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Generalized Form of Continuity by Using D_β-Closed Set

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Abstract

In this paper we introduce and study a new class of generalized closed sets called, D_β-closed sets, which are the generalization of D_α-closed sets defined in O.R. Sayed, A.M. Khalil, Some applications of D_α-closed sets in topological spaces, 3(2016), 26–34 [25], pre^*-closed set defined in T.Selvi, A. Punitha Dharani, Some new class of nearly closed and open sets, Asian Jour. of current engineering and maths, 5(2012), 305-307 [26] and $semi^*$-closed sets defined in A. Robert, S. Pious Missier, On $semi^*$-closed sets, Asian journal of current engineering and maths, 4(2012), 173–176 [23].

We establish the relationship of D_β-closed sets with some already existing generalized closed sets. We define D_β-continuous and D_β-irresolute functions and obtain their basic properties.

AMS subject classification: Primary 54A05, 54C05, 54C08; Secondary 54C10, 54D10, 54C99.
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1. Introduction

In general topology repeated application of interior and closure operators give rise to several different new classes of sets. Some of them are generalized form of open sets. These classes are found to have applications not only in mathematics but even in diverse fields outside the realm of mathematics ([20], [10], [24]). The most well known notion and inspiring sources are the notions of α–open set initiated by Njasted [21] in 1965, semiopen sets by Levine (See also [11], [12]) in 1963, preopen sets by Mashour et al. [16] in 1982 and β–open (semi–preopen) sets by Abd–El–Monsef et al. [19] (by Andrijevic (See also [1], [2])). Due to this, investigation of these sets have gained momentum in recent days. By originating the concept of generalized closed (g–closed) sets, Levine [12] provided an umbrella for the researchers working in the field of generalized closed sets. Levine [12] used the closure operator and the openness of the superset in the definition of g–closed sets. Levine discussed that compactness, normality and completeness in a uniform space are inherited by g–closed subsets. He used g–closed sets to define new separation axioms, called $T_{1/2}$–spaces in which the closed sets and g–closed sets coincide.

Balchandran et al. [4] introduced the notion of generalized continuous (g–continuous) functions by using g–closed sets and obtained some of their properties. Andrijevic [1] investigated some properties of topology of α–sets. Maheshwari et al. [13] defined and investigated the α–irresolute mapping as a function $f : (X, \tau) \to (Y, \sigma)$ is said to be α–irresolute if the preimage of every α–open sets in Y is α–open in X. Maki et al. [15] defined and investigated the concept of generalized α–closed sets in topological spaces as, a subset A of a space is said to be generalized α–closed set if α–Cl(A)$\subseteq U$, whenever $A \subseteq U$, U is α–open in X. Noiri [22] initiated the notion of weakly α–continuous functions in topological spaces and discussed very interesting results as, weakly α–continuous surjection preserves connected spaces and that weakly α–continuous functions into regular spaces are continuous.

Agashe et al. [3] introduced and studied the concept of immediate predecessor and immediate successor in the lattice of topologies and the adjacent topologies and also discussed their properties. In [23] Robert et al. originated the concept of $semi^*$–closed sets by using the closure operator C^* due to Dunham [6]. They investigated many fundamental properties of $semi^*$–closed sets. This class of set lies between closed sets and semi–closed sets. They also established $semi^*$–closure of any set. Missier [18] devised and studied the new notion of sets called α^*–open sets and α^*–closed sets and discussed the relationship of α^*–open sets and α^*–closed sets with some other sets. Selvi et al. [26] defined and investigated a new class of sets called pre^*–closed sets by using the generalized closure operator C^* due to Dunham [6].

Motivation and Contribution

Sayed et al. [25] devised a new class of generalized closed sets namely D_α–closed sets in topological spaces by using the generalized closure operator C^* due to Dunham [6]. They characterized the D_α–closed sets and D_α–open sets. The new concept and the way of representing results motivate us to generalize this concept of D_α–closed sets. In the present paper a new notion of generalized closed sets namely D_β–closed sets has been devised. A brief synopsis of the paper is as follows: The main objective of this paper is to introduce and study D_β–closed sets, which is the generalization of β–closed sets by using the generalized closure operator C^*. This class of sets are the generalization of D_α–closed sets, pre^*–closed sets and $semi^*$–closed sets.

This paper is organized as follows, section 1, gives basic notions which underpin our work. In section 2, we have define D_β–closed sets and discuss their characterization and basic properties and its relationships with already existing generalized closed sets. In section 3, we define D_β–open sets. In section 4, we define D_β–continuous and D_β–irresolute functions and investigate their fundamental properties.

1.1. Preliminaries

Throughout this paper (X, τ) will always denote a topological space on which no separation axioms are assumed, unless explicitly stated. If A is a subset of the space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A respectively. Here we recall the following known definitions and properties.

Definition 1.1. Let (X, τ) be a topological space. A subset A of the space X is said to be,

(i) preopen [21] if $A \subseteq Int(Cl(A))$ and preclosed if $Cl(Int(A)) \subseteq A$.

(ii) semi–open[11] if $A \subseteq Cl(Int(A))$ and semi–closed if $Int(Cl(A)) \subseteq A$.

(iii) α–open [21] if $A \subseteq Int(Cl(Int(A))$ and α–closed if $Cl(Int(Cl(A)) \subseteq A$.

(iv) β–open [19] if $A \subseteq Cl(Int(Cl(A)))$ and β–closed if $Int(Cl(Int(A))) \subseteq A$.

(v) generalized closed (briefly g–closed)[12] if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.generalized open(briefly g–open) if $X \setminus A$ is g–closed.

(vi) pre*–closed set [26] if $Cl^*(Int(A)) \subseteq A$ and pre^*–open set if $A \subseteq Int^*(Cl(A))$.

(vii) $semi^*$–closed set [23] if $Int * (Cl(A)) \subseteq A$ and $semi^*$–open set [18] if $A \subseteq Cl^*(Int(A))$.

$(viii)$ D_α–closed [25] if $Cl^*(Int(Cl^*(A))) \subseteq A$ and D_α–open if $X \setminus A$ is D_α–closed.

Definition 1.2. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be,

(i) α–continuous [17](resp.β–continuous [19]) if the inverse image of each open set in Y is α–open(resp.β–open) in X.

(ii) g–continuous [4] if the inverse image of each open set in Y is g–open in X.

(iii) D_α–continuous [25] if the inverse image of each open set in Y is D_α–open in X.

The α–closure [1] of a subset A of X is the intersection of all α–closed sets containing A and is denoted by $Cl_\alpha(A)$. The α–interior [1] of a subset A of X is the union of all α–open sets contained in A and is denoted by $Int_\alpha(A)$. The β–closure [19] of a subset A of X is the intersection of all β–closed sets containing A and is denoted by $Cl_\beta(A)$. The β–interior [19] of a subset A of X is the union of all β–open sets contained in A and is denoted by $Int_\beta(A)$. The intersection of all g–closed sets containing A [6] is called the g–closure of A and denoted by $Cl^*(A)$ and the g–interior of A [23] is the union of all g–open sets contained in A and is denoted by $Int^*(A)$.

The family of all D_β–closed (resp. D_α–closed, g–closed, β–closed) sets of X denoted by D_βC(X) (resp.D_αC(X), GC(X), βC(X)). The family of all D_β–open(resp. D_α–open, g–open, β–open)sets of X denoted by D_βO(X) (resp. D_αO(X), GO(X), βO(X)). $\beta O(X,x)$= {U: $U \in \alpha O(X,\tau)$}, $O(X,x)$= {U:$x \in U \in \tau$ }, $\beta C(X,x)$ = {U :$U \in \alpha C(X,\tau)$}, $D_\alpha O(X,x) = \{U : U \in \alpha O(X,\tau)\}$, $D_\alpha C(X,x) = \{U : U \in \alpha C(X,\tau)\}$.

Lemma 1.3. [6] Let $A \subset X$, then

(i) $X \setminus Cl^*(A) = Int^*(X \setminus A)$.

(i) $X \setminus Int^*(A) = Cl^*(X \setminus A)$.

2. D_β-Closed Set

In this section we introduce D_β-closed sets and investigate some of their basic properties.

Definition 2.1. A subset A of a topological space (X, τ) is called D_β-closed if $Int^*(Cl^*(Int^*(A))) \subseteq A$.

Example 2.2. Let $X = \{a,b,c,d\}$ be any set and $\tau = \{X, \phi, \{a,b,c\}, \{a,b\}\}$, then (X,τ) be a topological space. $C(X) = \{\phi, X, \{d\}, \{c,d\}\}$,
$GC(X) = \{\phi, X, \{d\}, \{c,d\}, \{a,b,d\}, \{a,d\}, \{b,d\}, \{a,c,d\}, \{b,c,d\}\}$,
$GO(X) = \{X, \phi, \{a,b,c\}, \{a,b\}, \{c\}, \{b,c\}, \{a,c\}, \{b\}, \{a\}\}$,
$D_\alpha C(X) = \{X, \phi, \{d\}, \{c,d\}, \{a,b,d\}, \{a,d\}, \{b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,c\}, \{c\}, \{b\}, \{a\}, \{b,c\}\}$
and $D_\beta C(X) = \{X, \phi, \{d\}, \{c,d\}, \{a,b,d\}, \{a,d\}, \{b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,c\}, \{c\}, \{b\}, \{a\}, \{b,c\}, \{a,b\}\}$.

Theorem 2.3. L et (X,τ) be a topological space, then

(i) Every β–closed subset of $(X,\tau)$is D_β–closed.

(ii) Every g–closed subset of (X,τ) is D_β–closed.

(iii) Every $semi^*$–closed subset of (X,τ) is D_β–closed.

(iv) Every pre^*–closed subset of (X,τ) is D_β–closed.

(v) Every D_α–closed subset of $(X,\tau)$is D_β–closed.

Proof. (i) Let A be any β–closed subset of the space X, then we have $Int(Cl(Int(A))) \subseteq A$. We know that $Int^*(A) \subseteq Int(A)$, then we have $Cl^*(Int^*(A) \subseteq Cl^*(Int(A)$
$Int^*(Cl^*(Int^*(A) \subseteq Int^*(Cl^*(Int(A) \subseteq$ and we get $Int(Cl(Int(A))) \subseteq A$.

(ii) Let A be any g–closed subset of the space X, then we have $Cl^*(A) = A$. Since $Int^*(A) \subseteq A$, then we have $Cl^*(Int^*(A)) \subseteq Cl^*(A) = A$
$Int^*(Cl^*(Int^*(A))) \subseteq Int^*(A) \subseteq A$
$Int^*(Cl^*(Int^*(A))) \subseteq A$ i.e. A is D_β–closed.

(iii) Let A be any $semi^*$–closed subset of the space X, then we have $Cl(Int^*(A)) \subseteq A$. Let $Int^*(A) \subseteq A$ and then we have $Cl^*(Int^*(A) \subseteq Cl^*(A) \subseteq Cl(A)$. Thus we get $Int^*(Cl^*(Int^*(A) \subseteq Int^*(Cl(A)) \subseteq A$.

[(iv)] Let A be any pre^*–closed subset of the space X, then we have $Cl^*(Int(A)) \subseteq A$. Let $Int^*(A) \subseteq Int(A)$, then we have $Cl^*(Int^*(A) \subseteq Cl^*(Int(A)) \subseteq A$. This implies that $Int^*(Cl^*(Int^*(A)))) \subseteq Int^*(A) \subseteq A$.

(v) It follows from (i). ■

Remark 2.4. The converse of Theorem 2.3 is not true as shown in the following example.

- (i) D$_\beta$–closed set need not be β–closed. (see Example 2.5 below)
- (ii) D_β–closed set need not be g–closed. (see Example 2.5 below)
- (iii) D_β–closed set need not be $semi^*$–closed. (see Example 2.5 below)
- (iv) D_β–closed set need not be pre^*–closed. (see Example 2.5 below)
- (v) D_β–closed set need not be D_α–closed. (see Example 2.5 below)

Example 2.5. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a, b\}, \{a, d\}, \{a\}, \{a, b, d\}\}$. Then (X, τ) be a topological space. $CX = \{\phi, X, \{c, d\}, \{b, c\}, \{b, c, d\}, \{c\}\}$, $GC(X) = \{\phi, X, \{c, d\}, \{b, c\}, \{b, c, d\}, \{c\}, \{a, b, c\}, \{a, c\}, \{a, c, d\}\}$, $GO(X) = \{X, \phi, \{a, b\}, \{a, d\}, \{a\}, \{a, b, d\}, \{d\}, \{b\}, \{b, d\}\}$,
$\beta C(X) = \{\phi, X, \{c, d\}, \{b, c\}, \{b, c, d\}, \{c\}, \{b, d\}, \{b\}, \{d\}\}$,
$\beta O(X) = \{X, \phi, \{a, b\}, \{a, d\}, \{a\}, \{a, b, d\}, \{d\}, \{a, c\}, \{a, c, d\}, \{a, b, c\}\}$,
$D_\alpha C(X) = \{phi, X, \{c, d\}, \{b, c\}, \{b, c, d\}, \{c\}, \{a, b, c\}, \{a, c\}, \{a, c, d\}, \{b\}, \{d\}, \{b, d\}\}$ and $D_\alpha O(X) = \{X, \phi, \{a, b\}, \{a, d\}, \{a\}, \{a, b, d\}, \{d\}, \{a, c\}, \{a, c.d\}, \{a, b, c\}\}$,
$D_\beta C(X) = \{phi, X, \{c, d\}, \{b, c\}, \{b, c, d\}, \{c\}, \{a, b, c\}, \{a, c\}, \{a, c, d\}, \{b\}, \{d\}, \{b, d\}, \{a, b\}, \{a\}, \{a, d\}\}$.
$D_\beta O(X) = \{phi, X, \{a, b\}, \{a, d\}, \{a\}.\{a, b, d\}, \{d\}, \{b, d\}, \{b\}, \{a, c, d\}, \{a, b, c\}, \{a, c\}, \{c, d\}, \{b, c, d\}, \{b, c\}\}$.
Let $A = \{a, b\}$ is a D_β–closed set of X but $\{a, b\}$ is not a β–closed neither g–closed nor a D_α–closed.

Theorem 2.6. Arbitrary intersection of D_β–closed sets is D_β–closed.

Proof. Let $\{G_\alpha : \alpha \in \triangle\}$ be a collection of D_β-closed sets in X. Then $Int^*(Cl^*(Int^*(G_\alpha))) \subseteq G_\alpha$ for each α. Since $\cap G_\alpha \subseteq G_\alpha$ for each α, $Int^*(\cap G_\alpha) \subseteq Int^*(G_\alpha)$ for each α. Therefore $Int^*(\cap G_\alpha) \subseteq \cap Intv(G_\alpha)$, $\alpha \in \triangle$.
Hence $Int^*(Cl^*(Int^*(\cap G_\alpha))) \subseteq Int^*(Cl^*(\bigcap Int^*(G_\alpha)$
$\subseteq Int^*(\cap(Cl^*(Int^*(G_\alpha)))$
$\subseteq \cap(Int^*(Cl^*(Int^*(G_\alpha)))$
$\subseteq \cap G_\alpha$. Therefore $\cap G_\alpha$ is D_β–closed. ■

Remark 2.7. The union of two D_β–closed sets need not be D_β–closed.

Example 2.8. In the Example 2.2, the sets $\{a, b\}$ and $\{b, d\}$ both are D_β–closed but their union $\{a, b\} \cup \{b, d\} = \{a, b, d, \}$ is not D_β–closed.

Remark 2.9. The collection of D_βC(X) does not form a topology.

Corollary 2.10. Let A and B are any two subsets of the space X, where A is D_β–closed and B is β–closed, then $A \cap B$ is D_β–closed.

Proof. It follows directly from the Theorems 2.3 and 2.6. ■

Corollary 2.11. If a subset A is D_β–closed and B is g–closed, then $A \cap B$ is D_β–closed.

Proof. It follows directly from Theorems 2.3 and 2.6. ■

Definition 2.12. Let A be any subset of a space X. The D_β–closure of A is the intersection of all D_β–closed sets in X containing A i.e. $Cl_{D_\beta}(A) = \bigcap \{G : A \subseteq G$ and $G \in D_\beta C(X)\}$. It is denoted by $Cl_{D_\beta}(A)$.

Theorem 2.13. Let A be a subset of X. Then A is D_β–closed set in X if and only if $Cl_{D_\beta}(A) = A$.

Proof. Suppose A is D_β–closed set in X. Since $Cl_{D_\beta}(A)$ is equal to the intersection of all D_β–closed sets in X containing A. Since $A \subseteq Cl_{D_\beta}(A)$, therefore $Cl_{D_\beta}(A) = A$. Let $Cl_{D_\beta}(A) = A$. Then A is D_β–closed set in X. ■

Theorem 2.14. Let A and B be subsets of X. Then the following results hold.

(i) $A \subseteq Cl_{D_\beta}(A) \subseteq Cl_\beta(A)$, $Cl_{D_\beta}(A) \subseteq Cl^*(A)$, $Cl_{D_\beta}(A) \subseteq Cl_{D_\alpha}(A)$.

(ii) $Cl_{D_\beta}(A) = \phi$ and $Cl_{D_\beta}(A) = X$.

(iii) If $A \subseteq B$, then $Cl_{D_\beta}(A) = Cl_{D_\beta}(B)$

(iv) $Cl_{D_\beta}(Cl_{D_\beta}(A)) = Cl_{D_\beta}(A)$

(v) $Cl_{D_\beta}(A) \cup Cl_{D_\beta}(B) \subseteq Cl_{D_\beta}(A \cup B)$

(vi) $Cl_{D_\beta}(A \cap B) \subseteq Cl_{D_\beta}(A) \cap Cl_{D_\beta}(B)$.

Proof.

(i) It follows directly from the Theorem 2.3.

(ii) It is trivially true.

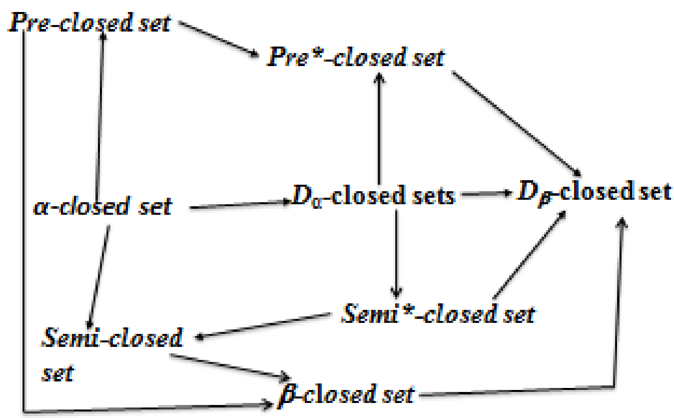
(iii) It is trivially true.

(iv) It follows from the facts that $Cl_{D_\beta}(A)$ is itself a D_β–closed set and D_β–closed set is a smallest closed sets containing A and from the Theorem 2.13.

(v) (v) and (vi) follows from (iii). ■

2.1. Interrelationship

The following diagram will describe the interrelations among D_β–closed set and other existing generalized–closed sets. None of these implications is reversible as shown by examples given below and well known facts.



3. D_β-open sets

In this section we introduce D_β-open sets and investigate some of their basic properties.

Definition 3.1. A subset A of a space (X, τ) is called an D_β-open if $X \setminus A$ is D_β-closed. Let D_βO(X) denotes the collection of all an D_β–open sets in X.

Lemma 3.2. L et $A \subseteq X$, then

(i) $X \setminus Cl^*(X \setminus A) = Int^*(A)$.

(ii) $X \setminus Int^*(X \setminus A) = Cl^*(A)$.

Proof. It is trivially true. ■

Theorem 3.3. A subset A of a space X is D_β-open if and only if $A \subseteq Cl^*(Int^*(Cl^*(A)))$.

Proof. Let A be any D_β-open set. Then $X \setminus A$ is D_β-closed and $Int^*(Cl^*(Int^*(X \setminus A))) \subseteq X \setminus A$. Therefore $A \subseteq (X \setminus Int^*(Cl^*(Int^*(A)))) = Cl^*(Int^*(Cl^*(A)))$. Thus, we have $A \subseteq Cl^*(Int^*(Cl^*(A)))$. ■

Theorem 3.4. Let (X, τ) be a topological space. Then

(i) Every β-open subset of (X, τ) is D_β-open.

(ii) Every g-open subset of (X, τ) is D_β-open.

(iii) Every $semi^*$-open subset of (X, τ) is D_β-open.

(iv) Every pre^*-open subset of (X, τ) is D_β-open.

(v) Every D_α-open subset of (X, τ) is D_β-open.

Proof. It is directly follows from the Theorem 2.3. ■

Remark 3.5. The converse of the above theorem is not true as seen from Example 2.2, the set {b,c} is D_β-open but it is not β-open nor a g-open and nor a D_α-open set.

Theorem 3.6. Arbitrary union of D_β-open set is D_β-open.

Proof. It follows from the Theorem 2.6. ■

Remark 3.7. The intersection of two D_β-open sets need not be D_β-open as seen from Example 2.5, in which two D_β-open sets are $A = \{a, c\}$ and $B = \{c, d\}$ but their intersection $A \cap B = \{c\}$ is not D_β-open set.

Corollary 3.8. If a subset A is D_β-open and B is β-open, then $A \cup B$ is D_β-open.

Proof. It follows from the Theorems 2.13 and 2.14. ■

Corollary 3.9. If a subset A is D_β-open and B is g-open, then $A \cup B$ is D_β-open.

Proof. It follows from the Theorems 2.13 and 2.14. ■

Definition 3.10. Let A be any subset of a space X. The D_β-interior of A is denoted by $Int_{D_\beta}(A)$, is the union of all the D_β-open sets in X, contained in A i.e. $Int_{D_\beta}(A) = \bigcup\{U : U \subset A, U \in D_\beta O(X)\}$.

Lemma 3.11. If A be any subset of X, then

(i) $X \setminus Cl_{D_\beta}(A) = Int_{D_\beta}(X \setminus A)$.

(ii) $X \setminus Int_{D_\beta}(A) = Cl_{D_\beta}(X \setminus A)$.

Proof. It is obvious. ∎

Theorem 3.12. Let A be any subset of X. Then A is D_β-open if and only if $Int_{D_\beta}(A) = A$.

Proof. It follows from Theorem 2.13 and Lemma 3.11. ∎

Theorem 3.13. Let A and B be subsets of X. Then the following results hold.

(i) $Int_\beta(A) \subseteq Int_{D_\beta}(A) \subseteq A$, $Int^*(A) \subseteq Int_{D_\beta}(A)$.

(ii) $Int_{D_\beta}(A) = X$,$Int_{D_\beta}(A) = \phi$

(iii) If $A \subseteq B$, then $Int_{D\beta}(A) \subseteq Int_{D_\beta}(B)$.

(iv) $Int_{D_\beta}(Int_{D\beta}(A)) = Int_{D_\beta}(A)$.

(v) $Int_{D_\beta}(A) \cup Int_{D_\beta}(B) \subseteq Int_{D_\beta}(A \cup B)$.

(vi) $Int_{D_\beta}(A) \cap Int_{D_\beta}(A) \subseteq Int_{D_\beta}(A \cap B)$.

Proof. It is obvious. ∎

Definition 3.14. Let X be any topological space and let $x \in X$, then a subset G_x of X is said to be D_β-neighborhood of x if there exists a D_β-open set U in X such that $x \in U \subset G_x$.

Theorem 3.15. Let $x \in X$, then $x \in Cl_{D_\beta}(A)$ if and only if $U \cap A = \phi$ for every D_β-open set U containing x.

Proof. Let $x \in Cl_{D_\beta}(A)$ and on contrary assume that, there exists a D_β-open set U containing x such that $U \cap A = \phi$. Then $A \subseteq X \setminus U$ and $X \setminus U$ is D_β-closed. Therefore $Cl_{D_\beta}(A) \subseteq Cl_{D_\beta}(X \setminus U) = X \setminus U$. Therefore $x \notin Cl_{D_\beta}(A) = X \setminus U$ i.e.$U \cap A \neq \phi$, which is contradiction to the assumption.
Conversely, suppose $U \cap A \neq \phi$,on contrary, we assume that for every D_β-open set U containing x and $x \notin Cl_{D_\beta}(A)$. Then there exists D_β-closed subset G containing A such that x not belongs to G. Therefore $x \in X \setminus G$ and $X \setminus G$ is D_β-open. Since $A \subseteq G$, $(X \setminus G) \cap A = \phi$, which is contradiction to the assumption. Hence the result. ∎

Definition 3.16. Let A be a subset of a space X. A point $x \in X$ is said to be a D_α-limit point of A if for each D_α-open set U containing x, we have $U \cap (A \setminus \{x\}) \neq \phi$. The set of all D_α-limit points of A is called the D_α-derived set of A and it is denoted by D_{D_α}(A).

Definition 3.17. Let A be a subset of a space X. A point $x \in X$ is said to be a D_β-limit point of A if for each D_β-open set U containing x, we have $U \cap (A \setminus \{x\}) \neq \phi$. The set of all D_β-limit points of A is called the D_β-derived set of A and is denoted by $D_{D_\beta}(A)$.

Remark 3.18. Since every open set is D_α-open, we have $D_{D_\alpha}(A) \subseteq D(A)$ and therefore $D_{D_\beta}(A) \subseteq D(A)$ for any subset $A \subseteq X$, where $D(A)$ is the derived set of A. Moreover, since every closed set is D_α-closed, we have $A \subseteq Cl_{D_\beta}(A) \subseteq Cl_{D_\alpha}(A) \subseteq Cl(A)$.

4. D_β-continuous and D_β-irresolute functions

In this section we introduce D_β-continuous functions and study some of their basic properties.

Definition 4.1. A function $f : (X, \tau) \to (Y, \sigma)$ is called D_β-continuous if the inverse image of each open set in Y is D_β-open in X.

Theorem 4.2.

(i) Every β-continuous function is D_β-continuous.

(ii) Every g-continuous function is D_β-continuous.

(iii) Every $semi^*$-continuous function is D_β-continuous.

(iv) Every pre^*-continuous function is D_β-continuous.

(v) Every D_α-continuous function is D_β-continuous.

Proof. It follows directly from the Theorem 2.13. ■

Remark 4.3.

(i) D_β-continuous function need not be β-continuous.
(see Example 4.4 below)

(ii) D_β–continuous function need not be g–continuous.
(see Example 4.4) below

(iii) D_β–continuous function need not be $semi^*$–continuous.
(see Example 4.4) below

(iv) D_β–continuous function need not be pre^*–continuous.
(see Example 4.4 below)

(v) D_β–continuous function need not be D_α–continuous.
(see Example 4.4 below)

Example 4.4. Let X={a, b, c}, τ={ϕ, X, {a}, {a, b}}, then (X, τ) is a topological space. $C(X)$={X, ϕ, {b, c}, {c}}. Let Y={1, 2, 3}, σ={ϕ, Y, {1}, {1, 2}}, then (Y,σ) be another topological space. $GC(X)$={X, ϕ, {b, c}, {c}, {a, c}},
$GO(X)$={X, ϕ, {a, b}, {a}, {b}},

$D_\alpha C(X)=\{X, \phi, \{b, c\}, \{c\}, \{a, c\}, \{b\}\}$,
$D_\alpha O(X)$=$\{X, \phi, \{a\}, \{a, b\}, \{b\}, \{a, c\}\}$, $\beta C(X)$=$\{X, \phi, \{b, c\}, \{c\}, \{b\}\}$,
$\beta O(X)$=$\{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$,
$D_\beta O(X)$=$\{X, \phi, \{a, b\}, \{a\}, \{b\}, \{a, c\}\}$,
$D_\beta C(X)$= $\{X, \phi, \{b, c\}, \{c\}, \{a, c\}, \{a\}, \{b\}\}$ and $D_\beta O(X)$ =$\{X, \phi, \{a, b\}, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by,

(i) $f(a) = 3$, $f(b) = 1$, $f(c) = 2$ is D_β-continuous, since the inverse image of each open set in Y is D_β-open in X. But it is not β-continuous since the preimage of an open set $A = \{1, 2\}$ in Y is {b,c}, which is not β-open set in X.

(ii) $f(a) = 3$,$f(b) = 1$,$f(c) = 2$ is D_β-continuous, since the inverse image of each open set in Y is D_β-open in X. But it is not g-continuous, since the preimage of an open set $A = \{1, 2\}$ in Y, is $\{b, c\}$, which is not g-open set in X.

(iii) $f(a) = 3$, $f(b) = 1$, $f(c) = 2$ is D_β-continuous, since the inverse image of each open set in Y is D_β-open in X. But it is not D_α-continuous, since the preimage of an open set $A = \{1, 2\}$ in Y, is $\{b, c\}$, which is not D_α-open set in X.

Theorem 4.5. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following are equivalent:

(i) f is D_β-continuous.

(ii) $f(Cl_{D_\beta}(A)) \subset Cl(f(A))$ for every subset A of X.

(iii) The inverse image of each closed set in Y is D_β-closed in X.

(iv) For each $x \in X$ and each open set $U \subset Y$ containing $f(x)$, there exists a D_β-open set $V \subset X$ containing x such that $f(V) \subset U$.

(v) $Cl_{D_\beta}(f^{-1}(B) \subseteq f^{-1}(Cl(B))$ for every subset B of Y.

(vi) $f^{-1}(Int(A)) \subset Int_{D_\beta}(f^{-1}(A))$ for every subset A of Y.

Proof. $(i) \Rightarrow (ii)$ Suppose f is D_β-continuous and let A be any subset of X. Let $x \in Cl_{D_\beta}(A)$, then $f(x) \in f(Cl_{D_\beta}(A))$. Suppose U be an open neighborhood of $f(x)$. Then $f^{-1}(U)$ is a D_β-open set of X containing x and it intersects A in the point y(other than x). Then the set U intersects $f(A)$ in the point $f(y)$, therefore $f(x) \in Cl(f(A))$ and we get $f(Cl_{D_\beta}(A)) \subset Cl(f(A))$.

$(ii) \Rightarrow (iii)$ Suppose the function f is D_β-continuous. Let A be any closed set in Y and let $B = f^{-1}(A)$. Since B is D_β–closed in X. We show that $Cl_{D_\beta}(B) = B$. For, $f(B) = f(f^{-1}(A)) \subset A$. Suppose $x \in Cl_{D_\beta}(B)$. Then we have $f(x) \in f(Cl_{D_\beta}(B)) \subset Cl(f(B)) \subset Cl(A) = A$. Thus $x \in f^{-1}(A) = B$. Therefore $Cl_{D_\beta}(B) \subset B$. Since $B \subset Cl_{D_\beta}(B)$. Hence we have $B = Cl_{D_\beta}(B)$ i.e.$B = f^{-1}(A)$ is a D_β–closed set in X.

$(iii) \Rightarrow (i)$ Since function f is D_β-continuous. Suppose U be any open set of Y. Let $A = Y \setminus U$ be any closed set in Y. Then $f^{-1}(A) = f^{-1}(Y) \setminus f^{-1}(U) = X \setminus f^{-1}(U)$ is a D_β-closed set in X. Therefore $f^{-1}(U)$ is open in X and hence f is D_β-continuous.
$(i) \Rightarrow (iv)$ Suppose f is D_β-continuous function. Suppose for each $x \in X$ and for each open set $U \subset Y$ containing $f(x)$, $f^{-1}(U) \in D_\beta O(X)$. We set $V = f^{-1}(U)$ containing x, we get $f(V) \subset U$.

$(iv) \Rightarrow (i)$ Let U be an open set in Y, containing $f(x)$ for each $x \in X$, then there exists a D_β-open set V_x(open neighborhood of x) containing x such that $f(V_x) \subset U$ and then $x \in V_x \subset f^{-1}(U)$, which shows that $f^{-1}(U)$ is open in X. Hence f is D_β-continuous.
$(ii) \Rightarrow (v)$ Let B be any subset of Y and $A = f^{-1}(B)$ is the subset of X. By hypothesis $f(Cl_{D_\beta}(A)) \subset Cl(f(A))$ for every subset A of X, then we have $f(Cl_{D_\beta}(f^{-1}(B))) \subset Cl(f(f^{-1}(B))) \subset Cl(B)$ and therefore we get $(Cl_{D_\beta}(f^{-1}(B)) \subset f^{-1}(Cl(B))$.

$(v) \Rightarrow (vi)$ Let F be any subset of Y. By hypothesis $(Cl_{D_\beta}(f^{-1}(Y \setminus F)) \subseteq f^{-1}(Cl(Y \setminus F))$. This Shows that $(Cl_{D_\beta}(X \setminus (f^{-1}(F)) \subseteq f^{-1}(Y \setminus Int(F))$. Therefore $X \setminus (Int_{D_\beta}(f^{-1}(F))) \subseteq X \setminus f^{-1}(Int(F)))$. Hence we get $f^{-1}(Int(F)) \subseteq Int_{D_\beta}(f^{-1}(F))$.
$(vi) \Rightarrow (i)$ We show that f is D_β-continuous. Let V be any open set in Y. Then $Int(V) = V$. By hypothesis $f^{-1}(Int(V)) \subseteq (Int_{D_\beta}(f^{-1}(V)))$. Thus we get $f^{-1}(V)) \subseteq (Int_{D_\beta}(f^{-1}(V)))$ and since $(Int_{D_\beta}(f^{-1}(V))) \subseteq f^{-1}(V))$. Hence we get $(Int_{D_\beta}(f^{-1}(V))) = f^{-1}(V)$, which implies that $f^{-1}(V)$ is D_β-open in X. ■

Theorem 4.6. Let $f : (X, \tau) \to (Y, \sigma)$ be D_β–continuous and $g : (Y, \sigma) \to (Z, \eta)$ be continuous functions. Then their composition $gof : (X, \tau) \to (Z, \eta)$ is D_β-continuous.

Proof. It is obvious. ■

Remark 4.7. Composition of two D_β-continuous functions need not be D_β-continuous.

Example 4.8. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a, c\}, \{c\}\}$, $C(X) = \{X,\phi, \{a, b\}, \{b\}\}$, then (X,τ) is a topological space.
$D_\beta C(X)$=$\{X,\phi,\{a, b\},\{b\},\{b, c\}, \{a\}, \{c\}\}$, $D_\beta C(X)$=$\{X, \phi,\{a, c\}, \{c\}, \{a\},\{b, c\},\{a, b\}\}$.
Let Y=$\{1, 2, 3\}$, σ=$\{Y, \phi,\{2, 3\}\}$, then (Y, σ) is a topological space. $C(Y)$=$\{Y, \phi,\{1\}\}$, $D_\beta C(Y)$=$\{Y, \phi, \{1, 3\}, \{1, 2\}, \{1\},\{2\}, \{3\}\}$, $D_\beta O(Y)$=$\{Y, \phi, \{2\}, \{2, 3\}, \{3\}, \{1, 2\}, \{1, 3\}\}$.
Let Z=$\{r, s, t\}$, η=$\{Z, \phi, \{s\}\}$, then (Z, η) is another topological space.
Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(a) = 2$, $f(b) = 3$ and $f(c) = 1$ and another function $g : (Y, \sigma) \to (Z, \eta)$ defined as $g(1) = r$, $g(2) = t$, $g(3) = s$. Here both the functions f and g are D_β–continuous. Let $A = \{s\}$ be any open set in Z, but $(gof)^{-1}(s) = f^{-1}(g^{-1}(s)) = f^{-1}(3) = \{b\}$, which is not a D_β–open set in X.

Theorem 4.9. If a function $f : (X, \tau) \to (Y, \sigma)$ is D_α-continuous and $g : (Y, \sigma) \to (Z, \eta)$ is continuous, then $g \circ f : (X, \tau) \to Z, \eta)$ is also D_α-continuous.

Proof. Let B be any open set in (Z, η). Since g is continuous, therefore $g^{-1}(B)$ is open in (Y, σ). Since f is D_α-continuous, we have $f^{-1}(g^{-1}(B)) = (g \circ f)^{-1}(B)$ is D_α-open in X. Thus $g \circ f$ is D_α-continuous. ■

Theorem 4.10. If a function $f : (X, \tau) \to (Y, \sigma)$ is D_α-continuous and $g : (Y, \sigma) \to (Z, \eta)$ is continuous, then $g \circ f : (X, \tau) \to Z, \eta)$ is D_β-continuous.

Proof. It is obvious. ■

Definition 4.11. A function $f : (X, \tau) \to (Y, \sigma)$ is called D_α-irresolute if the preimage of each D_α-closed(D_α-open) set in Y is D_α-closed (D_α-open) in X.

Definition 4.12. A function $f : (X, \tau) \to (Y, \sigma)$ is called D_β-irresolute if the preimage of each D_β-closed(D_β-open) set in Y is D_β-closed (D_β-open) in X.

Theorem 4.13. If a function $f : (X, \tau) \to (Y, \sigma)$ is D_α-irresolute and $g : (Y, \sigma) \to (Z, \eta)$ is D_α-irresolute, then $g \circ f : (X, \tau) \to Z, \eta)$ is D_α-irresolute.

Proof. Let A be any D_α-closed set in the space (Z, η). Since g is D_α-irresolute, therefore $g^{-1}(A)$ is D_α-closed set in Y. Since f is D_α-irresolute, then $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$ is D_α-closed in X. Hence $(g \circ f)$ is D_α-irresolute. ■

Theorem 4.14. If a function $f : (X, \tau) \to (Y, \sigma)$ is D_β-irresolute and $g : (Y, \sigma) \to (Z, \eta)$ is D_β-irresolute then $g \circ f : (X, \tau) \to Z, \eta)$ is also D_β-irresolute.

Proof. Its proof is similar to the Theorem 4.13. ■

Remark 4.15. Every D_α-irresolute function is D_β-irresolute but the converse is not true.

Example 4.16. Let X = $\{a, b, c, d\}$, $\tau = \{\phi, X, \{b\}, \{b, c\}\}$, then (X, τ) is a topological space. $C(X) = \{X, \phi, \{a, c, d\}, \{a, d\}\}$. Let $Y = \{1, 2, 3\}$, $\sigma = \{\phi, Y, \{1, 3\}\}$, then (Y, σ) is another topological space. $C(Y) = \{Y, \phi, \{2\}\}$.
$D_\alpha C(X) = \{X, \phi, \{a, c, d\}, \{a, d\}, \{a, b\}, \{a, b, c\}, \{d\}, \{a\}, \{a, c\}, \{b, c, d\}, \{a, b, d\}, \{b, d\}, \{c, d\}\}$,
$D_\alpha O(X) = \{X, \phi, \{b\}, \{b, c\}, \{c, d\}, \{d\}, \{a, b, c\}, \{b, c, d\}, \{b, d\}, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}$,
$D_\beta C(X) = \{X, \phi, \{a, c, d\}, \{a, d\}, \{a, b\}, \{a, b, c\}, \{d\}, \{a\}, \{a, c\}, \{b, c, d\}, \{a, b, d\}, \{c\}, \{b, d\}, \{b\}, \{c, d\}\}$ and $D_\beta O(X) = \{X, \phi, \{b\}, \{b, c\}, \{c, d\}, \{d\}, \{a, b, c\}, \{b, c, d\}, \{b, d\}, \{a\}, \{c\}, \{a, c, d\}, \{a, c\}, \{a, b, d\}, \{a, b\}\}$.
$D_\alpha C(Y) = \{Y, \phi, \{2\}, \{2, 3\}, \{1, 2\}\}$, $D_\alpha O(Y) = \{Y, \phi, \{1\}, \{1, 3\}, \{3\}\}$,
$D_\beta C(Y) = \{Y, \phi, \{2\}, \{2, 3\}, \{1, 2\}, \{1\}, \{3\}\}$, $D_\beta O(Y) = \{Y, \phi, \{1\}, \{1, 3\}, \{3\}, \{2, 3\}, \{1, 2\}\}$.
Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(a) = f(d) = 1, f(c) = 3, f(b) = 2$, is D_β-irresolute. Since the preimage of every D_β-closed set in X is D_β-closed in Y. But it is not D_α-closed, since the preimage of the D_α-closed set $A = \{2, 3\}$ is $\{b, c\}$, which is not D_α-closed in X.

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