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An approximate solution of boundary layer equations for third grade Non-Newtonian fluid

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Abstract

Two dimensional equations of steady motion for third order fluids are expressed in a special coordinate system generated by the potential flow corresponding to an inviscid fluid. For the inviscid flow around an arbitrary object, the streamlines are the φ-coordinates and the velocity potential lines are ψ-coordinates. The outcome, boundary layer equations, is then shown to be independent of the body shape immersed into the flow. The equations are transformed into an ordinary differential system. Numerical solutions to the out coming nonlinear coupled differential equations are found by spline collocation method.

Keywords: Coupled nonlinear system, steady motion for third order fluids, Quasilinearization, linear equations, spline collocation method.

INTRODUCTION

The flow of non- Newtonian has gained considerable importance to its application in engineering and industry like rubber, petroleum, soap and detergents, printing materials. Rivlin-Ericksen fluids gained much acceptance from both theorists and experimenters for a non-Newtonian fluid model. Special cases of the model, which is the fluid of the third grade, are extensively used, and a lot of works have been done on the subject. Several boundary layer equations are derived for different non-Newtonian models. [1] Acrivos et al. (1960) and Pakdemirli (1996) derived boundary layer equations for power-law fluids. [2-3] For the rate type of fluids, the works due to Beard and Walters (1964) and Astin et al. (1973) are of significant importance. [4] The multiple deck boundary layer concepts have been applied to the second and third grade fluids by Pakdemirli (1994). [5] Solution of third grade equation using special coordinate by Muhammet yurusoy (2003). [6-7] Solved coupled problem using numerical method.

Governing boundary layer equation of third grade as a non-Newtonian fluid model. It is shown by [8] Rivlin and Ericksen (1955).That the stress tensor is given by following relation

$$T = -\rho l + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 + \beta(trA_1^2)A_1 \qquad (1)$$

Where ρ is the pressure, μ is the viscosity, α_1 and α_2 are second grade fluid terms, β is the third grade fluid terms, and A_1 and A_2 are the first two Rivlin-Ericksen tensors given by the relations

$$\text{L= grad } v, \qquad A_1 = L + L^T, \qquad A_2 = \dot{A}_1 + A_1 L + L^T A_1$$

Where v is the velocity vector. Rivlin and Ericksen (1955) showed that making equation (1) compatible with the thermodynamics and minimizing the free energy when the fluid is at the rest, material constants should satisfy the relation,

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad \beta \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta}$$

The dimensionless from the equations of the motion for a third grade fluid are (Pakdemirli, 1992)

$$\frac{1}{2}\text{grad}|q|^2 + \omega \times q = -grad\,\rho + \varepsilon\nabla^2 \text{q} + \varepsilon_1(\nabla^2\omega \times \text{q}) +$$

$$\varepsilon_1 grad(q.\nabla^2\text{q}) + \frac{1}{4}(2\varepsilon_1 + \varepsilon_2)\text{grad}|A_1|^2 +$$

$$(\varepsilon_1 + \varepsilon_2)\left[A_1.\nabla^2 q + 2div(grad\ q(grad\ \text{q})^T)\right] +$$

$$+\varepsilon_3 A_1.grad|A_1|^2 + \varepsilon_3 |A_1|^2 \nabla^2\text{q}$$

$$div\ \text{q} = 0$$

Where q is the dimensionless velocity vector, ∇ denotes laplacian, ω=curl q and the dimensionless coefficients are defined as follows

$$\varepsilon = \frac{\mu}{\rho UL} = \frac{1}{\text{Re}}, \varepsilon_1 = \frac{\alpha_1}{\rho L^2}, \varepsilon_2 = \frac{\alpha_2}{\rho L^2}, \varepsilon_3 = \frac{\beta U}{\rho L^3}$$

Where L and U are some reference length and velocity, respectively, ρ is the density, Re is the Reynolds number.

The governing equations here are partial differential equations and use similarity transformation to convert partial differential equations into ordinary differential equations [5]. Here we have coupled nonlinear differential equations, which are solved by using the spline collocation method. In this way, the paper has been organized as follows. Section 2 and 3, spline method and approximate solution for the governing equations. Results and discussion are in section 4.

CUBIC SPLINE COLLOCATION METHOD

Consider equally spaced knots of partition:

$a = x_0 < x_1 < x_2 < .... < x_n = b$ on $[a,b]$. The cubic spline is defined by [9] Bickley (1968).

$$s(x) = a_0 + b_0(x - x_0) + \frac{1}{2}C_0(x - x_0)^2 + \frac{1}{6}\sum_{k=0}^{n-1} d_k (x - x_k)_+^3 \qquad (2)$$

Where the powers function $(x - x_k)_+$ is defined as

$$(x - x_k)_+ = \begin{cases} x - x_k, & x > x_k \\ 0, & x \le x_k \end{cases} \qquad (3)$$

and the boundary value problem is given by

$$y''(x) + p(x)\, y'(x) + q(x)\, y(x) = r(x) \;\; for\; a \le x \le b \quad (4)$$

Subject to boundary conditions

$$\alpha_o y_o + \beta_o y_o' = \gamma_o \qquad at \quad x = a$$

$$\alpha_n y_n - \beta_n y_n' = \gamma_n \qquad at \quad x = b$$

To solve this boundary value problem substitute s(x), s'(x), s"(x) from cubic spline, then the boundary value problem becomes

$$\sum_{k=0}^{n-1} d_k \left[(x_i - x_k)_+ + \frac{1}{2} p_i (x_i - x_k)_+^2 + \frac{1}{6} q_i (x_i - x_k)_+^3 \right]$$

$$+ c_0 \left[1 + p_i (x_i - x_0) + \frac{1}{2} q_i (x_i - x_0)^2 \right]$$

$$+ b_0 \left[p_i + q_i (x_i - x_0) \right] + a_0 [q_i] = r_i , \quad i = 0,1,....n$$

Thus for cubic spline and second order boundary value problem, we get linear algebraic equations in unknowns. $a_0, b_0, c_0, d_0, .., d_{n-1}$ The matrix form of this system is given by The matrix form of this system is given by

AX = B

Where $B = [\gamma_n, r_n \;....., r_1,\; r_0,\; \gamma_0]^T$

$$X = [d_{n-1},\; ....., d_1,\; d_0, c_0,\; b_0,\; a_0]^T$$

And the co-efficient matrix A is an upper Hessenberg matrix.
The co-efficient matrix A is an upper triangular Hessenberg matrix with a single lower sub diagonal, principal and upper diagonal having non-zero elements. Because of this nature of matrix A, the determination of the required quantities becomes simple and consumes less time. The values of these constants yield the cubic spline s(x) in equation (2).

3. SOLUTION BY USING COLLOCATION METHOD

We use a boundary layer equation of third grade as a non-Newtonian fluid model. It is shown by Rivlin and Ericksen [8].
The equation can be written in non-linear coupled equation as follow:

$$\frac{1}{2}(f^2 - 1) + 2gf' - \xi ff' - 2f'' - 12kf'' f'^2 = 0,$$
$$\xi f' - 2g' = 0 \qquad (5)$$
$$with \;\; f(0) = g(0) = 0 \;, f(10) = 1$$

Where k is the third grade fluid coefficient. We use quasilinearization technique to convert (5) into linear form with help of boundary conditions. We get linear form as

$$(-2 - 12kf_i^2) f_{i+1}'' + (2g_i - \xi f_i - 24kf_i'' f_i') f_{i+1}' +$$
$$(f_i - \xi f_i') f_{i+1} = -\frac{1}{2} f_i^2 + \frac{1}{2} - \xi f_i f_i' - 24kf_i'' f_i' f_i' + f_i f_i \;\; (6)$$

With boundary conditions

$$f_{i+1}(0) = 0, \quad f_{i+1}(10) = 1 \qquad (7)$$

Another linearize form for g is $g_{i+1}' = \frac{1}{2}(\xi f_i')$ (8)

With boundary conditions

$$g_{i+1}(0) = 0 \qquad (9)$$

The cubic spline collocation method and the quadratic spline collocation method are used to solve above (6) to (9) equations.
The Cubic spline is given by

$$s(\eta) = a_0 + b_0(\eta - \eta_0) + \frac{1}{2}c_0(\eta - \eta_0)^2 + \frac{1}{6}\sum_{k=0}^{n-1} d_k(\eta - \eta_k)_+^3 \qquad (10)$$

And the Quadratic spline is given by

$$s(\eta) = a_0 + b_0(\eta - \eta_0) + \frac{1}{2}\sum_{k=0}^{n-1} c_k(\eta - \eta_k)_+^2 \qquad (11)$$

Assume (10) be the approximate solution to $f(\eta)$ and (11) be the $g(\eta)$.
Solve above equation and we get collocation as

System of Collocation Equations for f:

$$\sum_{k=0}^{n-1} d_k[(-2-12kf_i^{'^2})(\eta_i - \eta_k) + \frac{1}{2}(2g_i - \xi f_i - 24kf_i^{''} f_i^{'})(\eta_i - \eta_k)^2 + \frac{1}{6}(f_i - \xi f_i^{'})(\eta_i - \eta_k)^3] + c_0[(-2-12kf_i^{'^2}) + (2g_i - \xi f_i - 24kf_i^{''} f_i^{'})(\eta_i - \eta_0) + \frac{1}{2}(f_i - \xi f_i^{'})(\eta_i - \eta_0)^2] + b_0[(2g_i - \xi f_i - 24kf_i^{''} f_i^{'}) + (f_i - \xi f_i^{'})(\eta_i - \eta_0)] + a_0[(f_i - \xi f_i^{'})] = -\frac{1}{2} f_i^2 + \frac{1}{2} - 24kf_i^{''} f_i^{'} f_i^{'} + f_i f_i - \xi f_i f_i^{'} \quad (12)$$

System of Collocation Equations for g:

$$b_0 + \sum_{k=0}^{n-1} c_k (n_i - n_k)^2 = \frac{1}{2}(\xi f_i^{'}) \qquad (13)$$

To obtain the spline solution, we begin with a curve $f(\eta) = \frac{1}{10}\eta$ and $g(\eta) = 0$, which satisfy given boundary conditions. To obtain the complete solution first we solve f using g . And after we solve g using spline f . These procedures do vice versa. We solve (12) and (13) using spline collocation method for similarity variable ξ and different values of $k = 0,10,30$.
Graphical solution of given problem:

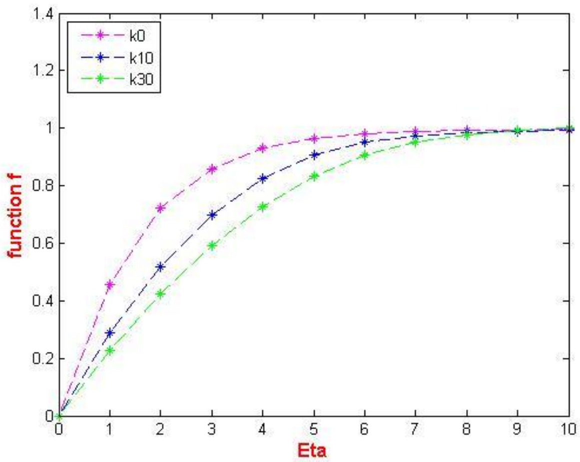

Figure-1: function f for various values of k

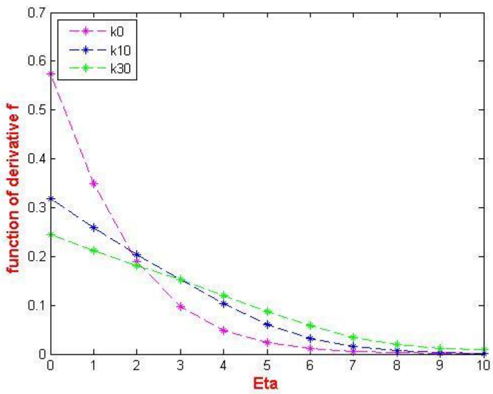

Figure-2: Derivative of f for various values of k

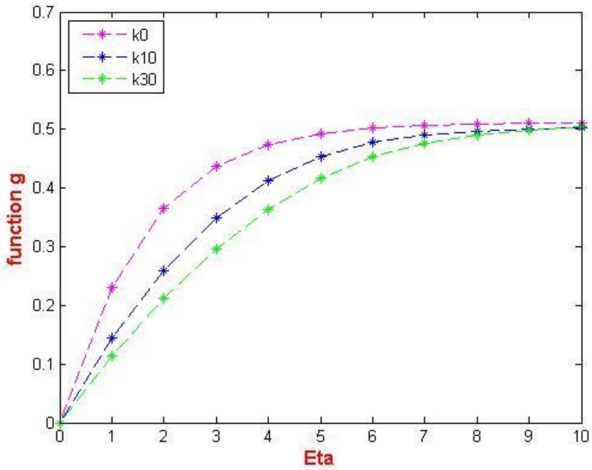

Figure-3: function g for various values of k

RESULT AND DISCUSSION

An increase in k yields an increase in the non-Newtonian behavior. From Figure-1 and Figure-3, we conclude that the boundary layer got thicker when the non-Newtonian aspect of the fluid behavior became more pronounced and fluid becomes more viscous. So its affect on flow of the fluid. Increase in viscosity, flow of fluid decrease.Figure-2, shows the behavior of derivative of f, when increase in k.

We solved the problem using quadratic and cubic spline collocation method. This shows the reliability of the method. Thus we can solve such type of problems using spline collocation method.

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