

## Spectra of some New Product of Graphs and its Applications

**Renny P. Varghese and Susha D.**

*Department of Mathematics,  
Catholicate College, Pathanamthitta,  
Kerala, India - 689 645.*

### Abstract

In this paper we define some operations in graphs and determine their Adjacency spectra. Here we define three products of graphs namely second duplication product (SDP), second subdivision product (SSP) and second subdivision edge product (SSEP) of graphs and determine their adjacency and Laplacian spectrum. Also we discuss some applications like the number of spanning tree, the Kirchhoff index and Laplace - energy - like invariant of such product of graphs.

**AMS subject classification:** 05C50.

**Keywords:** Spectrum, spanning tree, Kirchhoff index, Laplace - energy - like invariant.

### 1. Introduction

All graphs described in this paper are simple and undirected. Let  $G$  be a graph with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$ . The *adjacency matrix* of  $G$  denoted by  $A(G) = (a_{ij})_{n \times n}$  is an  $n \times n$  symmetric matrix defined as  $a_{ij} = 1$  if  $v_i$  and  $v_j$  are adjacent in  $G$ , 0 otherwise. Let  $d_i$  be the degree of the vertex  $v_i$  in  $G$  and  $D(G) = \text{diag}(d_1, d_2, \dots, d_n)$  be the diagonal matrix of  $G$ . The Laplacian matrix is defined as  $L(G) = D(G) - A(G)$ . The characteristic polynomial of  $A$  (or  $G$ ) is defined as  $f(A : x) = \det(xI_n - A)$  where  $I_n$  is the identity matrix of order  $n$ . The roots of the characteristic equation of  $A$  are called the *eigen values* of  $G$ . Let  $\lambda_1, \lambda_2, \dots, \lambda_{i_g}$  be the distinct eigenvalues of  $G$  with multiplicities  $m_1, m_2, \dots, m_{i_g}$  then the spectrum of  $G$  is defined as

$$\text{Spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_{i_g} \\ m_1 & m_2 & \dots & m_{i_g} \end{pmatrix}$$

and are called  $A$  - spectrum of  $G$  [3]. The eigenvalues of  $L(G)$  is denoted by  $0 = \mu_1(G) \leq \mu_2(G), \dots \leq \mu_n(G)$  and are called the  $L$  - spectrum or  $L\text{Spec}(G)$  of  $G$ . Since  $A(G)$  and  $L(G)$  are real and symmetric, their eigenvalues are all real numbers. A graph is  $A$  - integral, if the  $A$  - spectrum consists only of integers [2].

The incidence matrix of  $G$  is the  $0-1$  matrix  $R = (r_{ve})$  with rows indexed by vertices and column by edges where  $r_{ve} = 1$  when the vertex  $v$  is an end point of the edge  $e$  and  $0$  otherwise. The subdivision graph  $S(G)$  [3] of  $G$  is the graph obtained by inserting an additional vertex in each edge of  $G$ .

The characteristic polynomial and spectra of graphs helps to investigate some properties of graphs such as energy [8], number of spanning tree [9, 5], the Kirchhoff index [4, 6, 1], Laplace- energy - like invariants [1, 7] and so on.

The paper is organised as follows. In section 2 we use some basic results in spectral graph theory which are used in succeeding sections. In section 3 we define five new operations on graphs and calculating their adjacency spectrum. Then in section 4 and 5 we define three new product of graphs such as *second duplication product (SDP)*, *second subdivision product (SSP)* and *second subdivision edge product (SSEP)* of graphs and find their adjacency and Laplacian spectrum. Also we determine the number of spanning trees, Kirchhoff index and Laplace - energy - like invariant of such graph products.

## 2. Preliminaries

**Definition 2.1.** Let  $G$  be a graph with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$ . Take another set  $I(G) = \{u_1, u_2, \dots, u_n\}$ . Make  $u_i$  adjacent to all the vertices in  $N(v_i)$ , the neighbourhood set of  $v_i$ , in  $G$  for each  $i$  and remove the edges of  $G$  only. The resulting graph is called the *duplication graph* [10] of  $G$  and is denoted by  $DG$ .

In the graph  $G$  make  $u_i$  adjacent to all the vertices in the neighbourhood of  $v_i$  for each  $i = 1, 2, \dots, n$ . The resulting graph is called the *splitting graph* [11] of  $G$ .

**Proposition 2.2.** [3] Let  $M_1, M_2, M_3, M_4$  be respectively  $p \times p, p \times q, q \times p, q \times q$  matrix with  $M_1$  and  $M_4$  are invertible then

$$\begin{aligned} \det \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} &= \det(M_1) \det(M_4 - M_3 M_1^{-1} M_2) \\ &= \det(M_4) \det(M_1 - M_2 M_4^{-1} M_3) \end{aligned}$$

where  $M_4 - M_3 M_1^{-1} M_2$  and  $M_1 - M_2 M_4^{-1} M_3$  are called the schur complements of  $M_1$  and  $M_4$  respectively.

**Lemma 2.3.** [3] Let  $A = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}$  be a  $2 \times 2$  block symmetric matrix. Then the eigenvalues of  $A$  are those of  $A_1 + A_2$  together with  $A_1 - A_2$ .

**Lemma 2.4.** [3] Let  $G$  be a connected  $r$ -regular graph on  $n$  vertices with its adjacency matrix  $A$  having  $m$  distinct eigenvalues  $\lambda_1 = r, \lambda_2, \dots, \lambda_m$ . Then there exists a poly-

nomial  $P(x) = n \frac{(x - \lambda_2)(x - \lambda_3) \dots (x - \lambda_m)}{(r - \lambda_2)(r - \lambda_3) \dots (r - \lambda_m)}$ , such that  $P(A) = J$  where  $J$  is the square matrix of order  $n$  whose all entries are one, so that  $P(r) = n$  and  $P(\lambda_i) = 0$  for all  $\lambda_i \neq r$ .

**Lemma 2.5.** [3] Let  $G$  be an  $r$ -regular graph with adjacency matrix  $A$  and incidence matrix  $R$ , Then  $RR^T = A + rI$ .

### 3. Eigenvalues of some non - regular graphs

Let  $G$  be an  $r$ -regular graph with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$ . Let  $G_1$  be the duplication graph and  $G_2$  be the splitting graph of  $G$ . Consider the following operations on the duplication graph  $G_1$  and the splitting graph  $G_2$  and denote the the resultant non - regular graph by  $F_i, i = 1, 2, \dots, 5$ .

**Operation 1.** In the duplication graph  $G_1$  of  $G$ , take corona product of  $K_1$  with the vertices of  $G$  only.

**Operation 2.** In the duplication graph  $G_1$  introduce  $k$  vertices and make all of them adjacent to all the vertices of  $G$  only.

**Operation 3.** In the duplication graph  $G_1$ , take corona product of  $K_1$  with the vertices

Table 1: Spectrum of  $F_i, i = 1, 2, \dots, 5$

Op.	Adj. Matrix	Ch. Poly.	Eigen values
1	$\begin{bmatrix} 0 & A & I \\ A & 0 & 0 \\ I & 0 & 0 \end{bmatrix}$	$x^n \prod_{i=1}^n (x^2 - 1 - \lambda_i^2)$	$\begin{cases} 0, & n \text{ times} \\ \pm\sqrt{1 + \lambda_i^2} & \text{if } i=1,2,\dots,n \end{cases}$
2	$\begin{bmatrix} 0 & A & J \\ A & 0 & 0 \\ J & 0 & 0 \end{bmatrix}$	$x^k \prod_{i=1}^n (x^2 - kQ(\lambda_i) - \lambda_i^2)$	$\begin{cases} 0, & k \text{ times} \\ \pm\sqrt{kn + r^2} \\ \pm\lambda_i & \text{if } \lambda_i \neq r \end{cases}$
3	$\begin{bmatrix} 0 & A & I & J \\ A & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ J & 0 & 0 & 0 \end{bmatrix}$	$x^{n+k} \prod_{i=1}^n (x^2 - 1 - kQ(\lambda_i) - \lambda_i^2)$	$\begin{cases} 0, & n+k \text{ times} \\ \pm\sqrt{1 + kn + r^2} \\ \pm\sqrt{1 + \lambda_i^2} & \text{if } \lambda_i \neq r \end{cases}$
4	$\begin{bmatrix} 0 & A & A \\ A & 0 & 0 \\ A & 0 & 0 \end{bmatrix}$	$\prod_{i=1}^n (x - \lambda_i \cdot \alpha_k)$	$\lambda_i \cdot \alpha_k, k = 1, 2, 3 : i = 1, 2, \dots, n$
5	$\begin{bmatrix} A & A & I \\ A & 0 & 0 \\ I & 0 & 0 \end{bmatrix}$	$x^n \prod_{i=1}^n (x^2 - \lambda_i x - \lambda_i^2 - 1)$	$\begin{cases} 0, & n \text{ times} \\ \frac{\lambda_i \pm \sqrt{5\lambda_i^2 + 4}}{2}, & \text{if } i=1,2, \dots, n \end{cases}$

of  $G$  and introduce  $k$  distinct vertices and make all of them adjacent to all the vertices of  $G$  only.

**Operation 4.** In the duplication graph  $G_1$  make a copy of the original graph  $G$ . Join each vertex of the copy of  $G$  to the neighbourhood of the corresponding vertices of  $G$  only.

**Operation 5.** In the splitting graph  $G_2$ , take corona product of  $K_1$  with the vertices of  $G$  only.

Where  $A, J$  and  $I$  are respectively, the adjacency matrix of  $G$ , matrix of all entries one and identity matrix.  $\alpha_k$  are the roots of  $x^3 - x^2 - 2x + 1 = 0, k = 1, 2, 3$

### 4. Some New Product of Graphs

#### 4.1. Second Duplication Product of graphs (SDP)

Let  $D_1$  be the duplication graph of a graph  $G$  with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$  and let  $I(G) = \{u_1, u_2, \dots, u_n\}$  be the vertex set corresponding to  $V(G)$ . Take another copy of  $D_1$  with vertex set  $\{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_n\}$  and  $\{\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n\}$ . Now make  $v_i$  adjacent with  $\hat{v}_i$ . The new graph obtained is called second duplication product of graph, It is denoted by  $D^{(2)}(G)$ .

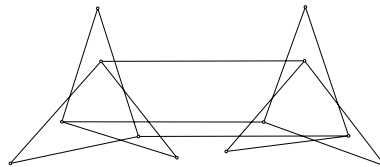


Figure 1:  $D^{(2)}(K_3)$ .

**Theorem 4.1.** Let  $G$  be an  $r$ -regular graph on  $n$  vertices with spectrum  $\{r = \lambda_1, \lambda_2, \dots, \lambda_n\}$ . Then the spectrum of  $D^{(2)}(G)$ ,

$$Spec(D^{(2)}(G)) = \left( \frac{1 \pm \sqrt{1 + 4\lambda_i^2}}{2}, \frac{-1 \pm \sqrt{1 + 4\lambda_i^2}}{2} \right) \text{ for } i = 1, 2, \dots, n$$

*Proof.* The adjacency matrix of  $D^{(2)}(G) = \begin{bmatrix} 0 & A & I & 0 \\ A & 0 & 0 & 0 \\ I & 0 & 0 & A \\ 0 & 0 & A & 0 \end{bmatrix}$

where  $A$  is the adjacency matrix of  $G$ .

Denote  $A_1 = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$  and  $A_2 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$

Then the adjacency spectrum of  $D^{(2)}(G) = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}$ . Hence by Lemma 2.3 the spectrum of  $D^{(2)}(G)$  are those of  $A_1 + A_2$  together with  $A_1 - A_2$ .

$$\begin{aligned} f(A_1 + A_2 : x) &= \det(xI - (A_1 + A_2)) \\ &= \det \begin{pmatrix} (x-1)I & -A \\ -A & xI \end{pmatrix} \\ &= \det(xI) \det((x-1)I - \frac{A^2}{x}) \\ &= \det((x(x-1)I - A^2)) \\ &= \prod_{i=1}^n (x^2 - x - \lambda_i^2) \end{aligned}$$

Similarly  $f(A_1 - A_2 : x) = \prod_{i=1}^n (x^2 + x - \lambda_i^2)$ . ■

**Corollary 4.2.** Let  $G$  be an  $r$ -regular graph on  $n$  vertices. Then  $D^{(2)}(G)$  is an integral if and only if either  $G = \overline{K_n}$  or  $\lambda = \sqrt{t(t+1)}$  for some integer  $t$ .

*Proof.* The eigen values of  $D^{(2)}(G)$  consist of  $\frac{1 \pm \sqrt{1 + 4\lambda_i^2}}{2}$ ,  $\frac{-1 \pm \sqrt{1 + 4\lambda_i^2}}{2}$  for  $i = 1, 2, \dots, n$ . Both are integers if and only if  $\sqrt{1 + 4\lambda_i^2}$  are odd integers. This is possible iff either  $\lambda_i = 0$  or  $\lambda_i = \sqrt{t(t+1)}$  for some integer  $t$ . ■

#### 4.2. Second Subdivision Product of graphs (SSP)

Let  $G$  be a graph with  $n$  vertices and  $m$  edges and  $S(G)$  be its subdivision graph. Let  $V(G) = \{v_1, v_2, \dots, v_n\}$  be the vertex set of  $G$  and  $I(G) = \{u_1, u_2, \dots, u_m\}$  be the vertex corresponding to the edges of  $G$ . Take another copy of  $S(G)$  with vertex set  $\{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_n\}$  and  $\{\hat{u}_1, \hat{u}_2, \dots, \hat{u}_m\}$ . Now make  $v_i$  adjacent with  $\hat{v}_i$  for each  $i = 1, 2, \dots, n$ . The resulting graph obtained is called second subdivision product of  $G$ . It is denoted by  $S^{(2)}(G)$ .

**Theorem 4.3.** Let  $G$  be an  $r$ -regular graph on  $n$  vertices and  $m$  edges with spectrum  $\{r = \lambda_1, \lambda_2, \dots, \lambda_n\}$ . Then the spectrum of  $S^{(2)}(G)$ ,

$$\text{Spec}(S^{(2)}(G)) = \begin{pmatrix} \frac{1 \pm \sqrt{1 + 4(\lambda_i + r)}}{2} & \frac{-1 \pm \sqrt{1 + 4(\lambda_i + r)}}{2} & 0 \\ 1 & 1 & 2(m-n) \end{pmatrix}$$

for  $i = 1, 2, \dots, n$ .

*Proof.* The adjacency matrix of  $S^{(2)}(G) = \begin{bmatrix} 0 & R & I & 0 \\ R^T & 0 & 0 & 0 \\ I & 0 & 0 & R \\ 0 & 0 & R^T & 0 \end{bmatrix}$

where  $A$  is the adjacency matrix of  $G$ .

Denote  $A_1 = \begin{bmatrix} 0 & R \\ R^T & 0 \end{bmatrix}$  and  $A_2 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ . Then the adjacency spectrum of  $S^{(2)}(G) = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}$ . Hence by Lemma 2.3 the spectrum of  $S^{(2)}(G)$  are those of  $A_1 + A_2$  together with  $A_1 - A_2$ . Here we use the fact that  $RR^T = A + rI$ . In a similar manner in Theorem 4.1, The characteristic polynomial of  $A_1 + A_2 = x^{m-n} \prod_{i=1}^n (x^2 - x - (r + \lambda_i))$

and those of  $A_1 - A_2 = x^{m-n} \prod_{i=1}^n (x^2 + x - (r + \lambda_i))$ . This proves the theorem. ■

### 4.3. Second Subdivision Edge Product of graphs(SSEP)

Let  $S(G)$  be the subdivision graph of a graph  $G$  with  $n$  vertices and  $m$  edges. Let  $V(G) = \{v_1, v_2, \dots, v_n\}$  be the vertex set of  $G$  and  $I(G) = \{u_1, u_2, \dots, u_m\}$  be the vertex corresponding to the edges of  $G$ . Take another copy of  $S(G)$  with vertex set  $\{\acute{v}_1, \acute{v}_2, \dots, \acute{v}_n\}$  and  $\{\acute{u}_1, \acute{u}_2, \dots, \acute{u}_m\}$ . Now make  $u_i$  adjacent with  $\acute{u}_i$  for each  $i = 1, 2, \dots, n$ . The resulting graph obtained is called second subdivision edge product of graph. It is denoted by  $S^{*(2)}(G)$ .

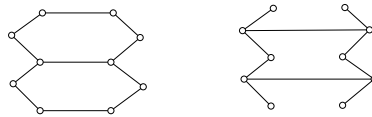
**Theorem 4.4.** Let  $G$  be an  $r$  - regular graph on  $n$  vertices and  $m$  edges with spectrum  $\{r = \lambda_1, \lambda_2, \dots, \lambda_n\}$ . Then the spectrum of  $S^{*(2)}(G)$  is

$$Spec(S^{*(2)}(G)) = \left( \begin{array}{cc|cc} -1 & 1 & \frac{1 \pm \sqrt{1 + 4(\lambda_i + r)}}{2} & \frac{-1 \pm \sqrt{1 + 4(\lambda_i + r)}}{2} \\ m - n & m - n & 1 & 1 \end{array} \right)$$

for  $i = 1, 2, \dots, n$ .

*Proof.* The adjacency matrix of  $S^{*(2)}(G) = \begin{bmatrix} 0 & R & 0 & 0 \\ R^T & 0 & 0 & I \\ 0 & 0 & 0 & R \\ 0 & I & R^T & 0 \end{bmatrix}$

where  $R$  is the vertex edge incidence matrix of  $G$ . The remaining proof is similar as the above theorem. ■

Figure 2:  $S^{(2)}(P_3)$  and  $S^{*(2)}(P_3)$ 

## 5. Laplacian Spectrum

**Theorem 5.1.** Let  $G$  be an  $r$ -regular graph on  $n$  vertices with spectrum  $\{0 = \mu_1, \mu_2, \dots, \mu_n\}$ . Then the Laplacian spectrum of  $D^{(2)}(G)$  is,

$$L\text{Spec}(D^{(2)}(G)) = \left( \begin{array}{ccc} \mu_i & 2r - \mu_i & (r+1) \pm \sqrt{1 + (r - \mu_i)^2} \\ 1 & 1 & 1 \end{array} \right) \text{ for } i = 1, 2, \dots, n$$

*Proof.* The Laplacian matrix of  $D^{(2)}(G) = \begin{bmatrix} (r+1)I & -A & -I & 0 \\ -A & rI & 0 & 0 \\ -I & 0 & (r+1)I & -A \\ 0 & 0 & -A & rI \end{bmatrix}$

where  $A$  is the adjacency matrix of  $G$ .

Denote  $A_1 = \begin{bmatrix} (r+1)I & -A \\ -A & rI \end{bmatrix}$  and  $A_2 = \begin{bmatrix} -I & 0 \\ 0 & 0 \end{bmatrix}$ .

Then  $A_1 + A_2 = \begin{bmatrix} rI & -A \\ -A & rI \end{bmatrix}$  and  $A_1 - A_2 = \begin{bmatrix} (r+2)I & -A \\ -A & rI \end{bmatrix}$

Hence by Lemma 2.3, the spectrum of  $D^{(2)}(G)$  are those of  $A_1 + A_2$  together with  $A_1 - A_2$ . The eigen values of  $A_1 + A_2$  are those of  $rI + A$  and  $rI - A$ . ie the eigen values of  $A_1 + A_2$  are  $r + \lambda_i, r - \lambda_i$  for  $i = 1, 2, \dots, n$ . Here we use the fact that for an  $r$ -regular graph  $\mu_i = r - \lambda_i$ , for  $i = 1, 2, \dots, n$ .

$$\begin{aligned} f(A_1 - A_2 : x) &= \det \begin{pmatrix} (x - r - 2)I & A \\ A & (x - r)I \end{pmatrix} \\ &= \det((x - r)I) \det\left((x - r - 2)I - \frac{A^2}{x - r}\right) \\ &= \det((x - r)(x - r - 2)I - A^2) \\ &= \prod_{i=1}^n (x^2 - 2(r+1)x + r(r+2) - \lambda_i^2) \\ &= \prod_{i=1}^n (x^2 - 2(r+1)x + 2r + 2r\mu_i - \mu_i^2) \end{aligned}$$

Hence the proof. ■

**Theorem 5.2.** Let  $G$  be an  $r$ -regular graph on  $n$  vertices and  $m$  edges with Laplacian spectrum  $\{0 = \mu_1, \mu_2, \dots, \mu_n\}$ . Then the Laplacian spectrum of  $S^{(2)}(G)$ ,

$$LSpec(G) = \left( \begin{array}{cc} 2 & \frac{(r+2) \pm \sqrt{(r+2)^2 - 4\mu_i}}{2} & \frac{(r+4) \pm \sqrt{r^2 + 8r - 4\mu_i}}{2} \\ 2(m-n) & 1 & 1 \end{array} \right)$$

for  $i = 1, 2, 3, \dots, n$ .

*Proof.* The Laplacian matrix of  $S^{(2)}(G) = \begin{bmatrix} (r+1)I & -R & -I & 0 \\ -R^T & 2I & 0 & 0 \\ -I & 0 & (r+1)I & -R \\ 0 & 0 & -R^T & 2I \end{bmatrix}$

where  $R$  is the incidence matrix of  $G$ .

Denote  $A_1 = \begin{bmatrix} (r+1)I_n & -R_{n \times m} \\ -R_{m \times n}^T & 2I_m \end{bmatrix}$  and  $A_2 = \begin{bmatrix} -I_n & 0_{n \times m} \\ 0_{m \times n} & 0_m \end{bmatrix}$

Then  $A_1 + A_2 = \begin{bmatrix} rI & -R \\ -R^T & 2I \end{bmatrix}$  and  $A_1 - A_2 = \begin{bmatrix} (r+2)I & -R \\ -R^T & 2I \end{bmatrix}$  Hence by Lemma

2.3, the spectrum of  $S^{(2)}(G)$  are those of  $A_1 + A_2$  together with  $A_1 - A_2$ . The characteristic polynomial of  $A_1 + A_2 = (x - 2)^{m-n} \prod_{i=1}^n ((x - 2)(x - r) - (r + \lambda_i))$  and those

of  $A_1 - A_2 = (x - 2)^{m-n} \prod_{i=1}^n ((x - r - 2)(x - 2) - (r + \lambda_i))$ . Hence the result. ■

**Theorem 5.3.** Let  $G$  be an  $r$ -regular graph on  $n$  vertices and  $m$  edges with Laplacian spectrum  $\{0 = \mu_1, \mu_2, \dots, \mu_n\}$ . Then the Laplacian spectrum of  $S^{*(2)}(G)$  is,

$$\left( \begin{array}{ccc} 2 & 4 & \frac{(r+2) \pm \sqrt{(r+2)^2 - 4\mu_i}}{2} & \frac{(r+4) \pm \sqrt{r^2 - 4\mu_i + 16}}{2} \\ m-n & m-n & 1 & 1 \end{array} \right)$$

for  $i = 1, 2, 3, \dots, n$ .

*Proof.* The Laplacian matrix of  $S^{*(2)}(G) = \begin{bmatrix} rI & -R & 0 & 0 \\ -R^T & 3I & 0 & -I \\ 0 & 0 & rI & -R \\ 0 & -I & -R^T & 3I \end{bmatrix}$

where  $R$  is the incidence matrix of  $G$ .

Denote  $A_1 = \begin{bmatrix} rI_n & -R_{n \times m} \\ -R_{m \times n}^T & 3I_m \end{bmatrix}$  and  $A_2 = \begin{bmatrix} 0_{n \times n} & 0_{n \times m} \\ 0_{m \times n} & -I_m \end{bmatrix}$

The remaining proof is similar that of the above theorem. ■

### 5.1. Applications

Let  $t(G)$  denote the number of spanning tree of the graph  $G$ , the total number of distinct spanning subgraphs of  $G$  that are trees. If  $G$  is a connected graph with  $n$  vertices and the



Laplacian spectrum  $0 = \mu_1(G) \leq \mu_2(G) \leq \dots \leq \mu_n(G)$  then [5]

$$t(G) = \frac{\mu_2(G)\mu_3(G)\dots\mu_n(G)}{n}$$

**Corollary 5.4.** Let  $G$  be an  $r$ -regular graph on  $n$  vertices with Laplacian spectrum  $\{0 = \mu_1, \mu_2, \dots, \mu_n\}$ . Then

$$t(D^{(2)}(G)) = \frac{t(G)}{4} \prod_{i=1}^n (2r - \mu_i) \prod_{i=1}^n (2r + 2r\mu_i - \mu_i^2)$$

**Corollary 5.5.** Let  $G$  be an  $r$ -regular graph on  $n$  vertices and  $m$  edges with Laplacian spectrum  $\{0 = \mu_1, \mu_2, \dots, \mu_n\}$ . Then

$$t(S^{(2)}(G)) = \frac{2^{2m-2n-1}(2+r)n \prod_{i=1}^n (\mu_i + 4)}{m+n} t(G)$$

*Proof.* From Theorem 5.2 the  $L$ -spectrum consist of,

1. 0
2.  $2 + r$
3. 2, repeats  $2(m - n)$  times
4. Two roots of the equation  $x^2 - (2 + r)x + \mu_i = 0$  for  $i = 2, 3, \dots, n$
5. Two roots of the equation  $x^2 - (4 + r)x + 4 + \mu_i = 0$  for  $i = 1, 2, \dots, n$ . ■

**Corollary 5.6.** Let  $G$  be an  $r$ -regular graph on  $n$  vertices and  $m$  edges with Laplacian spectrum  $\{0 = \mu_1, \mu_2, \dots, \mu_n\}$ . Then

$$t(S^{*(2)}(G)) = \frac{n(2+r)8^{m-n} \prod_{i=1}^n (\mu_i + 2r)}{2(m+n)} t(G)$$

*Proof.* From Theorem 5.3 the  $L$ -spectrum consist of,

1. 0
2.  $2 + r$
3. 2, repeats  $(m - n)$  times
4. 4, repeats  $(m - n)$  times
5. Two roots of the equation  $x^2 - (2 + r)x + \mu_i = 0$  for  $i = 2, 3, \dots, n$
6. Two roots of the equation  $x^2 - (4 + r)x + 2r + \mu_i = 0$  for  $i = 1, 2, \dots, n$  ■

Klein and Randić [6] introduced a new distance function named *resistance distance*. The resistance distance between two vertices of a graph  $G$  is defined to be the effective electrical resistance between them when unit resistors are placed on every edge of  $G$ . The electric resistance is calculated by means of the Kirchhoff laws called *kirchhoff index*.

Kirchhoff index of a connected graph  $G$  with  $n$  ( $n \geq 2$ ) vertices is defined as

$$Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i}$$

**Corollary 5.7.** Let  $G$  be an  $r$ -regular graph on  $n$  vertices with Laplacian spectrum  $\{0 = \mu_1, \mu_2, \dots, \mu_n\}$ . Then

$$Kf(D^{(2)}(G)) = 4n \left[ \sum_{i=1}^n \frac{1}{2r - \mu_i} + \sum_{i=1}^n \frac{2(r+1)}{r^2 + 2r\mu_i - \mu_i^2} \right] + 4Kf(G)$$

**Corollary 5.8.** Let  $G$  be an  $r$ -regular graph on  $n$  vertices and  $m$  edges with Laplacian spectrum  $\{0 = \mu_1, \mu_2, \dots, \mu_n\}$ . Then

$$Kf(S^{(2)}(G)) = 2(m+n) \left[ (m-n) + \frac{1}{r+2} + \sum_{i=1}^n \frac{r+4}{\mu_i+4} \right] + \frac{2(m+n)(r+2)}{n} Kf(G)$$

**corollary 5.9.** Let  $G$  be an  $r$ -regular graph on  $n$  vertices and  $m$  edges with Laplacian spectrum  $\{0 = \mu_1, \mu_2, \dots, \mu_n\}$ . Then

$$Kf(S^{*(2)}(G)) = 2(m+n) \left[ \frac{1}{r+2} + \frac{3}{4}(m-n) + \sum_{i=1}^n \frac{r+4}{2r+\mu_i} \right] + \frac{2(r+2)(m+n)}{n} Kf(G)$$

Another Laplacian spectrum based on graph invariant was defined by Liu and Liu [7] called the Laplacian - energy - like invariant.

The Laplacian - energy - like invariant(LEL) of a graph  $G$  of  $n$  vertices is defined as

$$LEL(G) = \sum_{i=2}^n \sqrt{\mu_i}.$$

**Corollary 5.10.** Let  $G$  be an  $r$ -regular graph on  $n$  vertices with Laplacian spectrum  $\{0 = \mu_1, \mu_2, \dots, \mu_n\}$ . Then

$$\begin{aligned} LEL(D^{(2)}(G)) &= LEL(G) + \sum_{i=1}^n \sqrt{2r - \mu_i} \\ &\quad + \sum_{i=1}^n \left( 2(r+1) + 2\sqrt{2r + 2r\mu_i - \mu_i^2} \right)^{1/2} \end{aligned}$$

*Proof.* Using Theorem 5.1 and the identity  $(\sqrt{x} + \sqrt{y})^2 = (x + y) + 2\sqrt{xy}$  we get the result. ■

**Corollary 5.11.** Let  $G$  be an  $r$ -regular graph on  $n$  vertices and  $m$  edges with Laplacian spectrum  $\{0 = \mu_1, \mu_2, \dots, \mu_n\}$ . Then

$$\begin{aligned} LEL(S^{(2)}(G)) &= 2\sqrt{2}(m - n) + \sqrt{2 + r} + \sum_{i=2}^n (2 + r + 2\sqrt{\mu_i})^{1/2} \\ &\quad + \sum_{i=1}^n (4 + r + 2\sqrt{4 + \mu_i})^{1/2} \end{aligned}$$

**Corollary 5.12.** Let  $G$  be an  $r$ -regular graph on  $n$  vertices and  $m$  edges with Laplacian spectrum  $\{0 = \mu_1, \mu_2, \dots, \mu_n\}$ . Then

$$\begin{aligned} LEL(S^{*(2)}(G)) &= \sqrt{2}(m - n) + 2(m - n) + \sum_{i=2}^n (2 + r + 2\sqrt{\mu_i})^{1/2} \\ &\quad + \sum_{i=1}^n (4 + r + 2\sqrt{\mu_i + 2r})^{1/2} \end{aligned}$$

### Acknowledgments

The author thankful to the University Grants Commission of Government of India for providing financial support under the FDP in the  $XII^{th}$  plan.

### References

- [1] B.Arsic, I. Gutman, K. Ch. Das, K. Xu *Relation between Kirchoff Index and Laplacian - Energy - like Invariant*, Bulletin T.CXLIV de l'Academic serbe des sciences et des arts - 2012 Classe des Sciences mathematiques et naturelles Sciences mathematiques, No 37.
- [2] K. Balinska, D. Cvetkovic, Z. Radosavljevic, S. Simic, and D. Stevanovic, *A survey on integral graphs*, Publ. Elektrotehn. Fak. Ser. Mat. 13 (2002), 42–65.
- [3] D. M. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs, Theory and Applications*, Third edition, Johann Ambrosius Barth Heidelberg, 1995.
- [4] X. Gao, Y.F. Luo, W.W. Liu, *Kirchhoff index in line, subdivision and total graphs of a regular graph*, Discrete Appl. Math. 160(2012) 560–565.
- [5] Jianxi Li, Wai Chee Shiu, An Chang, *The number of spanning tree of a graph*, Applied Mathematics Letters 23 (2010) 286–290.
- [6] D. J. Klein, M. Randic, *Resistance distance*, J. Math. Chem. 12 (1993) 81–95.
- [7] J.Liu, B. Liu, *A Laplacian - energy - like invariant of a graph*, MATCH Commun. Math. comput. Chem 59(2008) 397–413.

- [8] X.L. Li, Y.T. Shi, I. Gutman, *Graph Energy*, Springer-Verlag New York Inc., 2012.
- [9] K. Ozeki, T. Yamashita, *Spanning trees: A survey*, *Graphs and Combinatorics* 27(1) (2011) 1–26.
- [10] E. Sampathkumar, *On Duplicate Graphs*, *Journal of the Indian Math. Soc.* 37(1973) 285–293.
- [11] E. Sampathkumar and H.B. Walikar, *On Splitting Graph of a Graph*, *J. Karnatak Univ. Sci.*, 25(13) (1980), 13–16.