Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 10 (2017), pp. 7427-7439

http://www.ripublication.com

Blow-up Phenomena and Global Existence for a Semilinear Heat Equation with Nonlinear Neumann Boundary Condition

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Abstract

This article deal with the blow-up and global solution for a semilinear heat equation under the nonlinear Neumann boundary condition. we prove that under condition the blow-up will occur at some finite time, an upper and lower bound time.

Keyword: Blow-up; Global existence; semilinear heat equation

1 INTRODUCTION

In this paper, we consider the following a semilinear heat equation under the nonlinear boundary conditions problem

$$\begin{cases} u_t = \nabla\left(|\nabla u|^p \nabla u\right) - (p+2)u^r, (x,t) \in \Omega \times \left(0, t^*\right) \\ |\nabla u|^p \dfrac{\partial u}{\partial n} = f(u), (x,t) \in \partial\Omega \times \left(0, t^*\right) \\ u(x,0) = u_0(x) \geq 0, x \in \Omega \end{cases} \tag{1}$$

Project Name: The Natural Science Fund of Department of Education of Sichuan Province(09ZA119).

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where $p \geq 0$, Ω is a bounded domain in $R^N(N \geq 2)$ with smooth boundary $\partial\Omega$ and $u_0(x)$ is the initial data , $\frac{\partial u}{\partial n}$ is the outward normal derivative of u on the boundary $\partial\Omega$, t^* is the blow-up time when blow-up occurs.

The phenomenon of blow-up and global existence for the nonlinear Neumann boundary conditions can be some result in the paper. This problem is worth studying. In the past decades, there have been many works dealing with global existence or nonexistence and the blow-up in finite time of solutions to semilinear heat equation (one can see[1-7]). we also refer the reader to[8-10]and the references therein.

In recent years, Ahmed, Mu and Zhang in[11]considered the following initial boundary value problem

$$\begin{cases} u_t = \operatorname{div}\left(|\nabla u|^{r-2}\nabla u\right) - f(u), (x,t) \in \Omega \times (0,t^*) \\ |\nabla u|^{r-2}\frac{\partial u}{\partial n} = g(u), (x,t) \in \partial\Omega \times (0,t^*) \\ u(x,0) = u_0(x) > 0, x \in \Omega \end{cases} \tag{2}$$

where $r \geq 2$, they establish sufficient condition the solution $u(x,t)$ exists globally and an upper bound establish for the blow up in finite time t^*.

In particular[12], Pavne et al. also studied the following semilinear heat equation problem

$$\begin{cases} u_t = \nabla\left(|\nabla u|^{2p}\nabla u\right), x \in \Omega, t \in (0,t^*), \\ |\nabla u|^{2p}\frac{\partial u}{\partial n} = f(u), x \in \partial\Omega, t \in (0,t^*), \\ u(x,0) = u_0(x) \geq 0, x \in \Omega. \end{cases} \tag{3}$$

they by using the suitable techniques of differential inequalities , derived blow-up occurs upper and lower bounds of the blow-up time.

The rest of this paper is orgainzed as follows.

2 THE GLOBAL EXISTENCE

In this section, we establish the sufficient condition on the function f to guarantee that $u(x,t)$ exists globally. we get the following result.

Theorem1. Let $u(x,t)$ be the solution of problem (1) and assume that the nonnegative functions f satisfy conditions

$$f(\xi)\geq k_1\xi^q, \qquad \xi\geq 0 \tag{4}$$

where $k\geq 0$, $q>1$ and $q-1<p<r+1$. Then the nonnegative solution $u(x,t)$ of problem (1) exists globally for all time $t>0$.

Proof. We set

$$\Phi(t)=\int_\Omega u^2dx \tag{5}$$

Taking the derivative of $\Phi(t)$ with respect to t, then it follows from (1),(4) and (5), we have

$$\Phi'(t)\leq 2k_1\int_{\partial\Omega}u^{q+1}ds-2\int_\Omega|\nabla u|^{p+2}dx-2(p+2)\int_\Omega u^{r+1}dx, \tag{6}$$

By Lemma2.1 in[11], we get $$\int_{\partial\Omega}u^{q+1}ds\leq\frac{N}{\rho_0}\int_\Omega u^{q+1}dx+\frac{(q+1)d}{\rho_0}\int_\Omega u^q|\nabla u|dx, \tag{7}$$

where $\rho_0=\min_{x\in\partial\Omega}(|x\cdot n|)>0$, $d=\max_{x\in\partial\Omega}|x|$. Combining (6) and (7), we obtain

$$\Phi'(t)\leq\frac{2k_1N}{\rho_0}\int_\Omega u^{q+1}dx+\frac{2k_1(q+1)d}{\rho_0}\int_\Omega u^q|\nabla u|dx-2\int_\Omega|\nabla u|^{p+2}dx-2(p+2)\int_\Omega u^{r+1}dx, \tag{8}$$

By using Young's inequality, we

get $$\int_\Omega u^q|\nabla u|dx\leq\frac{(p+1)\varepsilon}{p+2}\int_\Omega u^{\frac{(p+2)q}{p+1}}dx+\frac{1}{(p+2)\varepsilon}\int_\Omega|\nabla u|^{p+2}dx, \tag{9}$$

for $\varepsilon > 0$. Combining (8) and (9), we have

$$\Phi'(t) \le \frac{2k_1 N}{\rho_0}\int_\Omega u^{q+1}dx + \frac{2k_1(q+1)(p+1)d\varepsilon}{\rho_0(p+2)}\int_\Omega u^{\frac{(p+2)q}{p+1}}dx - \left[2 - \frac{2k_1(q+1)d}{\rho_0(p+2)\varepsilon}\right]\int_\Omega |\nabla u|^{p+2}dx - 2(p+2)\int_\Omega u^{r+1}dx \quad , \tag{10}$$

Since $\left|\nabla u^{\frac{p}{2}+1}\right|^2 = \left(\frac{p}{2}+1\right)^2 u^p |\nabla u|^2$. Make using of Hölder inequality ,we get

$$\int_\Omega \left|\nabla u^{\frac{p}{2}+1}\right|^2 dx \le \left(\frac{p}{2}+1\right)^2 \left(\int_\Omega |\nabla u|^{p+2}dx\right)^{\frac{2}{p+2}} \cdot \left(\int_\Omega u^{p+2}dx\right)^{\frac{p}{p+2}}, \tag{11}$$

By membrane inequality $\lambda_1 \int_\Omega \varpi^2 dx \le \int_\Omega |\nabla \varpi|^2 dx$ in [13]. where λ_1 is the first eigenvalue in the fixed membrane problem $\Delta\varpi + \lambda_1\varpi = 0$, $\varpi > 0$ in Ω, $\varpi = 0$ on $\partial\Omega$.

And follows from upper, we have $\int_\Omega u^{p+2}dx \le \left[\frac{(p+2)^2}{4\lambda_1}\right]^{\frac{p}{2}+1} \cdot \int_\Omega |\nabla u|^{p+2}dx$, (12)

By Hölder inequality, we have $\int_\Omega u^{\frac{(p+2)q}{p+1}}dx \le \left(\int_\Omega u^{q+1}dx\right)^\alpha \cdot \left(\int_\Omega u^{r+1}dx\right)^{1-\alpha}$, (13)

with $\alpha = \frac{(p+2)q-(p+1)(r+1)}{(p+1)(q-r)}$. We applying the following inequality

$$a_1^{s_1}a_2^{s_2} \le s_1a_1 + s_2a_2,\ a_1, a_2 > 0,\ s_1, s_2 > 0 \text{ and } s_1 + s_2 = 1. \tag{14}$$

From (14), we have

$$\int_\Omega u^{\frac{(p+2)q}{p+1}}dx \le \left(\kappa^{\frac{\alpha-1}{\alpha}}\int_\Omega u^{q+1}dx\right)^\alpha \cdot \left(\kappa\int_\Omega u^{r+1}dx\right)^{1-\alpha} \le \alpha\kappa^{\frac{\alpha-1}{\alpha}}\int_\Omega u^{q+1}dx + (1-\alpha)\kappa\int_\Omega u^{r+1}dx, \tag{15}$$

for $\kappa > 0$. By inserting (15) in (10), we obtain (16)as follows

$$\Phi'(t)\le\left\{\left[\frac{2k_1N}{\rho_0}+\frac{2k_1(q+1)(p+1)d\varepsilon}{\rho_0(p+2)}\alpha\kappa^{\frac{\alpha-1}{\alpha}}\right]\int_\Omega u^{q+1}dx-2\left[\frac{4\lambda_1}{(p+2)^2}\right]^{\frac{p}{2}+1}\int_\Omega u^{p+2}dx\right\}$$

$$+\left\{\frac{2k_1(q+1)d}{\rho_0(p+2)\varepsilon}\cdot\left[\frac{4\lambda_1}{(p+2)^2}\right]^{\frac{p}{2}+1}\int_\Omega u^{p+2}dx-\left[2(p+2)-\frac{2k_1(q+1)(p+1)d\varepsilon}{\rho_0(p+2)}(1-\alpha)\kappa\right]\int_\Omega u^{r+1}dx\right\}$$ B

$$=I_1+I_2$$

y Hölder inequality, we have

$$\int_\Omega u^{q+1}dx\le\left(\int_\Omega u^{p+2}dx\right)^{\frac{q+1}{p+2}}\cdot|\Omega|^{\frac{p-q+1}{p+2}} \quad\text{and}\quad \Phi(t)=\int_\Omega u^2dx\le\left(\int_\Omega u^{q+1}dx\right)^{\frac{2}{q+1}}\cdot|\Omega|^{\frac{q-1}{q+1}}, \tag{17}$$

Combining (16) and (17), we have
$$I_1\le\int_\Omega u^{q+1}dx\left[M_1-M_2\Phi^{\frac{p-q+1}{2}}(t)\right], \tag{18}$$

where
$$M_1=\frac{2k_1N}{\rho_0}+\frac{2k_1(q+1)(p+1)d\varepsilon}{\rho_0(p+2)}\alpha\kappa^{\frac{\alpha-1}{\alpha}}>0 \quad M_2=2\left[\frac{4\lambda_1}{(P+2)^2}\right]^{\frac{P}{2}+1}\cdot|\Omega|^{-\frac{p-q+1}{2}}>0 \tag{19}$$

By again Hölder inequality, we have

$$\int_\Omega u^{p+2}dx\le\left(\int_\Omega u^{r+1}dx\right)^{\frac{p+2}{r+1}}\cdot|\Omega|^{\frac{r-p-1}{r+1}} \quad\text{and}\quad \Phi(t)=\int_\Omega u^2dx\le\left(\int_\Omega u^{r+1}dx\right)^{\frac{2}{r+1}}\cdot|\Omega|^{\frac{r-1}{r+1}} \tag{20}$$

Combining(16) and (20), we have
$$I_2\le\left(\int_\Omega u^{r+1}dx\right)^{\frac{p+2}{r+1}}\left[M_3-M_4\Phi^{\frac{r-p-1}{2}}(t)\right], \tag{21}$$

Where
$$M_3=\frac{2k_1(q+1)d}{\rho_0(p+2)\varepsilon}\cdot\left[\frac{4\lambda_1}{(p+2)^2}\right]^{\frac{p}{2}+1}\cdot|\Omega|^{\frac{r-p-1}{r+1}}>0, \tag{22}$$

$$M_4=\left[2(p+2)-\frac{2k_1(q+1)(p+1)d\varepsilon}{\rho_0(p+2)}(1-\alpha)\kappa\right]\cdot|\Omega|^{\frac{(1-r)(r-p-1)}{2(r+1)}}, \tag{23}$$

We can choose κ small enough to have $M_4>0$. From (16),(18) and (21), we obtain

$$\Phi'(t) \le \int_\Omega u^{q+1}dx\left[M_1 - M_2\Phi^{\frac{p-q+1}{2}}(t)\right] + \left(\int_\Omega u^{r+1}dx\right)^{\frac{p+2}{r+1}}\left[M_3 - M_4\Phi^{\frac{r-p-1}{2}}(t)\right], \tag{24}$$

Hence, from the type we can get the following conclusion

(1). If $\Phi(t) < \min\left\{\left(\frac{M_1}{M_2}\right)^{\frac{2}{p-q+1}}, \left(\frac{M_3}{M_4}\right)^{\frac{2}{r-p-1}}\right\}$,The conclusion is true. $\Phi(t)$ remains bounded.

(2). If $\Phi(t) \ge \max\left\{\left(\frac{M_1}{M_2}\right)^{\frac{2}{p-q+1}}, \left(\frac{M_3}{M_4}\right)^{\frac{2}{r-p-1}}\right\}$,we can conclude that $\Phi(t)$ is decreasing in each time interval. At this time, we have $\Phi(t) < \Phi(0)$. So that $\Phi(t)$ remains bounded for all time under the conditions in Theorem1. This completes the proof.

3 BLOW-UP AND UPPER BOUND OF t^*

In this section, we establish the conditions on the data of problem (1) under which the solution will blow-up in finite time and an upper bound for t^*. In fact, we prove the following result.

Theorem2. Let $u(x,t)$be the solution of (1), assume nonnegative function f satisfy conditions

$$\xi f(\xi) \ge (p+2)F(\xi) \text{ and } F(\xi) = \int_0^\xi f(\eta)d\eta, \tag{25}$$

Moreover we assume $\Psi(0) \ge 0$ with $\Psi(t) = 2\int_{\partial\Omega} F(u)ds - \int_\Omega |\nabla u|^{p+2}dx - 2\int_\Omega u^r dx$, (26)

Then the solution $u(x,t)$of problem (1) blows up at some time $t^* < T$, with

$$T=\frac{\Phi(0)}{p\Psi(0)} \tag{27}$$

where $\Phi(t)$ is defined in (5). If $p=0$, we have $T=\infty$.

Proof. It follows (6), we obtain

$$\Phi'(t)\geq 2(p+2)\int_{\partial\Omega}F(u)ds-2\int_{\Omega}|\nabla u|^{p+2}dx-2(p+2)\int_{\Omega}u^{r+1}dx\geq (p+2)\Psi(t), \tag{28}$$

Differentiating (26), we have

$$\Psi'(t)=2\int_{\partial\Omega}u_t f(u)ds-\int_{\Omega}\left(|\nabla u|^{p+2}\right)_t dx-2\int_{\Omega}\left(u^{r+1}\right)_t dx=2\int_{\Omega}(u_t)^2dx\geq 0, \tag{29}$$

which with $\Psi(0)>0$, then $\Psi(t)>0$, for all $t\in(0,t^*)$. Making use of Cauchy inequality,

we get
$$(\Phi'(t))^2=4\left(\int_{\Omega}uu_t dx\right)^2\leq 4\int_{\Omega}u^2dx\int_{\Omega}(u_t)^2dx=2\Phi(t)\Psi'(t), \tag{30}$$

It follows from (29) and (30), we have
$$\Phi(t)\Psi'(t)\geq\frac{1}{2}(\Phi'(t))^2=\frac{p+2}{2}\Phi'(t)\Psi(t), \tag{31}$$

So, we have
$$\left(\Phi^{-\frac{p+2}{2}}\Psi\right)'(t)\geq 0, \tag{32}$$

Integrating from 0 to t, we obtain
$$\Phi^{-\frac{p+2}{2}}(t)\Psi(t)\geq\Phi^{-\frac{p+2}{2}}(0)\Psi(0)=:M>0. \tag{33}$$

Combining (28) and (33), we have
$$\Phi'(t)\geq(p+2)\Psi(t)\geq(p+2)M\Phi^{\frac{p+2}{2}}(t), \tag{34}$$

If $p>0$, again integrating over $(0,t)$, we have
$$\Phi(t)\geq\left[\Phi^{-\frac{p}{2}}(0)-\frac{p(p+2)}{2}Mt\right]^{-\frac{p}{2}}, \tag{35}$$

which implies $\Phi(t)\to+\infty$, as
$$t\to T=\frac{2\Phi^{-\frac{p}{2}}(0)}{p(p+2)M}=\frac{2\Phi(0)}{p(p+2)\Psi(0)}, \tag{36}$$

So, for $p>0$, we obtain $$t^* \le \frac{2\Phi(0)}{p(p+2)\Psi(0)}, \tag{37}$$

If $p=0$, from (35)we have $$\Phi(t) \ge \Phi(0)e^{2Mt}, \tag{38}$$

for all $t>0$, Which implies $t^*=\infty$. This completes the proof.

4 LOWER BOUNDS FOR t^*

In this section, we seek a lower bound for the blow-up time t^* under some assumptions and assuming that Ω is a star shaped domain, we establish the result as follows.

Theorem3. Let $u(x,t)$ be the nonnegative solution of problem (1) and $u(x,t)$ blow up at t^*, and that the data f satisfy the conditions $f(\xi) \ge k_2 \xi^s$, $\xi \ge 0$ (39)

With $k_2>0$, $s>1, n>1$. we defined the auxiliary function $\varphi(t) := \int_\Omega u^{2n} dx$, (40)

and show that $\varphi(t)$ satisfies inequality $$\varphi'(t) \le \Gamma(\varphi), \tag{41}$$

for some function $\Gamma(\varphi)$. It follows that t^* is bounded from below by

$$t^* \ge \int_{\varphi(0)}^{\infty} \frac{d\eta}{\Gamma(\eta)} \tag{42}$$

Proof. Taking the derivative of $\varphi(t)$ with respect to t, making use of the boundary condition together with the condition (39), we have

$$\varphi'(t) \le 2nk_2 \int_{\partial\Omega} u^{2n+s-1} ds - 2n(2n-1)\int_\Omega u^{2n-2}|\nabla u|^{p+2} dx - 2n(p+2)\int_\Omega u^{2n+r-1} dx, \tag{43}$$

By using of (7), we have

$$\int_{\partial\Omega} u^{2n+s-1} ds \le \frac{N}{\rho_0}\int_\Omega u^{2n+s-1} dx + \frac{(2n+s-1)d}{\rho_0}\int_\Omega u^{2n+s-2}|\nabla u| dx, \tag{44}$$

Substituting (44) into (43), we have

$$\begin{aligned}\varphi'(t) \le & \frac{2nk_2 N}{\rho_0}\int_\Omega u^{2n++s-1} dx + \frac{2nk_2(2n+s-1)d}{\rho_0}\int_\Omega u^{2n+s-2}|\nabla u| dx \\ & -2n(2n-1)\int_\Omega u^{2n-2}|\nabla u|^{p+2} dx - 2n(p+2)\int_\Omega u^{2n+r-1} dx\end{aligned} \tag{45}$$

Making use of Young inequality, we have

$$\int_\Omega u^{2n+s-2}|\nabla u| dx \le \frac{\mu}{p+2}\int_\Omega u^{2n-2}|\nabla u|^{p+2} + \frac{(p+1)}{(p+2)\mu}\int_\Omega u^{\frac{p(2n+s-2)+2(n+s-1)}{p+1}} dx, \tag{46}$$

for all $\mu > 0$, by Hölder inequality, we have

$$\int_\Omega u^{\frac{p(2n+s-2)+2(n+s-1)}{p+1}} dx \le \left(\int_\Omega u^2\right)^{\frac{p(2n+s-2)+2(n+s-1)}{2(p+1)}} \cdot |\Omega|^{\frac{2(p+1)(2-n)-s(p+2)}{2(p+1)}}, \tag{47}$$

Combining (45),(46) and (47), we obtain

$$\begin{aligned}\varphi'(t) \le & \frac{2nk_2 N}{\rho_0} J_1(t) - \left[2n(2n-1) - \frac{2nk_2(2n+s-1)d}{\rho_0}\cdot\frac{\mu}{p+2}\right] J_2(t) \\ & + \frac{(p+1)}{(p+2)\mu}\cdot|\Omega|^{\frac{2(p+1)(2-n)-s(p+2)}{2(p+1)}} \varphi^{\frac{p(2n+s-2)+2(n+s-1)}{2(p+1)}}(t) - 2n(p+2)\int_\Omega u^{2n+r-1} dx\end{aligned} \tag{48}$$

Where $J_1(t) = \int_\Omega u^{2n++s-1} dx, \quad J_2(t) = \int_\Omega u^{2n-2}|\nabla u|^{p+2} dx,$ (49)

Again using Sobolev type inequality derived by Payne et al. [10] we have

$$J_1(t) = \int_\Omega u^{2n+s-1} dx \le \left[\frac{3}{\rho_0}\int_\Omega u^{\frac{2}{3}(2n+s-1)} dx + \frac{(2n+s-1)(\rho_0+d)}{3\rho_0}\int_\Omega u^{2n+s-2}|\nabla u| dx\right]^{\frac{3}{2}}, \tag{50}$$

By using Hölder inequality, we have $\int_\Omega u^{2n+s-1}|\nabla u|dx \le \left(\int_\Omega u^{2n}dx\right)^{\delta\cdot\frac{p+1}{p+2}} \cdot J_2^{\frac{1}{P+2}}(t)$, (51)

where $\delta = \dfrac{(p+1)(2n-2)+s(p+2)}{2n(p+1)}$, (52)

Apply Hölder inequality, we have $\int_\Omega u^{\frac{2}{3}(2n+s-1)}dx \le \varphi^{\frac{2n+s-1}{3n}}(t)\cdot|\Omega|^{\frac{n-s+1}{3n}}$, (53)

From (50), (51) and (53), we get

$$J_1(t) \le \left[\frac{3}{\rho_0}\cdot|\Omega|^{\frac{n-s+1}{3n}}\varphi^{\frac{2n+s-1}{3n}}(t) + \frac{(2n+s-1)(\rho_0+d)}{3\rho_0}\varphi^{\frac{\delta(p+1)}{p+2}}(t)J_2^{\frac{1}{p+2}}(t)\right]^{\frac{3}{2}}, \tag{54}$$

we make use of the following inequality $(a_1+a_2)^{\frac{3}{2}} \le \sqrt{2}\left(a_1^{\frac{3}{2}}+a_2^{\frac{3}{2}}\right)$, (55)

Combining (54) and (55), we obtain

$$J_1(t) = \tilde{c}_1(t)\varphi^{\frac{2n+s-1}{2n}}(t) + \tilde{c}_2(t)\varphi^{\frac{3\delta(p+1)}{2(p+2)}}(t)J_2^{\frac{3}{2(p+2)}}(t), \tag{56}$$

where $\tilde{c}_1 = \sqrt{2}\left(\dfrac{3}{\rho_0}\right)^{\frac{3}{2}}\cdot|\Omega|^{\frac{n-s+1}{2n}} > 0$, $\tilde{c}_2 = \sqrt{2}\left[\dfrac{(2n+s-1)(\rho_0+d)}{3\rho_0}\right]^{\frac{3}{2}} > 0$, (57)

By again Hölder inequality, we have $\int_\Omega u^{2n}dx \le \left(\int_\Omega u^{2n+r-1}dx\right)^{\frac{2n}{2n+r-1}}\cdot|\Omega|^{\frac{r-1}{2n+r-1}}$, (58)

from (58), we get $\int_\Omega u^{2n+r-1}dx \ge \varphi^{\frac{2n+r-1}{2n}}\cdot|\Omega|^{\frac{1-r}{2n}}$, (59)

Inserting (56) and (59) in (48), we obtain

$$\varphi'(t) \le \tilde{c}_1 \frac{2nk_2N}{\rho_0} \varphi^{\frac{2n+s-1}{2n}}(t) + \tilde{c}_2 \frac{2nk_2N}{\rho_0} \varphi^{\frac{3\delta(p+1)}{2(n+2)}}(t) J_2^{\frac{3}{2(p+2)}}(t) - \left[2n(2n-1) - \frac{2nk_2(2n+s-1)d}{\rho_0} \cdot \frac{\mu}{p+2} \right] J_2(t) + \frac{(p+1)}{(p+2)\mu} \cdot |\Omega|^{\frac{2(p+1)(2-n)-s(p+2)}{2(p+1)}} \varphi^{\frac{p(2n+s-2)+2(n+s-1)}{2(p+1)}}(t) - 2n\varphi^{\frac{2n+r-1}{2n}}(t) \cdot |\Omega|^{\frac{1-r}{2n}} \tag{60}$$

Next, we need to eliminate $J_2(t)$.

By using Young inequality technique

$$\varphi^{\beta_1}(t) J_2^{\beta_2}(t) = (\gamma J_2(t))^{\beta_2} \left[\frac{\varphi^{\frac{\beta_1}{1-\beta_2}}(t)}{\gamma^{\frac{\beta_2}{1-\beta_2}}} \right]^{1-\beta_2} \le \gamma\beta_2 J_2(t) + (1-\beta_2)\gamma^{\frac{\beta_2}{\beta_2-1}} \varphi^{\frac{\beta_1}{1-\beta_2}}, \tag{61}$$

for $0 < \beta_2 < 1$, where γ is an arbitrary positive constant, then we have

$$\varphi^{\frac{3\delta(p+1)}{2(p+2)}}(t) J_2^{\frac{3}{2(p+2)}}(t) \le \frac{3\gamma}{2(p+2)} J_2(t) + \frac{2p+1}{2(p+2)} \gamma^{-\frac{3}{2p+1}} \varphi^{\frac{3\delta(p+1)}{2p+1}}(t), \tag{62}$$

Inserting (62) in (60), and choosing the arbitrary position constant γ such that

$$\tilde{c}_2 \frac{3nk_2N\gamma}{\rho_0(p+2)} - \left[2n(2n-1) - \frac{2nk_2(2n+s-1)d}{\rho_0} \cdot \frac{\mu}{p+2} \right] = 0, \tag{63}$$

So, we have

$$\varphi'(t) \le \tilde{c}_1 \frac{2nk_2N}{\rho_0} \varphi^{\frac{2n+s-1}{2n}}(t) + c_2\varphi^{\frac{3\delta(p+1)}{2p+1}}(t) + c_3\varphi^{\frac{p(2n+s-2)-2(n+s-1)}{2(p+1)}}(t) - 2n|\Omega|^{\frac{1-r}{2n}} \varphi^{\frac{2n+r-1}{2n}}(t), \tag{64}$$

where $$c_2 = \tilde{c}_2 \frac{nk_2N(2p+1)}{\rho_0(p+2)} \gamma^{-\frac{3}{2p+1}}, \quad c_3 = \frac{(p+1)}{(p+2)\mu} \cdot |\Omega|^{\frac{2(p+1)(2-n)-s(p+2)}{2(p+1)}}, \tag{65}$$

In the particular case $s = n+1, r = n+1$, the equality (64) can write to

$$\varphi'(t) \le \bar{c}_1 \varphi^{\frac{3}{2}}(t) + c_2 \varphi^{\frac{3\delta(p+1)}{2p+1}}(t) + c_3 \varphi^{\frac{p(3n-1)-4n}{2(p+1)}}(t), \tag{66}$$

With $\bar{c}_1 = \tilde{c}_1 \dfrac{2nk_2 N}{\rho_0} - 2n|\Omega|^{-\frac{1}{2}}$ (67).

Integrating (79) over $[0,t]$, we obtain

$$t^* \ge \int_{\varphi(0)}^{\infty} \frac{1}{\bar{c}_1 \eta^{\frac{3}{2}} + c_2 \eta^{\frac{3\delta(p+1)}{2p+1}} + c_3 \varphi^{\frac{p(3n-1)-4n}{2(p+1)}}(t)} \tag{68}$$

This completes the proof.

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