}}\right) \quad (2.8)$$

Where, c_1 and c_2 are constants independent of N .

The proof exists in [12]

3. SINC-METHOD FOR TWO-DIMENSIONAL FREDHOLM INTEGRAL EQUATION

In this section we will use the Sinc basis functions for approximating the unknown function $f(x, y)$ of the integral equations (1.1) and (1.2). Firstly, inserting equation (2.7) into equation (1.1), we have

$$g(x, y) = \int_a^b \int_a^b k(x, y, s, t) \sum_{k=-N}^N \sum_{k'=-N}^N f_{k,k'} S_k(\varphi(s)) S_{k'}(\varphi(t)) ds dt \quad (3.1)$$

where $f_{k,k'}$, $k, k' = -N, \dots, N$ are unknowns that need be to determined. Consequently, substituting the Sinc collocation points $x_i, i = -N, \dots, N$ and $y_j, j = -N, \dots, N$ into equation (3.1) to have the system of linear equations

$$g(x_i, y_j) = \sum_{k=-N}^N \sum_{k'=-N}^N f_{k,k'} \left(\int_a^b \int_a^b k(x_i, y_j, s, t) S_k(\varphi(s)) S_{k'}(\varphi(t)) ds dt \right) \quad (3.2)$$

$i, j = -N, \dots, N$

For simplicity, let $A_{k,k'}(x, y) = \int_a^b \int_a^b k(x, y, s, t) S_k(\varphi(s)) S_{k'}(\varphi(t)) ds dt$, then equation (3.2) becomes as

$$g(x_i, y_j) = \sum_{k=-N}^N \sum_{k'=-N}^N f_{k,k'} A_{k,k'}(x_i, y_j), \quad i, j = -N, \dots, N \tag{3.3}$$

The system (3.3) can be written in the matrix equation

$$G = AF \tag{3.4}$$

where

$$G = \begin{bmatrix} g(x_{-N}, y_{-N}), g(x_{-N}, y_{-N+1}), \dots, g(x_{-N}, y_N), \\ g(x_{-N+1}, y_{-N}), \dots, g(x_{-N+1}, y_N), \dots, g(x_N, y_N) \end{bmatrix} \tag{3.5}$$

$$F = \begin{bmatrix} f(x_{-N}, y_{-N}), f(x_{-N}, y_{-N+1}), \dots, \\ f(x_{-N}, y_N), f(x_{-N+1}, y_{-N}), \dots, f(x_{-N+1}, y_N), \dots, f(x_N, y_N) \end{bmatrix} \tag{3.6}$$

and

$$A = \begin{bmatrix} A_{-N,-N}(x_{-N}, y_{-N}) & \dots & A_{-N,N}(x_{-N}, y_{-N}) & A_{-N+1,-N}(x_{-N}, y_{-N}) & A_{-N+1,N}(x_{-N}, y_{-N}) \dots A_{N,N}(x_{-N}, y_{-N}) \\ A_{-N,-N}(x_{-N+1}, y_{-N}) & \dots & A_{-N,N}(x_{-N+1}, y_{-N}) & A_{-N+1,-N}(x_{-N+1}, y_{-N}) & A_{-N+1,N}(x_{-N}, y_{-N}) \dots A_{N,N}(x_{-N+1}, y_{-N}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{-N,-N}(x_N, y_N) & \dots & A_{-N,N}(x_N, y_N) & A_{-N+1,-N}(x_N, y_N) & A_{-N+1,N}(x_N, y_N) \dots A_{N,N}(x_N, y_N) \end{bmatrix} \tag{3.7}$$

By solving equation (3.4) using inverse method as $F = A^{-1}G$, then we will obtain the approximation solution for the unknown function $f(x, y)$. In fact, if the matrix A is singular, then an approximation inverse could be evaluated by using the Pseudo inverse matrix.

Now, the same process could be used to solve equation (1.2) where the unknown function $f(x, y)$ of equation (1.2) is approximated by Sinc function interpolation (2.7), substituting equation (2.7) into equation (1.2), then substituting the Sinc collocation points $(x_k, y_{k'})$, $k, k' = -N, \dots, N$ to have a linear system of the unknowns $f_{k,k'}$, $k, k' = -N, \dots, N$ of the form

$$g(x_i, y_j) = \sum_{k=-N}^N \sum_{k'=-N}^N f_{k,k'} \left(\begin{matrix} S_k(\varphi(x_i)) S_{k'}(\varphi(y_j)) \\ -\int_a^b \int_a^b k(x_i, y_j, s, t) S_k(\varphi(s)) S_{k'}(\varphi(t)) ds dt \end{matrix} \right) \tag{3.8}$$

$$i, j = -N, \dots, N$$

let

$$B_{k,k'}(x,y) = S_k(\varphi(x))S_{k'}(\varphi(y)) - \int_a^b \int_a^b k(x,y,s,t)S_k(\varphi(s))S_{k'}(\varphi(t))dsdt$$

Then the system (3.8) can be written in matrix equation

$$BG = F \quad (3.9)$$

where. $B = [B_{i,j}]$

4. CONVERGE ANALYSIS

In this section, we provide the necessary conditions for the convergent of our method for solving integral equations (1.1) and (1.2), where we proved that the rate of convergent is $O(N)$, the proof of the next theorem based on [13]

Theorem 1. If $f(x,y)$ is the exact solution of (1.1) and $k(x,y,s,t)$ continuous on $[a,b]^2 \times [a,b]^2$, let

$$P_N^N(f)(x,y) = \sum_{k=-N}^N \sum_{k'=-N}^N f_{k,k'}^N S_k(\varphi(x))S_{k'}(\varphi(y)) \quad (4.1)$$

Is the Sinc function interpolation of $f(x,y)$, where $f_{k,k'}^N, k, k' = -N, \dots, N$ are the coefficients that are determined by solving the matrix equation (3.4), then

$$\|f(x,y) - P_N^N(f)\|_{\infty} \leq (c_2 + c_3 \log N) N^{\frac{1}{2}} \exp(-(\pi d \alpha N)^{\frac{1}{2}}) \left(1 + \frac{4}{\pi^2} (3 + \log N)^2\right)$$

where c_1 and c_2 are constants independent of N .

Proof: let

$$P_N(f(x,y)) = \sum_{k=-N}^N \sum_{k'=-N}^N f_{k,k'} S_k(\varphi(x))S_{k'}(\varphi(y)) \quad (4.2)$$

Be the Sinc interpolation for the function $f(x,y)$ where $f_{k,k'} = f(\varphi^{-1}(kh), \varphi^{-1}(k'h))$ the exact values of the function $f(x,y)$ at the Sinc collocation points $(x_k = \varphi^{-1}(kh), y_{k'} = \varphi^{-1}(k'h))$. Then

$$\|f - P_N^N(f)\| \leq \|f - P_N(f)\| + \|P_N(f) - P_N^N(f)\| \quad (4.3)$$

Now, by substituting equations (4.1) and (4.2) into the integral equation (1.1) we have the following equations

$$g(x, y) = \int_a^b \int_a^b k(x, y, s, t) P_N^N(f(s, t)) ds dt \tag{4.4}$$

$$\xi(x, y) = \int_a^b \int_a^b k(x, y, s, t) P_N(f(s, t)) ds dt \tag{4.5}$$

To obtain the coefficients $f_{k,k'} = f(\varphi^{-1}(kh), \varphi^{-1}(k'h))$ and $f_{k,k'}^N$ of equations (4.2) and (4.1) respectively, we substitute the Sinc collocation points $(x_k, y_{k'})$ to have a linear system of equations of the unknowns $f_{k,k'}$ that can be solved using inverse method as in previous section such that

$$\begin{bmatrix} f_{-N,N} & \dots & f_{N,N} \end{bmatrix} = A^{-1} \left[\xi(x_i, y_j) \right]_{i,j=-N}^N$$

and

$$\begin{bmatrix} f_{-N,N}^N & \dots & f_{N,N}^N \end{bmatrix} = A^{-1} \left[g(x_i, y_j) \right]_{i,j=-N}^N$$

where A is defined in equation (3.7). So that

$$\sup_{k,k' \in \text{In}[-N,N]} |f_{k,k'}^N - f_{k,k'}| \leq \|A^{-1}\| \sup_{i,j \in \text{In}[-N,N]} |g(x_i, y_j) - \xi(x_i, y_j)| \tag{4.6}$$

where $\text{In}[-N, N]$ is the set of all integers in $[-N, N]$. Now subtracting equation (1.1) from equation (4.5), then we get

$$\zeta(x, y) - g(x, y) = \int_a^b \int_a^b k(x, y, s, t) [f(s, t) - P_N(f(s, t))] ds dt$$

Then

$$\begin{aligned} \sup_{i,j \in \text{In}[-N,N]} |\zeta(x_i, y_j) - g(x_i, y_j)| &= \sup_{i,j \in \text{In}[-N,N]} \left| \int_a^b \int_a^b k(x_i, y_j, s, t) [f(s, t) - P_N(f(s, t))] ds dt \right| \\ &\leq \int_a^b \int_a^b \sup_{\substack{i,j \in \text{In}[-N,N] \\ s,t \in [-N,N]}} |k(x_i, y_j, s, t)| \|f(s, t) - P_N(f(s, t))\| ds dt \\ &\leq (b-a)^2 M \|f - P_N\| \end{aligned} \tag{4.7}$$

where $M = \sup |k|$, by using theorem 2 and (4.7) equation (4.6) becomes as

$$\sup_{k,k' \in \text{In}[-N,N]} |f_{k,k'}^N - f_{k,k'}| \leq \|A^{-1}\| (b-a)^2 M (c_1 + c_2 \log N) N^{\frac{1}{2}} \exp(-\pi \alpha d N)^{\frac{1}{2}} \tag{4.8}$$

Now

$$\begin{aligned} \sup \left| (P_N^N - P_N)(f(x, y)) \right| &= \sup \left| \sum_{k=-N}^N \sum_{k'=-N}^N (f_{k,k'}^N - f_{k,k'}) S_k(x) S_{k'}(y) \right| \\ &\leq \|A^{-1}\| \left\| (b-a)^2 M(c_1 + c_2 \log(N)) N^{\frac{1}{2}} \exp(-(\pi\alpha dN)^{\frac{1}{2}}) \sup \left| \sum_{k=-N}^N \sum_{k'=-N}^N S_k(x) S_{k'}(y) \right| \right\| \\ &\leq \|A^{-1}\| \left\| (b-a)^2 M(c_1 + c_2 \log(N)) N^{\frac{1}{2}} \exp(-(\pi\alpha dN)^{\frac{1}{2}}) \frac{4}{\pi^2} (3 + \log N)^2 \right\| \end{aligned} \quad (4.9)$$

Finally, equation becomes (4.3)

$$\|f - P_N^N\| \leq (c_2 + c_3 \log N) N^{\frac{1}{2}} \exp(-(\pi d \alpha N)^{\frac{1}{2}}) \left(1 + \frac{4}{\pi^2} (3 + \log N)^2 \right).$$

The proof is completed.

Theorem: If $f(x, y)$ is the exact solution of (1.2) and $k(x, y, s, t)$ continuous on $[a, b]^2 \times [a, b]^2$, let

$$P_N^N(f)(x, y) = \sum_{k=-N}^N \sum_{k'=-N}^N f_{k,k'}^N S_k(\varphi(x)) S_{k'}(\varphi(y)) \quad (4.10)$$

Is the Sinc function interpolation of $f(x, y)$, where $f_{k,k'}^N, k, k' = -N, \dots, N$ are the coefficients that are determined by solving the matrix equation (3.4), then

$$\|f(x, y) - P_N^N(f)\|_{\infty} \leq (c_2 + c_3 \log N) N^{\frac{1}{2}} \exp(-(\pi d \alpha N)^{\frac{1}{2}}) \left(1 + \frac{4}{\pi^2} (3 + \log N)^2 \right)$$

where c_1 and c_2 are constants independent of N .

Proof: According to equations (4.1) and (4.2) which are the Sinc function approximation and Sinc interpolation of the function $f(x, y)$, then we consider the equations

$$g(x, y) = P_N^N(f(x, y)) - \int_a^b \int_a^b k(x, y, s, t) P_N^N(f(s, t)) ds dt \quad (4.11)$$

$$\xi(x, y) = P_N(f(x, y)) - \int_a^b \int_a^b k(x, y, s, t) P_N(f(s, t)) ds dt \quad (4.12)$$

Where the coefficients $f_{k,k'}$ and $f_{k,k'}^N$ are obtained by substituting the Sinc collocation points into the above equations, then solving the linear systems as in previous theorem, then

$$\sup_{k,k' \in \text{In}[-N,N]} |f_{k,k'}^N - f_{k,k'}| \leq \|B^{-1}\| \sup_{i,j \in \text{In}[-N,N]} |g(x_i, y_j) - \xi(x_i, y_j)| \tag{4.13}$$

By subtracting equation (1.2) from (4.12) to get

$$\begin{aligned} \xi(x, y) - g(x, y) &= f(x, y) \\ &\quad - P_N(f(x, y)) \int_a^b \int_a^b k(x, y, s, t) [f(s, t) - P_N(f(s, t))] ds dt \end{aligned} \tag{4.14}$$

Then

$$\begin{aligned} \sup_{i,j \in \text{In}[-N,N]} |\zeta(x_i, y_j) - g(x_i, y_j)| &= \sup_{i,j \in \text{In}[-N,N]} \left| \left(f(x_i, y_j) - P_N(f(x_i, y_j)) \right) - \right. \\ &\quad \left. \int_a^b \int_a^b k(x_i, y_j, s, t) [f(s, t) - P_N(f(s, t))] ds dt \right| \\ &\leq \|f(s, t) - P_N(f(s, t))\| + \\ &\quad \int_a^b \int_a^b \sup_{\substack{i,j \in \text{In}[-N,N] \\ s,t \in [-N,N]}} |k(x_i, y_j, s, t)| \|f(s, t) - P_N(f(s, t))\| ds dt \\ &\leq ((b-a)^2 M + 1) \|f - P_N\| \\ &\leq ((b-a)^2 M + 1) (c_1 + c_2 \log N) N^{\frac{1}{2}} \exp\left(-(\pi\alpha dN)^{\frac{1}{2}}\right) \end{aligned} \tag{4.15}$$

so

$$\sup_{k,k' \in \text{In}[-N,N]} |f_{k,k'}^N - f_{k,k'}| \leq \|B^{-1}\| ((b-a)^2 M + 1) (c_1 + c_2 \log N) N^{\frac{1}{2}} \exp(-(\pi\alpha dN)^{\frac{1}{2}}) \tag{4.16}$$

and

$$\begin{aligned} \sup \left| (P_N^N - P_N)(f(x, y)) \right| &= \sup \left| \sum_{k=-N}^N \sum_{k'=-N}^N (f_{k,k'}^N - f_{k,k'}) S_k(x) S_{k'}(y) \right| \\ &\leq \|B^{-1}\| ((b-a)^2 M + 1) (c_1 + c_2 \log(N)) N^{\frac{1}{2}} \exp \\ &\quad - (\pi\alpha dN)^{\frac{1}{2}} \frac{4}{\pi^2} (3 + \log N)^2 \end{aligned}$$

Finally,

$$\begin{aligned} \|f - P_N^N(f)\| &\leq \|f - P_N(f)\| + \|P_N(f) - P_N^N(f)\| \\ &\leq \left(\|B^{-1}\| \left((b-a)^2 M + 1 \right) \frac{4}{\pi^2} (3 + \log N)^2 + 1 \right) (c_1 + c_2 \log(N)) N^{\frac{1}{2}} \exp(-\pi\alpha dN)^{\frac{1}{2}} \end{aligned}$$

5. NUMERICAL EXAMPLES

In this section, we are applying our method for solving equations(1.1) and (1.2). The examples have been solved with $\alpha = 1$, $d = \frac{\pi}{2}$ and different values of N .

Example 1. For equation (1.1), let $g(x, y) = \cos xy$, $k(x, y, s, t) = \sin xy(1-st)$, the exact solution $f(x, y) = \frac{3}{4} \cos xy \sin xy$, $x, y \in [0, 1]$ and let

$$|E_N| = \max_{-N \leq x_k, y_{k'} \leq N} |f(x_k, y_{k'}) - P^N(x_k, y_{k'})|, \text{ where } x_k, y_{k'} \text{ are the Sinc collocation points.}$$

Table 1 shows the values for the error with different values of N .

Example 2: For equation(1.2), let $e^{x+y} - \frac{1}{2}(-1+e^2)xy$, $k(x, y, s, t) = xy e^{s-t}$, the exact solution $f(x, y) = e^{x+y}$, $x, y \in [0, 1]$ and let $|E_N| = \max_{-N \leq x_k, y_{k'} \leq N} |f(x_k, y_{k'}) - P^N(x_k, y_{k'})|$, where $x_k, y_{k'}$ are the Sinc collocation points. Table 1 shows the values for the error with different values of N .

Table 1. The results for examples 1 and 2.

	N	5	10	20	30
Example1	$ E_N $	1.09813×10^{-4}	1.0042×10^{-4}	5.4571×10^{-8}	1.01845×10^{-9}
Example 2	$ E_N $	3.9552×10^3	2.1978×10^{-4}	7.08913×10^{-7}	4.4354×10^{-8}

6. CONCLUSION

In this paper, we have investigated the application Sinc basis function for solving the Fredholm integral equations of first and second kinds. This method is very simple and involves less computation. Also we can expand this method to higher dimensional problems and other classes of integral equations such as integral-algebraic equation of

index-1 and index-2. Note that, for applying this method we need to solve the linear system of equations, and the Pseudo inverse could be use if the coefficient matrix is singular.

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