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Oscillation of Second-Order Nonlinear Functional Dynamic Equations on Time Scales

S. Padmavathi
Associate Professor,
Pachaiyappa's College,
Chennai 600 030, India.

Abstract

This paper deals with oscillatory behavior of second order nonlinear functional dynamic equation of the form

$$\left[r(t)(x^{\Delta}(t))^{\gamma}\right]^{\Delta} + p(t)\left[x^{\sigma}(\tau(t))\right]^{\beta} = 0,$$

on an arbitrary time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$ and $\lim_{t\to\infty} \tau(t) = \infty$. We establish some new sufficient conditions which ensure that every solution oscillates or converges to zero. Some examples are given to illustrate our main results.

AMS subject classification: 34K11, 39A10, 39A99.
Keywords: Oscillation, Second order, Nonlinear functional dynamic equations, Time scales.

1. Introduction

In this paper, we discuss the oscillation of the second order nonlinear functional dynamic equation of the form

$$\left[r(t)(x^{\Delta}(t))^{\gamma}\right]^{\Delta} + p(t)\left[x^{\sigma}(\tau(t))\right]^{\beta} = 0, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$, $t_0 \in \mathbb{T}$, $[t_0, \infty)_{\mathbb{T}} := \{t \in \mathbb{T} : t \geqslant t_0\}$ is a time scale interval in $\mathbb{T}$. $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ is the forward jump operator on $\mathbb{T}$ and $x^{\sigma} := x \circ \sigma$, Through out this paper we will assume the following hypotheses:

(H_1) γ and β are ratios of odd positive integers, $\tau : \mathbb{T} \longrightarrow \mathbb{T}$ *and* $\lim_{t\to\infty} \tau(t) = \infty$;

(H_2) r and p are real valued $rd-$continuous positive functions defined on $\mathbb{T}$.

We will consider the following two cases:

$$\int_{t_0}^{\infty} \left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}} \Delta t = \infty, \tag{1.2}$$

and

$$\int_{t_0}^{\infty} \left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}} \Delta t < \infty, \tag{1.3}$$

Equation (1.1) is called a delay dynamic equation if $\tau(t) < t$ and is called an advance dynamic equation if $\tau(t) > t$ and ordinary if $\tau(t) = t$. Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. Throughout this paper this assumption will be supposed to hold: Let $T_0 = \min\{\tau(t) : t \geqslant 0\}$ and $\tau_{-1}(t)$ is the inverse of $\tau(t)$ when the latter exists. By a solution of (1.1) we mean a nontrivial real valued function $x \in C^1_{rd}[\tau_{-1}(t_0), \infty)$ which has the property that $x^{[1]} \in C^1_{rd}[\tau_{-1}(t_0), \infty)$, where C_{rd} is the space of rd-continuous functions and

$$x^{[1]} := r(x^{\Delta})^{\gamma}, \text{ and } x^{[2]} := (x^{[1]})^{\Delta}. \tag{1.4}$$

Our attention is restricted to those solutions of (1.1) which exist on $[t_x, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)| : t > t_1\} > 0$ for any $t_1 \geqslant t_x$. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. A solution x(t) of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the reals $\mathbb{R}$. The study of dynamic equations on time scales is a fairly new topic, and work in this area is rapidly growing. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a time scale. In this way results not only related to the set of real numbers $\mathbb{R}$ or the set of integers $\mathbb{Z}$ but those pertaining to more general time scales are obtained. Dynamic equations on time scales have many applications in biology, engineering, economics, physics, neural networks, social sciences and so on. A book on the subject of time scales, by Bohner and Peterson [10], summarizes and organizes much of time scale calculus, see also the book by Bohner and Peterson [11] for advances in dynamic equations on time scales.

In the last few years, there has been increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations on time scales. For contribution, we refer the reader to the papers [1]–[28] and the references cited therein.

As a special case of (1.1) Akm-Bohner and Hoffacker [9] considered the equation

$$x^{\Delta\Delta}(t) + p(t)(x^{\sigma})^{\gamma} = 0, \tag{1.5}$$

and established some necessary and sufficient conditions for oscillation of all solutions when $\gamma > 1$ and $0 < \gamma < 1$. Their results cannot be applied in the case when $\gamma = 1$ and applied only on discrete time scales.

In [1], Agarwal et al. considered the second order delay dynamic equations on time scales

$$x^{\Delta\Delta}(t) + p(t)x(\tau(t)) = 0, \tag{1.6}$$

and established some sufficient conditions for oscillation of (1.6) when

$$\int_{t_0}^{\infty} \tau(t)p(t)\Delta t = \infty. \tag{1.7}$$

Saker [25] examines oscillation for half-linear dynamic equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)x^{\gamma}(t) = 0, \tag{1.8}$$

on time scales, where $\gamma > 1$ is an odd positive integer which cannot be applied when $0 < \gamma \leqslant 1$.

Erbe et al. [16] considered (1.8) and the equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)x^{\gamma}(\sigma(t)) = 0,$$

when

$$r^{\Delta}(t) \geqslant 0, \text{ and } \int_{t_0}^{\infty} \sigma^{\gamma}(t)p(t)\Delta t = \infty, \tag{1.9}$$

and established some necessary and sufficient conditions for nonoscillation of Hille-Kneser type. Erbe et al. [20] considered the half-linear delay dynamic equations on time scales

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)x^{\gamma}(\tau(t)) = 0, \tag{1.10}$$

where $\gamma > 1$ is the quotient of odd positive integers and

$$r^{\Delta}(t) \geqslant 0, \text{ and } \int_{t_0}^{\infty} \tau^{\gamma}(t)p(t)\Delta t = \infty, \tag{1.11}$$

and utilized a Riccati transformation technique and established some oscillation criteria for (1.10). In [22], Grace et al. considered the equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)x^{\beta}(t) = 0, \tag{1.12}$$

where γ and $\beta > 0$ are ratios of odd positive integers, r and p are positive rd-continuous functions on $\mathbb{T}$ and established several sufficient for oscillation.

Saker et al. [28] considered the equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)x^{\beta}(\tau(t)) = 0, \tag{1.13}$$

where γ and $\beta > 0$ are ratios of odd positive integers, r, p and τ are positive rd-continuous functions on $\mathbb{T}$ and established several sufficient for oscillation.

Motivated by the above observations, in this paper we establish some sufficient conditions for oscillation of (1.1). In Section 2, we present some basic definitions and results concerning the calculus on time scales. In Section 3, we give several useful lemmas. In Section 4, we state and prove main results. In the last section, some examples are considered to illustrate our main results.

2. Preliminaries on time scales

In this section, we recall the following concepts and results related to the notation of time scales.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above, that is, it is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$. On any time scale, we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \;\; \rho(t) := \sup\{s \in \mathbb{T} : s < t\}. \tag{2.1}$$

A point $t \in \mathbb{T}$ is said to be left dense if $\rho(t) = t$, right dense if $\sigma(t) = t$, left scattered if $\rho(t) < t$, and right scattered if $\sigma(t) > t$. The graininess function μ of the time scale is defined by $\mu(t) := \sigma(t) - t$.

For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may actually be replaced with any Banach space), the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \tag{2.2}$$

if f is continuous at t and t is right scattered. If t is not right scattered, then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s\to t} \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}, \tag{2.3}$$

provided that this limit exists.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$ and f is said to be differentiable if its derivative exists. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous function is denoted by $C^1_{rd}(\mathbb{T}, \mathbb{R})$.

The derivative and the shift operator σ are related by the formula

$$f^{\sigma}(t) = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{2.4}$$

Let f be a real-valued function defined on an interval $[a, b]$. We say that f is increasing, decreasing, nondecreasing and nonincreasing on $[a, b]$ if $t_1, t_2 \in [a, b]$ and $t_2 > t_1$ imply $f(t_2) > f(t_1)$, $f(t_2) < f(t_1)$, $f(t_2) \geqslant f(t_1)$, and $f(t_2) \leq f(t_1)$, respectively. Let f be a differentiable function on $[a, b]$. Then f is increasing, decreasing, nondecreasing and nonincreasing on $[a, b]$ if $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geqslant 0$, and $f^{\Delta}(t) \leq 0$ for all $t \in [a, b)$, respectively.

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $g(t)g(\sigma(t)) \neq 0$) of two differentiable functions f and g.

$$\left.\begin{aligned} (fg)^{\Delta}(t) &= f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)), \\ \left(\frac{f}{g}\right)^{\Delta}(t) &= \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}. \end{aligned}\right\} \tag{2.5}$$

For $a, b \in \mathbb{T}$, and a (delta) differentiable function f, the Cauchy (delta) integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a). \tag{2.6}$$

The integration by parts formula reads

$$\int_a^b f^{\Delta}(t)g(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t, \tag{2.7}$$

and infinite integral is defined as

$$\int_a^{\infty} f(s)\Delta s = \lim_{t\to\infty} \int_a^t f(s)\Delta s. \tag{2.8}$$

3. Some Preliminary Lemmas

In this section, we present some lemmas which are used in the following sections.

Lemma 3.1. ([28, Lemma 2.1]) Assume that (H_1), (H_2), (1.2) hold and (1.1) has a nonoscillatory solution x on $[t_0, \infty)$. Then there exists $T \geqslant t_0$ such that $x(t)x^{[1]}(t) > 0$ for $t \geqslant T$.

Lemma 3.2. ([14,Lemma 2.3]) Suppose that following condition holds:

(H_3) $\tau^{\Delta}(t) > 0$ is rd-continuous on $\mathbb{T}$, $\tilde{\mathbb{T}} := \tau(\mathbb{T}) = \{\tau(t) : t \in \mathbb{T}\} \subset \mathbb{T}$ is a time scale, and $(\tau^{\sigma})(t) = (\sigma \circ \tau)(t)$ for all $t \in \mathbb{T}$, where $(\tau^{\sigma})(t) = (\tau \circ \sigma)(t)$.

Let $y : \mathbb{T} \to \mathbb{R}$. If $y^{\Delta}(t)$ exists for all sufficiently large $t \in \mathbb{T}$, then $(y \circ \tau)^{\Delta}(t) = (y^{\Delta} \circ \tau)(t)\tau^{\Delta}(t)$ for all sufficiently large $t \in \mathbb{T}$.

Lemma 3.3. ([14, Lemma 2.4]) Let $F : \mathbb{T} \to \mathbb{R}$ and $\gamma > 0$ be a constant. Furthermore, assume $F^{\Delta}(t) > 0$ and $F(t) > 0$ for all sufficiently large $t \in \mathbb{T}$. Then we have the following:

(i) If $0 < \gamma < 1$, then $(F^{\gamma})^{\Delta}(t) \geqslant \gamma(F^{\sigma})^{\gamma-1}(t)F^{\Delta}(t)$ for all sufficiently large $t \in \mathbb{T}$, where $F^{\sigma} := F \circ \sigma$;

(ii) If $\gamma \geqslant 1$, then $(F^{\gamma})^{\Delta}(t) \geqslant \gamma F^{\gamma-1}(t)F^{\Delta}(t)$ for all sufficiently large $t \in \mathbb{T}$.

Lemma 3.4. ([23]) If X and Y are nonnegative, then

$$\lambda XY^{\lambda-1} - X^{\lambda} \leq (\lambda - 1)Y^{\lambda} \text{ for } \lambda > 1,$$

where the equality holds if and only if $X = Y$.

4. Main Results

In this section, we state and prove our main results. We establish several new oscillation criteria for (1.1).

Theorem 4.1. Assume that (H_1,) (H_2) and (1.2) hold. Furthermore, assume that

$$\int_{t_0}^{\infty} p(s)\Delta s = \infty. \tag{4.1}$$

Then every solution of (1.1) oscillates.

Proof. Suppose to the contrary that x is a nonoscillatory solution of (1.1). Without loss of generality we may assume that $x(t) > 0$, so that $x(\tau(t)) > 0$, and $x^{\sigma}(\tau(t)) > 0$ for $t \geqslant t_0$ sufficiently large. Therefore, from (1.1) we have

$$\left(x^{[1]}(t)\right)^{\Delta} = -p(t)\left[x^{\sigma}(\tau(t))\right]^{\beta} < 0, \;\; for\ t \geqslant t_0. \tag{4.2}$$

Then $x^{[1]}(t)$ is strictly decreasing for $t \geqslant t_0$ and of one sign. We claim that $x^{[1]}(t) > 0$ for $t_1 \geqslant t_0$. Assume not, then there is a $t_1 \geqslant t_0$ such that $x^{[1]}(t) =: c < 0$. Therefore, $x^{[1]}(t) \leq x^{[1]}(t_2) = c > 0$, for $t_1 \geqslant t_0$. This implies from the definition $x^{[1]}(t)$ that $x^{\Delta}(t) \leq ar^{\frac{-1}{\gamma}}(t)$, for $t \geqslant t_1$ where $a := c^{\frac{1}{\gamma}} < 0$. Integrating, we find that

$$x(t) = x(t_1) + \int_{t_1}^{t} x^{\Delta}(s)\Delta s \leq x(t_1) + a\int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} \longrightarrow -\infty \; as\ t \longrightarrow \infty,$$

which contradicts the fact that $x(t) > 0$ for all $t \geqslant t_0$. Hence $r(t)(x^{\Delta}(t))^{\gamma} > 0$ for $t \geqslant t_1$ and so

$$x^{\Delta}(t) > 0 \text{ for } t \geqslant t_1. \tag{4.3}$$

Take $t_2 \geqslant t_1$ such that $\tau(t) \geqslant t_1$ for $t \geqslant t_2$ then we have $\sigma(\tau(t)) \geqslant \tau(t) \geqslant t_1$ for $t \geqslant t_2$. From (4.3) we conclude $x^{\sigma}(\tau(t)) \geqslant x(t_1) > 0$ for $t \geqslant t_2$. Hence from (4.2) we get

$$\left(x^{[1]}(t)\right)^{\Delta} = \left[r(t)(x^{\Delta}(t))^{\gamma}\right]^{\Delta} \leq -p(t)x^{\beta}(t_1) := -c_1 p(t), \ \ for\ t \geqslant t_2. \tag{4.4}$$

where $c_1 := x^{\beta}(t_1) > 0$. Integrating (4.4) from t to v, we obtain

$$r(t)(x^{\Delta}(t))^{\gamma} \geqslant r(v)(x^{\Delta}(v))^{\gamma} + c_1 \int_t^v p(u)\Delta u > c_1 \int_t^v p(u)\Delta u \text{ for } v \geqslant t \geqslant t_2. \tag{4.5}$$

Take $t = t_2$ then from (4.5) we conclude

$$\int_{t_2}^{v} p(u)\Delta u < c_1^{-1} r(t_2)(x^{\Delta}(t_2))^{\gamma} \ \ for\ v \geqslant t_2.$$

Letting $v \longrightarrow \infty$, we get

$$\int_{t_2}^{\infty} p(u)\Delta u < c_1^{-1} r(t_2)(x^{\Delta}(t_2))^{\gamma} < \infty,$$

which contradicts (4.1). The proof is complete. ■

In the following, we consider the case when

$$\int_{t_0}^{\infty} p(s)\Delta s < \infty. \tag{4.6}$$

Theorem 4.2. Assume that (H_1) - (H_3), (1.2) and the following conditions hold:

(H_4) $\tau(t) \leq t$, for $t \in [t_0, \infty)_{\mathbb{T}}$;

(H_5) There exists a positive rd-continuous Δ-differentiable function $\delta(t)$

such that

$$\limsup_{t \to \infty} \int_{t_0}^{t} \left[\delta(s)p(s) - \frac{(\gamma/\beta)^{\gamma} r(\tau(s))(\delta_+^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left[\tau^{\Delta}(s)\delta(s)\eta^{\sigma}(s)\right]^{\gamma}} \right] \Delta s = \infty, \tag{4.7}$$

for all $t \geqslant T > t_0$ sufficiently large, where

$$\eta^\sigma(t) := \begin{cases} c_1 \text{ is any positive constant}, & if\, \beta > \gamma, \\ 1 & if\, \beta = \gamma, \\ c_2 \left(\int_T^{\tau^\sigma(t)} \frac{1}{r^{\frac{1}{\gamma}}(s)} \Delta s \right)^{\beta/\gamma - 1}, c_2 \text{ is any positive constant}, & if\, \beta < \gamma, \end{cases} \tag{4.8}$$

and $\delta_+^\Delta := \max\left\{\delta^\Delta(s), 0\right\}$. Then (1.1) is oscillatory.

Proof. Assume that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t)$ is an eventually solution of (1.1) such that $x(t) > 0,\ x(\tau(t)) > 0,\ x^\sigma(\tau(t)) > 0$ for $t \geqslant t_0$. Then from Lemma (3.1), since (1.2) holds, there exists $t_1 > t_0$ such that $x(t) > 0\ x^{[1]}(t) > 0$, and $x^{[2]}(t) < 0$, for $t \geqslant t_1$. Define the function $w(t)$ by the generalized Riccati substitution

$$w(t) = \delta(t) \frac{x^{[1]}(t)}{x^\beta(\tau(t))} \ for\ t \geqslant t_1. \tag{4.9}$$

Then $w(t) > 0$ for $t \geqslant t_1$. By the product and quotient rule (2.5) and the definition of $w(t)$, we obtain

$$\begin{aligned} w^\Delta &= \left[x^{[1]}\right]^\Delta \frac{\delta}{(x \circ \tau)^\beta} + \left(x^{[1]}\right)^\sigma \left[\frac{\delta}{(x \circ \tau)^\beta}\right]^\Delta \\ &= x^{[2]} \frac{\delta}{(x \circ \tau)^\beta} + \left(x^{[1]}\right)^\sigma \left[\frac{\delta^\Delta}{\left((x \circ \tau)^\beta\right)^\sigma} - \frac{\delta\left[x \circ \tau)^\beta\right]^\Delta}{(x \circ \tau)^\beta \left((x \circ \tau)^\beta\right)^\sigma}\right] \\ &= x^{[2]} \frac{\delta}{(x^\sigma \circ \tau)^\beta} \frac{(x^\sigma \circ \tau)^\beta}{(x \circ \tau)^\beta} + \frac{\delta^\Delta}{\delta^\sigma} w^\sigma - \frac{\delta \left(x^{[1]}\right)^\sigma \left[x \circ \tau)^\beta\right]^\Delta}{(x \circ \tau)^\beta \left((x \circ \tau)^\beta\right)^\sigma} \\ &\leq -\delta p + \frac{\delta_+^\Delta}{\delta^\sigma} w^\sigma - \frac{\delta \left(x^{[1]}\right)^\sigma \left[x \circ \tau)^\beta\right]^\Delta}{(x \circ \tau)^\beta (x \circ \tau^\sigma)^\beta}, \end{aligned} \tag{4.10}$$

where δ_+^Δ is defined as in Theorem 4.2 and $x^\sigma(\tau(t)) \geqslant x(\tau(t))$. By Lemma 3.2, we get

$$(x \circ \tau)^\Delta = (x^\Delta \circ \tau)\tau^\Delta > 0 \tag{4.11}$$

If $0 < \beta < 1$, then by taking $F = x \circ \tau$ and by Lemma 3.3(i) and (4.11) we get

$$\begin{aligned} \left[(x \circ \tau)^\beta\right]^\Delta &\geqslant \beta(x \circ \tau^\sigma)^{\beta - 1}(x \circ \tau)^\Delta \\ &= \beta(x \circ \tau^\sigma)^{\beta - 1}(x^\Delta \circ \tau)\tau^\Delta. \end{aligned} \tag{4.12}$$

From (4.10) and (4.12), it follows that

$$
\begin{aligned}
w^{\Delta} &\leq -\delta p + \frac{\delta_{+}^{\Delta}}{\delta^{\sigma}} w^{\sigma} - \frac{\delta \left(x^{[1]}\right)^{\sigma} \beta (x \circ \tau^{\sigma})^{\beta-1} (x^{\Delta} \circ \tau) \tau^{\Delta}}{(x \circ \tau)^{\beta} (x \circ \tau^{\sigma})^{\beta}} \\
&= -\delta p + \frac{\delta_{+}^{\Delta}}{\delta^{\sigma}} w^{\sigma} - \beta \tau^{\Delta} \delta \frac{\left(x^{[1]}\right)^{\sigma}}{(x \circ \tau^{\sigma})^{\beta+1}} \frac{(x \circ \tau^{\sigma})^{\beta}}{(x \circ \tau)^{\beta}} (x^{\Delta} \circ \tau). \qquad (4.13)
\end{aligned}
$$

If $\beta \geqslant 1$, then by taking $F = x \circ \tau$ and by Lemma 3.3(ii) and (4.11) we get

$$
\begin{aligned}
\left[(x \circ \tau)^{\beta}\right]^{\Delta} &\geqslant \beta (x \circ \tau)^{\beta-1} (x \circ \tau)^{\Delta} \\
&= \beta (x \circ \tau)^{\beta-1} (x^{\Delta} \circ \tau) \tau^{\Delta}. \qquad (4.14)
\end{aligned}
$$

It follows from (4.10) and (4.14) that

$$
\begin{aligned}
w^{\Delta} &\leq -\delta p + \frac{\delta_{+}^{\Delta}}{\delta^{\sigma}} w^{\sigma} - \frac{\delta \left(x^{[1]}\right)^{\sigma} \beta (x \circ \tau)^{\beta-1} (x^{\Delta} \circ \tau) \tau^{\Delta}}{(x \circ \tau)^{\beta} (x \circ \tau^{\sigma})^{\beta}} \\
&= -\delta p + \frac{\delta_{+}^{\Delta}}{\delta^{\sigma}} w^{\sigma} - \beta \tau^{\Delta} \delta \frac{\left(x^{[1]}\right)^{\sigma}}{(x \circ \tau^{\sigma})^{\beta+1}} \frac{(x \circ \tau^{\sigma})}{(x \circ \tau)} (x^{\Delta} \circ \tau). \qquad (4.15)
\end{aligned}
$$

From (H_4) we see that $\tau(t)$ is increasing on $\mathbb{T}$. Since $t \leq \sigma(t)$, we have $\tau(t) \leq \tau^{\sigma}(t)$. Since $x^{\Delta}(t) > 0$, we obtain $(x \circ \tau)(t) \leq (x \circ \tau^{\sigma})(t)$. Thus for all $\beta > 0$, from (4.13) and (4.15) we get

$$
w^{\Delta} \leq -\delta p + \frac{\delta_{+}^{\Delta}}{\delta^{\sigma}} w^{\sigma} - \beta \tau^{\Delta} \delta \frac{\left(x^{[1]}\right)^{\sigma}}{(x \circ \tau^{\sigma})^{\beta+1}} (x^{\Delta} \circ \tau). \qquad (4.16)
$$

Since $x^{[1]}(t)$ is decreasing and $\tau(t) \leq t \leq \sigma(t)$, we have

$$
\left((r \circ \tau)(x^{\Delta} \circ \tau)^{\gamma}\right)(t) \geqslant (r (x^{\Delta})^{\gamma})^{\sigma}(t)
$$

and

$$
(x^{\Delta} \circ \tau)(t) \geqslant \left[(r (x^{\Delta})^{\gamma})^{\sigma}(t)\right]^{1/\gamma} / \left[(r \circ \tau)(t)\right]^{1/\gamma}. \qquad (4.17)
$$

From (4.16) and (4.17), we conclude

$$
w^{\Delta}(t) \leq -\delta(t) p(t) + \frac{\delta_{+}^{\Delta}(t)}{\delta^{\sigma}(t)} w^{\sigma}(t) - \beta \tau^{\Delta}(t) \delta(t) \frac{\left[\left(x^{[1]}\right)^{\sigma}(t)\right]^{1+1/\gamma}}{(x \circ \tau^{\sigma})^{\beta+1} (r \circ \tau)^{1/\gamma}(t)}. \qquad (4.18)
$$

From (4.18) and definition of $w(t)$ we have

$$
w^{\Delta}(t) \leq -\delta(t) p(t) + \frac{\delta_{+}^{\Delta}(t)}{\delta^{\sigma}(t)} w^{\sigma}(t) - \beta \tau^{\Delta}(t) \delta(t) \frac{(x \circ \tau^{\sigma})^{\beta/\gamma - 1}(t)}{(r \circ \tau)^{1/\gamma}(t)} \left(\frac{w^{\sigma}(t)}{\delta^{\sigma}(t)}\right)^{1+1/\gamma} \quad t \geqslant t_1. \qquad (4.19)
$$

Next, we consider the three cases: $\beta > \gamma,\ \beta = \gamma$ and $\beta < \gamma$.

Case (i). Let $\beta > \gamma$. Since $\tau(t)$ is increasing on $\mathbb{T}$ and $\sigma(t) \geqslant t \geqslant t_2$ on $[t_2, \infty)_{\mathbb{T}}$, we get $\tau^{\sigma}(t) \geqslant \tau(t_2)$ and $(x \circ \tau^{\sigma})(t) \geqslant (x \circ \tau)(t_2) := c$. Therefore, for $t \in [t_2, \infty)_{\mathbb{T}}$ we conclude

$$(x \circ \tau^{\sigma})^{\beta/\gamma - 1}(t) \geqslant c^{\beta/\gamma - 1} := c_1 > 0 \tag{4.20}$$

Case (ii). Let $\beta = \gamma$. Then we have

$$(x \circ \tau^{\sigma})^{\beta/\gamma - 1}(t) = 1 \; for \; t \in [t_2, \infty)_{\mathbb{T}} \tag{4.21}$$

Case (iii). Let $\beta < \gamma$. From Lemma 3.1, since $x^{[1]}(t)$ is positive and decreasing, we see that $x^{[1]}(t) \leq x^{[1]}(t_2) = c$, for $t \in [t_2, \infty)_{\mathbb{T}}$. This implies that

$$x^{\Delta}(t) \leq \frac{c^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)}. \tag{4.22}$$

Integrating both sides of above equation from t_2 to $\tau^{\sigma}(t)$ we get

$$(x \circ \tau^{\sigma})(t) \leq x(t_2) + c^{\frac{1}{\gamma}} \int_{t_2}^{\tau^{\sigma}(t)} \frac{1}{r^{\frac{1}{\gamma}}(s)} \Delta s.$$

Therefore, we get

$$\big((x \circ \tau^{\sigma})(t)\big)^{\beta/\gamma - 1} > c_2 \left(\int_{t_2}^{\tau^{\sigma}(t)} \frac{1}{r^{\frac{1}{\gamma}}(s)} \Delta s \right)^{\beta/\gamma - 1} \tag{4.23}$$

where $c_2 = (c^{1/\gamma})^{\beta/\gamma - 1}$. Using these three cases in (4.19) and using the definition of $\eta^{\sigma}(t)$, we get

$$w^{\Delta}(t) \leq -\delta(t)p(t) + \frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)} w^{\sigma}(t) - \beta \tau^{\Delta}(t)\delta(t) \frac{\eta^{\sigma}(t)}{(r \circ \tau)^{1/\gamma}(t)} \left(\frac{w^{\sigma}(t)}{\delta^{\sigma}(t)} \right)^{1+1/\gamma} \quad t \geqslant t_2. \tag{4.24}$$

Taking $\lambda = 1 + 1/\gamma$,

$$X = \left[\beta \tau^{\Delta}(t)\delta(t) \frac{\eta^{\sigma}(t)}{(r \circ \tau)^{1/\gamma}(t)} \right]^{1/\lambda} \left(\frac{w^{\sigma}(t)}{\delta^{\sigma}(t)} \right)$$

and

$$Y = \frac{\left[\delta_+^{\Delta}(t)\right]^{\gamma}}{\lambda^{\gamma}} \left[\beta \tau^{\Delta}(t)\delta(t) \frac{\eta^{\sigma}(t)}{(r \circ \tau)^{1/\gamma}(t)} \right]^{-\gamma/\lambda} \; for \; t \geqslant t_2,$$

by Lemma 3.4, and (4.24) we have

$$w^{\Delta}(t) \leq -\delta(t)p(t) + \frac{(\gamma/\beta)^{\gamma} r(\tau(t))(\delta_+^{\Delta}(t))^{\gamma+1}}{(\gamma+1)^{\gamma+1} \left[\tau^{\Delta}(t)\delta(t)\eta^{\sigma}(t)\right]^{\gamma}}, \; t \geqslant t_2.$$

Integrating both sides of the last inequality from t_2 to t, we obtain

$$w(t) - w(t_2) \leq - \int_{t_2}^{t} \left[\delta(s)p(s) - \frac{(\gamma/\beta)^{\gamma} r(\tau(s))(\delta_+^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left[\tau^{\Delta}(s)\delta(s)\eta^{\sigma}(s)\right]^{\gamma}} \right] \Delta s.$$

Since $w(t) > 0$ for $t \geqslant t_2$, we have

$$\int_{t_2}^{t} \left[\delta(s)p(s) - \frac{(\gamma/\beta)^{\gamma} r(\tau(s))(\delta_+^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left[\tau^{\Delta}(s)\delta(s)\eta^{\sigma}(s)\right]^{\gamma}} \right] \Delta s \leq w(t_2) - w(t) < w(t_2) \; for \, t \geqslant t_2.$$

Therefore, we get

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[\delta(s)p(s) - \frac{(\gamma/\beta)^{\gamma} r(\tau(s))(\delta_+^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left[\tau^{\Delta}(s)\delta(s)\eta^{\sigma}(s)\right]^{\gamma}} \right] \Delta s \leq w(t_2) < \infty,$$

which contradicts condition (4.7). Then every solution of (1.1) oscillates. The proof is complete. ■

From Theorem 4.2, we can obtain different conditions for the oscillation of (1.1) by using different choices of $\delta(t)$. For instance, if $\delta(t) = t$, we have the following result.

Corollary 4.3. Assume that (H_1,) - (H_5) and (1.2) hold. Furthermore assume that

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[sp(s) - \frac{(\gamma/\beta)^{\gamma} r(\tau(s))}{(\gamma+1)^{\gamma+1}\left[\tau^{\Delta}(s)s\eta^{\sigma}(s)\right]^{\gamma}} \right] \Delta s = \infty, \tag{4.25}$$

then every solution of (1.1) oscillates.

The following Theorem gives the extension of the Kamenev-type oscillation criterion for equation (1.1).

First, let us introduce the class of functions $\Re$ which will be extensively used in the sequel. Let $\mathbb{D}_0 = \left\{(t,s) \in \mathbb{T}^2;\ t > s \geqslant t_0\right\}$ and $\mathbb{D} = \left\{(t,s) \in \mathbb{T}^2;\ t \geqslant s \geqslant t_0\right\}$. The function $H \in C_{rd}(\mathbb{D}, \mathbb{R})$ is said belongs to the class $\Re$ if

(i) $H(t,t) = 0,\ t \geqslant t_0,\ H(t,s) > 0 \; on \; \mathbb{D}_0.$

(ii) H has a continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ on $\mathbb{D}_0$ with respect to the second variable. (H is rd-continuous function if H is rd-continuous function in t and s.)

Theorem 4.4. Assume that (H_1,)–(H_5) and (1.2) hold. Let $h, H : D \to R$ be rd-continuous functions such that H belongs to the class $\Re$, and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\delta(s)p(s) - \frac{(\gamma/\beta)^{\gamma} r(\tau(s))(h_+(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left[\tau^{\Delta}(s)\delta(s)\eta^{\sigma}(s)\right]^{\gamma}} \right] \Delta s = \infty, \tag{4.26}$$

where

$$H^{\Delta_s}(t,s)+H(t,s)\frac{\delta_+^{\Delta}(s)}{\delta^{\sigma}(s)}=\frac{h(t,s)}{\delta^{\sigma}(s)}H^{\frac{\gamma}{\gamma+1}}(t,s)\ and\ h_+(t,s):=\max\{h(t,s),0\} \tag{4.27}$$

then every solution of (1.1) oscillates.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality we may assume that $x(t)>0,\ x(\tau(t))>0,\ and\ x^{\sigma}(\tau(t))>0$ for $t\geqslant t_0$ sufficiently large. We proceed as in the proof of Theorem 4.2, to prove that there exists $t_1\geqslant t_0$ such that (4.24) holds for $t\geqslant t_1$. From (4.24), it follows that

$$\delta(t)p(t)\leq -w^{\Delta}(t)+\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t)-\frac{\beta\tau^{\Delta}(t)\delta(t)\eta^{\sigma}(t)}{(r\circ\tau)^{1/\gamma}(t)}\left(\frac{w^{\sigma}(t)}{\delta^{\sigma}(t)}\right)^{1+1/\gamma}. \tag{4.28}$$

Multiplying (4.28) by $H(t,s)$ and integrating from t_2 to t, we have

$$\begin{aligned}\int_{t_2}^{t}H(t,s)\delta(s)p(s)\Delta s &\leq -\int_{t_2}^{t}H(t,s)w^{\Delta}(s)\Delta s+\int_{t_2}^{t}H(t,s)\frac{\delta_+^{\Delta}(s)}{\delta^{\sigma}(s)}w^{\sigma}(s)\Delta s\\ &- \int_{t_2}^{t}H(t,s)\frac{\beta\tau^{\Delta}(s)\delta(s)\eta^{\sigma}(s)}{(r\circ\tau)^{1/\gamma}(s)}\left(\frac{w^{\sigma}(s)}{\delta^{\sigma}(s)}\right)^{1+1/\gamma}\Delta s. \quad (4.29)\end{aligned}$$

Using integration by parts formula, we have

$$\begin{aligned}\int_{t_2}^{t}H(t,s)w^{\Delta}(s)\Delta s &= H(t,s)w(s)|_{t_2}^{t}-\int_{t_2}^{t}H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s\\ &= -H(t,t_2)w(t_2)-\int_{t_2}^{t}H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s \quad (4.30)\end{aligned}$$

Substitute (4.30) in (4.29), we get

$$\begin{aligned}\int_{t_2}^{t}H(t,s)\delta(s)p(s)\Delta s &\leq H(t,t_2)w(t_2)+\int_{t_2}^{t}\left[\left(H^{\Delta_s}(t,s)+H(t,s)\frac{\delta_+^{\Delta}(s)}{\delta^{\sigma}(s)}\right)w^{\sigma}(s)\right.\\ &- \left.H(t,s)\frac{\beta\tau^{\Delta}(s)\delta(s)\eta^{\sigma}(s)}{(r\circ\tau)^{1/\gamma}(s)}\left(\frac{w^{\sigma}(s)}{\delta^{\sigma}(s)}\right)^{1+1/\gamma}\right]\Delta s. \quad (4.31)\end{aligned}$$

In view of (4.27), we have

$$\begin{aligned}\int_{t_2}^{t} H(t,s)\delta(s)p(s)\Delta s &\le H(t,t_2)w(t_2) + \int_{t_2}^{t}\left[\frac{h(t,s)}{\delta^\sigma(s)}H^{\frac{\gamma}{\gamma+1}}(t,s)w^\sigma(s)\right.\\ &- \left. H(t,s)\frac{\beta\tau^\Delta(s)\delta(s)\eta^\sigma(s)}{(r\circ\tau)^{1/\gamma}(s)}\left(\frac{w^\sigma(s)}{\delta^\sigma(s)}\right)^{1+1/\gamma}\right]\Delta s\\ &\le H(t,t_2)w(t_2) + \int_{t_2}^{t}\left[\frac{h_+(t,s)}{\delta^\sigma(s)}H^{\frac{\gamma}{\gamma+1}}(t,s)w^\sigma(s)\right.\\ &- \left. H(t,s)\frac{\beta\tau^\Delta(s)\delta(s)\eta^\sigma(s)}{(r\circ\tau)^{1/\gamma}(s)}\left(\frac{w^\sigma(s)}{\delta^\sigma(s)}\right)^{1+1/\gamma}\right]\Delta s.\end{aligned} \tag{4.32}$$

for $t \geqslant t_2$ where h_+ is defined in Theorem 4,3.

Taking $\lambda = 1 + 1/\gamma$,

$$X = \left[H(t,s)\frac{\beta\tau^\Delta(s)\delta(s)\eta^\sigma(s)}{(r\circ\tau)^{1/\gamma}(s)}\right]^{\frac{1}{\lambda}}\frac{w^\sigma(s)}{\delta^\sigma(s)}$$

and

$$Y = \frac{\left[h_+(t,s)H^{\frac{\gamma}{\gamma+1}}(t,s)\right]^\gamma}{\lambda^\gamma}\left[H(t,s)\beta\tau^\Delta(s)\delta(s)\frac{\eta^\sigma(s)}{(r\circ\tau)^{1/\gamma}(s)}\right]^{-\gamma/\lambda} \text{ for } t \geqslant t_2,$$

by Lemma 3.4, and (4.32) we have

$$\int_{t_2}^{t} H(t,s)\delta(s)p(s)\Delta s \le H(t,t_2)w(t_2) + \int_{t_2}^{t}\frac{(\gamma/\beta)^\gamma r(\tau(s))(h_+(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left[\tau^\Delta(s)\delta(s)\eta^\sigma(s)\right]^\gamma}\Delta s \text{ for } t \geqslant t_2.$$

Therefore, we obtain

$$\frac{1}{H(t,t_2)}\int_{t_2}^{t}\left[H(t,s)\delta(s)p(s) - \frac{(\gamma/\beta)^\gamma r(\tau(s))(h_+(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left[\tau^\Delta(s)\delta(s)\eta^\sigma(s)\right]^\gamma}\right]\Delta s \le w(t_2) \text{ for } t \geqslant t_2$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_2}^{t}\left[H(t,s)\delta(s)p(s) - \frac{(\gamma/\beta)^\gamma r(\tau(s))(h_+(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left[\tau^\Delta(s)\delta(s)\eta^\sigma(s)\right]^\gamma}\right]\Delta s \le w(t_2) < \infty,$$

which contradicts (4.26). The proof is complete. ∎

Corollary 4.5. Let the assumption (4.26) in Theorem 4.3, be replaced by

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\delta(s)p(s)\Delta s=\infty,$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{(\gamma/\beta)^{\gamma}r(\tau(s))(h_+(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left[\tau^{\Delta}(s)\delta(s)\eta^{\sigma}(s)\right]^{\gamma}}\Delta s<\infty.$$

Then every solution of equation (1.1) oscillates on $[t_0,\infty)$.

As a special case by choosing $H(t,s)=(t-s)^m$ for $m\geqslant 1$. We have from Theorem 4.3 the following Kamenev-type oscillation criterion.

Corollary 4.6. Assume that (H_1,)–(H_5) and (1.2) hold. If for $m>1$,

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_0}^{t}\left[(t-s)^m\delta(s)p(s)-\frac{(\gamma/\beta)^{\gamma}r(\tau(s))(\delta^{\sigma})^{\gamma+1}K^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}(t-s)^{m\gamma}\left[\tau^{\Delta}(s)\delta(s)\eta^{\sigma}(s)\right]^{\gamma}}\right]\Delta s=\infty, \tag{4.33}$$

where

$$K(t,s)=(t-s)^m\frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}-m(t-\sigma(s))^{m-1}$$

then every solution of (1.1) is oscillatory on $[t_0,\infty)$.

Theorem 4.7. Assume that (H_1)–(H_5) and (1.3) hold. Furthermore, assume that

$$\int_{t_0}^{\infty}\left[\frac{1}{r(t)}\int_{t_0}^{t}p(s)\Delta s\right]^{\frac{1}{\gamma}}\Delta t=\infty, \tag{4.34}$$

hold. If one of the conditions (4.7) and (4.26) holds, then every solution of (1.1) oscillates or converges to zero.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality we may assume that $x(t)$ is eventually positive. Proceeding as in the proof of Theorem 4.1, we see that there exists $t\geqslant t_2$ such that (4.4) holds. Integrating both sides of (4.4) from t_3 to t, we have

$$\begin{aligned} r(t)(x^{\Delta}(t))^{\gamma} &\le r(t_3)(x^{\Delta}(t_3))^{\gamma}-c_1\int_{t_3}^{t}p(u)\Delta u\\ &\le -c_1\int_{t_3}^{t}p(u)\Delta u. \end{aligned} \tag{4.35}$$

It follows from this last inequality that

$$x^{\Delta}(t) \leq \left[-c_1 \frac{1}{r(t)} \int_{t_3}^{t} p(u)\Delta u\right]^{\frac{1}{\gamma}}$$

again integrating from t_3 to t, we get

$$x(t) - x(t_3) \leq -c_2 \int_{t_3}^{t} \left[\frac{1}{r(s)} \int_{t_3}^{s} p(u)\Delta u\right]^{\frac{1}{\gamma}} \Delta s.$$

Hence by (4.1), we have $\lim_{t\to\infty} x(t) = -\infty$ which contradicts the fact that x is a positive solution of (1.1). Thus $c_2 = 0$ and then $x(t) \to 0$ as $t \to \infty$. The proof is complete. ■

5. Examples

In this section we present some examples to illustrate the main results.

Example 5.1. Consider the following second order delay dynamic equation

$$(\sqrt{t}(x^{\Delta}(t))^{\frac{5}{9}})^{\Delta} + \frac{1}{t}(x^{\sigma}(t-a))^{\frac{3}{13}} = 0 \; for \; t \in [t_0, \infty)_{\mathbb{T}} \tag{5.1}$$

where $\mathbb{T}$ is a time scale, a is arbitrary positive real number and $t_0 = a$.

In (5.1), $\gamma = \frac{5}{9}$, $\beta = \frac{3}{13}$, $r(t) = \sqrt{t}$, $p(t) = \frac{1}{t}$ and $\tau(t) = t - a$. Then we have

$$\int_{t_0}^{\infty} r^{-\frac{1}{\gamma}}(t)\Delta t = \int_{a}^{\infty} \frac{1}{t^{9/10}}\Delta t = \infty$$

and

$$\int_{t_0}^{\infty} p(t)\Delta t = \int_{a}^{\infty} \frac{1}{t}\Delta t = \infty.$$

Therefore, the conditions (H_1), (H_2), (1.2) and (4.1) are satisfied. By Theorem 4.1, every solution of (5.1) is oscillatory.

Example 5.2. Consider the Δ-differential Euler equation

$$x^{\Delta\Delta}(t) + \frac{\lambda}{t\sigma(t)}x^{\sigma}(t) = 0 \; t \in [1, \infty). \tag{5.2}$$

Here $\tau(t)=t,\ \gamma=\beta=1,\ r(t)=1,\ \lim_{t\to\infty}\tau(t)=\infty$, and $p(t)=\dfrac{\lambda}{t\sigma(t)}$ where λ is a constant. If $\mathbb{T}=\mathbb{R}$ then (5.2) is the second order Euler differential equation

$$x''(t)+\frac{\lambda}{t^2}x(t)=0,\ t\geqslant 1,$$

then $\mu(t)=0,\ and\ \sigma(t)=t$. It is easy to see that assumption (1.2) holds and also (4.6) is satisfied, since

$$\int_{t_0}^{\infty}p(s)\Delta s=\lambda\int_{t_0}^{\infty}\frac{1}{s^2}\Delta s<\infty.$$

To apply Corollary 4.1, it remains to discuss condition (4.25). Note

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[sp(s)-\frac{(\gamma/\beta)^{\gamma}r(\tau(s))}{(\gamma+1)^{\gamma+1}(\tau^{\Delta}(s)s\eta^{\sigma}(s))^{\gamma}}\right]\Delta s$$

$$=\limsup_{t\to\infty}\int_{1}^{t}\left[\frac{\lambda}{s}-\frac{1}{4s}\right]\Delta s=\infty$$

If $\mathbb{T}=\mathbb{Z}$ then (5.2) is the second order discrete Euler difference equation

$$\Delta^2x_t+\frac{\lambda}{t(t+1)}x_{t+1}=0,\ t=1,1,...$$

then $\mu(t)=1,\ and\ \sigma(t)=t+1$. It is easy to see that assumption (1.2) holds and also (4.6) is satisfied, since

$$\int_{t_0}^{\infty}p(s)\Delta s=\lambda\int_{t_0}^{\infty}\frac{1}{s(s+1)}\Delta s<\infty.$$

To apply Corollary 4.1, it remains to discuss condition (4.25). Note

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[sp(s)-\frac{r(s)}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right]\Delta s$$

$$=\limsup_{t\to\infty}\int_{1}^{t}\left[\frac{\lambda}{s+1}-\frac{1}{4s}\right]\Delta s=\infty$$

Hence, by Corollary 4.1, every solution of (5.2) oscillates.

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References

[1] Agarwal, R.P., Bohner, M., and Saker, S.H., 2005, "Oscillation of second order delay dynamic equations," Canad. Appl. Math. Quart., 13, pp. 1–17.

[2] Agarwal, R.P., O'Regan, D., and Saker, S.H., 2004, "Oscillation criteria for second order nonlinear neutral delay dynamic equations," J. Math. Anal. Appl., 300, pp. 203–217.

[3] Agarwal, R.P., O'Regan, D., and Saker, S.H., 2006, "Oscillation criteria for nonlinear perturbed dynamic equations of second order on time scales," J. Appl. Math. Comput., 20, pp. 133–147.

[4] Agarwal, R.P., O'Regan, D., and Saker, S.H., 2006, "Properties of bounded solutions of nonlinear dynamic equations on time scales," Canad. Appl. Math. Quart., 14, pp. 1–10.

[5] Agarwal, R.P., O'Regan, D., and Saker, S.H., 2007, "Oscillation results for second order nonlinear neutral delay dynamic equations on time scales," Appl. Anal., 86, pp. 1–17.

[6] Agarwal, R.P., O'Regan, D., and Saker, S.H., 2007, "Oscillation of second order damped dynamic equations on time scales," J. Math. Anal. Appl., 330, pp. 1317–1337.

[7] Agarwal, R.P., O'Regan, D., and Saker, S.H., 2008, "Oscillation theorems for second order nonlinear neutral delay dynamic equations on time scales," Acta. Math. Sinica, 24, pp. 1409–1432.

[8] Akinn-Bohner, E., Bohner, M., and Saker, S.H., 2007, "Oscillation for a certain of class of second order Emden-Fowler dynamic equations," Electron. Trans. Numer. Anal., 27, pp. 1–12.

[9] Akinn-Bohner, E., and J. Hoffacker, J., 2003, "Oscillation properties of an Emden-Fowler type equations on discrete time scales," J. Difference Equ. Appl., 9, pp. 603–612.

[10] Bohner, M., and Peterson, A., 2001, "Dynamic Equations on Time Scales: An Introduction with Applications," Birkhäuser: Boston.

[11] Bohner, M., and Peterson, A., 2003, "Advances in Dynamic Equations on Time Scales," Birkhäuser: Boston.

[12] Bohner, M., and Saker. S.H., 2004, "Oscillation of second order nonlinear dynamic equations on time scales," Rocky Mountain J. Math., 34, pp. 1239–1254.

[13] Bohner, M., and Saker. S.H., 2004, "Oscillation criteria for perturbed nonlinear dynamic equations," Math. Comput. Modelling, 40, pp. 249–260.

[14] Chen, D.X., 2010, "Oscillation of second order Emden-Fowler neutral delay dynamic equations on time scales," Math. Comput. Modelling, 51, pp. 1221–1229.

[15] Chen, D.X., and Nie, C.Y., 2016, "Oscillation of even order nonlinear functional dynamic equations on time scales," J. Appl. Math., 46, pp. 535–544.

[16] Erbe, L., Hassan, T.S., Peterson, A., and Saker, S.H., 2006, "Hille-Kneser type criteria for second order dynamic equations on time scales," Adv. Differential Equations, 2006, pp. 1–18.

[17] Erbe, L., Hassan, T.S., Peterson, A., and Saker, S.H., 2007, "Oscillation criteria for second order nonlinear delay dynamic equations," J. Math. Anal. Appl., 333, pp. 505–522.

[18] Erbe, L., Hassan, T.S., Peterson, A., and Saker, S.H., 2008, "Oscillation criteria for sublinear halflinear delay dynamic equations on time scales," Int. J. Difference Equations, 3, pp. 227–245.

[19] Erbe, L., Hassan, T.S., Peterson, A., and Saker, S.H., 2008, "Oscillation criteria for a forced second order nonlinear dynamic equation," J. Difference Eqn. Appl., 14, pp. 997–1009.

[20] Erbe, L., Hassan, T.S., Peterson, A., and Saker, S.H., 2009, "Oscillation criteria for halflinear delay dynamic equations on time scales," Nonlinear Dyn. Syst. Theory, 9, pp. 51–68.

[21] Erbe, L., Peterson, A., and Saker, S.H., 2003, "Oscillation criteria for second order nonlinear dynamic equations on time scales," J. London Math. Soc., 76 (2), pp. 701–714.

[22] Grace, S., Agarwal, R.P., Kaymak¸calan, B., and Sae-Jie. W., 2009, "On the oscillation of certain second order nonlinear dynamic equations," Math. Comput. Modelling, 50, pp. 273–286.

[23] Hardy, G.H., Littlewood, J.E., and Pólya, G., 1988, "Inequalities," second ed., Cambridge: Cambridge University Press.

[24] Saker, S.H., 2004, "Oscillation of nonlinear dynamic equations on time scales," Appl. Math. Comput., 148, pp. 81–91.

[25] Saker, S.H., 2005, "Oscillation criteria of second order half-linear dynamic equations on time scales," J. Comput. Appl. Math., 177, pp. 375–387.

[26] Saker, S.H., 2005, "Oscillation of second order forced nonlinear dynamic equations on time scales," Electron. J. Qual. Theory Differ, Equ., 23, pp. 1–17.

[27] Saker, S.H., 2007, "Oscillation of second order nonlinear neutral delay dynamic equations on time scales," J. Comput. Appl. Math., 187, pp. 125–141.

[28] Saker, S.H., and Grace, S.R., 2012, "Oscillation criteria for quasi-linear functional dynamic equations on time scales," Math. Slovaca, 62, pp. 501–524.