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On Product of Disemigraphs

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Abstract

In this paper we present the product of semigraphs. Taking product of two semigraphs generates a new Semigraph. Concepts of the different operations on graphs due to Iiwoo is referred to define the different types of products of disemigraphs. Further we study the effect of different types of products on the disemigraphs.

AMS subject classification: 05C99, 05C76.
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1. Preliminaries

For the basic concepts in graphs and semigraphs we use references [1, 4, 5].

Definition 1.1. A semigraph G is a pair (V, X) where V is a nonempty set whose elements are called vertices of G and X is a set of $n$tuples called edges of G of distinct vertices for $n \geq 2$ satisfying the following conditions.
(1) Any two edges have atmost one vertex in common.

Two edges $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_m)$ are considered to be equal, if and only if, A semigraph G may be drawn as a set of points representing the vertices. An edge $E = (v_1, v_2, \ldots, v_n)$ is represented by a Jordan curve joining the points corresponding to the vertices $(v_1, v_2, \ldots, v_n)$ in the same order as they appear in E. The end points of the curve are denoted by thick dots. The vertices in between the end points are the middle vertices denoted by small circles. If an end vertex of some edge E, is the middle vertex of some other edge E', a small tangent is drawn to the circle at the end of E.

All the vertices on an edge of a semigraph are considered to be adjacent to one another for obvious reasons.The vertices are divided into four types namely end vertices, middle vertices, middle-end vertices and isolated vertices.

Example 1.2. $G = (V, X)$ where $V = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and

$$X = \{(u_1, u_2, u_3), (u_1, u_6), (u_3, u_6, u_5), (u_4, u_5), (u_3, u_4), (u_4, u_6)\}.$$

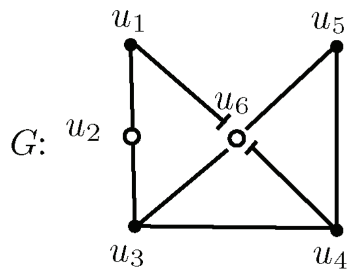

Figure 1:

Definition 1.3. A *subedge* of an edge $E = (v_1, v_2, \ldots, v_n)$ is a k-tuple $E' = (v_{i_1}, v_{i_2}, \ldots, v_{i_k})$ where $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ or $1 \leq i_k < i_{k-1} < i_{k-2} \ldots < i_1 \leq n$. E' is said to be induced by the set of vertices $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$.

Definition 1.4. *A partial edge E* is a $(j - i + 1)$-tuple. $E(v_i, v_j) = (v_i, v_{i+1}, \ldots, v_j)$ where $1 \leq i \leq n$. Thus a subedge E' of an edge E is a partial edge if and only if any two consecutive vertices in E' are also consecutive vertices of E.

Definition 1.5. A semigraph $G' = (V', X')$ is a subsemigraph of a semigraph $G = (V, X)$ if $V' \subseteq V$ and the edges in G' are subedges of G.

Definition 1.6. A *directed semigraph* or *disemigraph D* is a finite set of objects called vertices together with a (possibly empty) set of ordered n-tuples of distinct vertices of D for various $n \geq 2$ called *directed edges* or *arcs*, satisfying the condition C given below.

Suppose $a = (u_1, u_2, \ldots, u_n)$ is an arc. Then for $1 \leq i < j \leq n$, u_i is adjacent to u_j and u_j is adjacent from u_i. Thus each u_i is adjacent to u_j, $1 \leq i < j \leq n$ and each u_j is adjacent from u_i, $1 \leq i < j \leq n$.

C: for any two distinct vertices u and v in a disemigraph D there is at most one arc containing u and v such that u is adjacent to v and at most one arc containing u and v such that v is adjacent to u.

Definition 1.7. A disemigraph D_1 is a *subdisemigraph* of a disemigraph D if $V(D_1) \subseteq V(D)$ and $E(D_1) \subseteq E(D)$ This is denoted by $D_1 \subseteq D$, where $V(D)$ is the vertex set of

D and $E(D)$ is the edge set of D. A subdisemigraph D_1 is a spanning subdisemigraph of D if D_1 has the same vertex set as D.

Definition 1.8. A disemigraph D is *simple* if any two arcs either contain at most one vertex or all vertices in common.

Two vertices u and v are adjacent if u is either adjacent to or adjacent from v. It may happen that u is adjacent to v in one arc and v is adjacent to u in another arc.

Example 1.9. The disemigraph D with $V(D) = \{v_1, v_2, v_3, v_4, v_5\}$ and the arc set $E(D) = \{(v_5, v_4), (v_4, v_3, v_2), (v_5, v_3), (v_4, v_1), (v_1, v_2), (v_3, v_1, v_5)\}$.

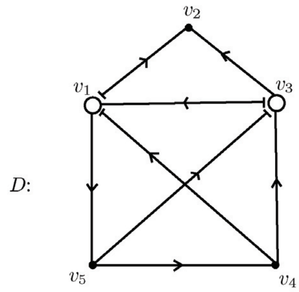

Figure 2:

Now the different products of the graphs considered here are taken from [2, 3] the definition of edge product and vertex product are given below.

Definition 1.10. [3] If G_1 is the graph $(V(G_1), E(G_1))$ and G_2 is the graph $(V(G_2), E(G_2))$ then the *edge product* of G_1 and G_2 denoted by $G_1 \times_E G_2$ is defined as $G = G_1 \times_E G_2$ where

$$E(G) = E(G_1) \times E(G_2) = \{(e_1, e_2)/e_1 \in G_1, e_2 \in G_2\}$$

The vertex set $V(G)$ is the collection of all pairs $(v_1, v_2) \in V(G_1) \times V(G_2)$ satisfying the vertex property V, where vertex property V is given below.

$V : (v_1, v_2) \in V(G) \subseteq V(G_1) \times V(G_2)$ if and only if there exists an edge $(e_1, e_2) \in E(G)$ such that $(e_1, e_2) = (v_1, v_2)(e_1, e_2) = (v_1e_1, v_2e_2)$. or $(e_1, e_2) = (e_1, e_2(v_1, v_2)) = (e_1v_1, e_2v_2)$. That is $(v_1, v_2) \in V(G) = V(G_1) \times V(G_2)$ if and only if there exists an edge $(e_1, e_2) \in E(G)$ such that $e_1 = v_1e_1$ or $e_1 = e_1v_1$ and $e_2 = v_2e_2$ respectively $e_2 = e_2v_2$.

Definition 1.11. [3] If G_1 is the graph $(V(G_1), E(G_1))$ and G_2 is the graph $(V(G_2), E(G_2))$ then the *vertex product* of G_1 and G_2 denoted by $G_1 \times_V G_2$ is defined as $G = G_1 \times_V G_2$

where

$$V(G) = V(G_1) \times V(G_2)$$
$$= \{(v_1, v_2)/v_1 \in V(G_1), v_2 \in V(G_2)\}$$

and its edge set $E(G)$, determined by the following edge property E. E : If (v_1, v_1') and (v_2, v_2') be vertices in $V(G)$ then there exists an edge $(e_1, e_2) \in E(G) \subseteq E(G_1) \times E(G_2)$ only if there exists edges $e_1 = v_1 e_1 v_2 \in E(G_1)$ and $e_2 = v_1' e_2 v_2' \in E(G_2)$.

From the definition of vertex product and edge product of graphs defined above, the edge in the product graph represented by (e, e') is a single directed edge from some vertex (v_1, v_2) to some other vertex (v_1', v_2').

Definition 1.12. Let G_1 and G_2 be two disemigraphs and f be a one-to-one mapping from $V(G_1)$ onto $V(G_2)$ then G_1 is isomorphic to G_2 if $(u_1, u_2, \ldots, u_n) \in E(G_1)$ if and only if

$$(f(u_1), f(u_2), \ldots, f(u_n)) \in E(G_2).$$

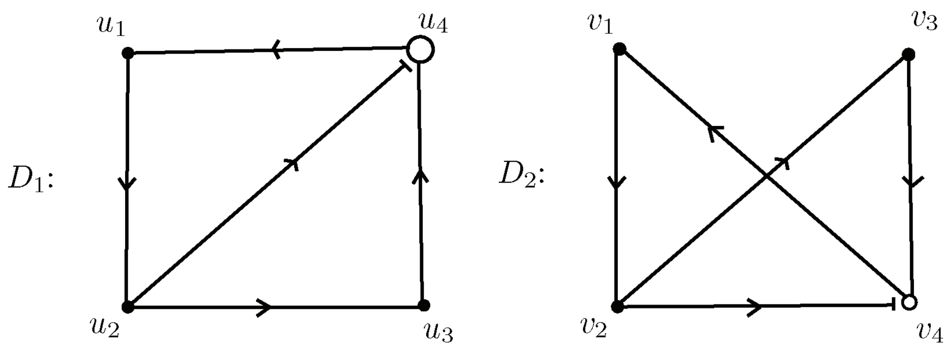

Figure 3:

Definition 1.13. A collection $\{E_1, E_2, E_3, \ldots, E_n\}$ of edges of a semigraph $G = (V, X)$ is said to be a *chain* if (i) $|E_i \cap E_{i+1}| = 1$ and (ii) $|E_i \cap E_j| = 0$, for every $i \neq j$, $j \neq i + 1$, i.e., no edges are adjacent except the consecutive ones.

2. Product of two semigraphs

Operations on semigraphs produce new semigraphs. Here be consider some of the different possible products on semigraphs producing new semigraphs.

2.1. The Cartesian product of semigraphs G_1 and G_2

Definition 2.1.1. Let G_1 be the semigraph $(V(G_1), E(G_1))$ and G_2 be the semigraph $(V(G_2), E(G_2))$. Define the cartesian product, $G_1 \times G_2$ as the semigraph with vertex set $V = (V(G_1) \times V(G_2))$ and edge set E where $e \in E$ is an edge between vertices (v_1, v_1') and (v_2, v_2') if and only if:

1. $v_1 = v_2$ and there exists a partial edge or edge (v_1', v_2') in $E(G_2)$ or
2. $v_1' = v_2'$ and there exists a partial edge or edge (v_1, v_2) in $E(G_1)$

and a vertex (v_1, v_2) in $G_1 \times G_2$ is a middle vertex if and only if both v_1 and v_2 are middle vertices in G_1 and G_2 respectively, otherwise the vertex is an end vertex in $G_1 \times G_2$.

If G_1 and G_2 are disemigraphs then the direction of the edges in $G_1 \times G_2$ is the same as the direction of the edges in inducing graph G_1 or G_2.

Example 2.1.2. The cartesian product of semigraphs G_1 and G_2 in Fig. 4 is given in Fig. 5.

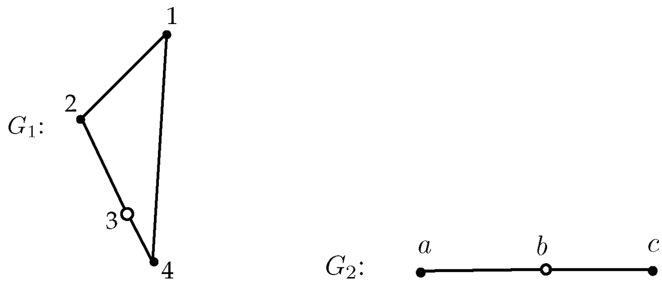

Figure 4:

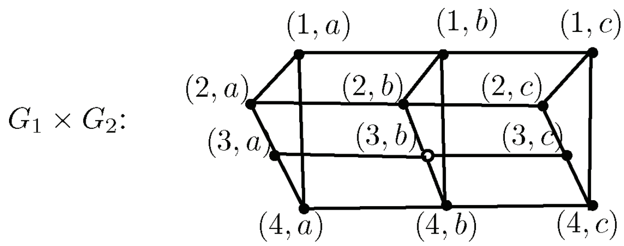

Figure 5:

Example 2.1.3. The cartesian products of disemigraphs G_1 and G_2 in Fig. 6 is given in Fig. 7.

Proposition 2.1.4. The Cartesian product of two semigraphs is connected if and only if both factors are connected.
The proof is similar to the proof of that of graphs.

Remark 2.1.5. The underlying graph of the product of two digraphs and the product of the underlying graphs of disemigraphs will be the same.

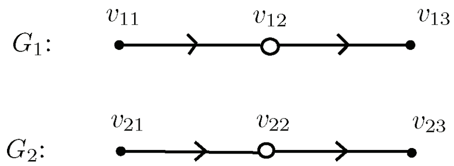

Figure 6:

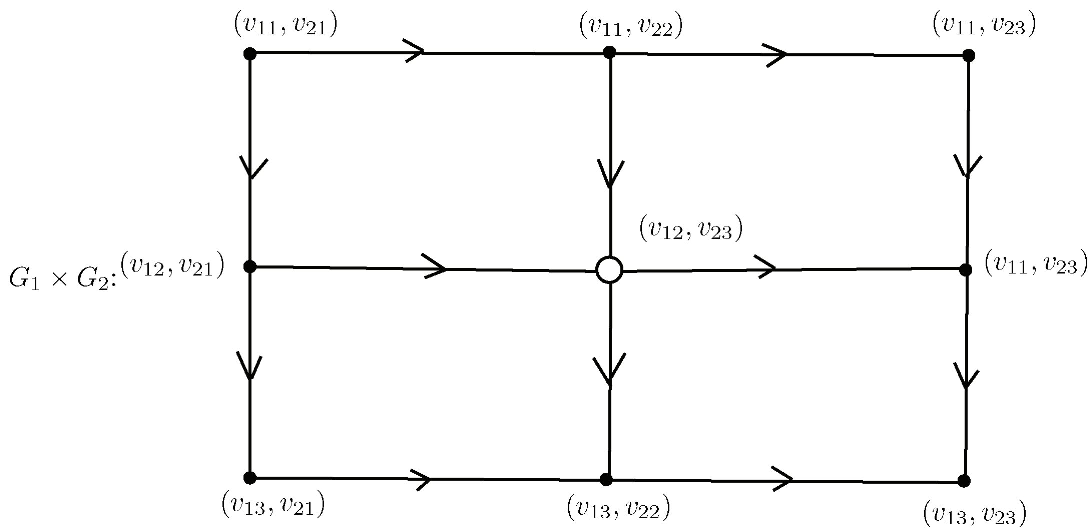

Figure 7:

Theorem 2.1.6. If G_1 and G_2 are two disemigraphs then their Cartesian product $G_1 \times G_2$ is isomorphic to $G_2 \times G_1$.

Proof. Let $G = G_1 \times G_2$ and $G' = G_2 \times G_1$ define $f_V : V(G) \to V(G')$ as $f_V(v_1, v_2) = (v_2, v_1)$. Then f_V is a bijection since, for every $(v_2, v_1) \in G'$ there exists $(v_1, v_2) \in G$ such that $f_V(v_1, v_2) = (v_2, v_1)$ and

$$\begin{aligned} f_V(v_1, v_2) = f_V(v_1', v_2') \\ &\Rightarrow (v_2, v_1) = (v_2', v_1') \\ &\Rightarrow v_2 = v_2' \text{ and } v_1 = v_1' \\ &\Rightarrow (v_1, v_2) = (v_1', v_2'). \end{aligned}$$

Now by definition of isomorphism on disemigraphs, if $e \in E(G)$ is a two tuple

$$\begin{aligned} &\text{i.e.; } e = ((v_{11}, v_{12}), (v_{21}, v_{22})) \\ &\text{i.e.; either } v_{11} = v_{21} \text{ and there exists an edge } (v_{12}, v_{22}) \text{ in } G_2 \\ &\text{or } v_{12} = v_{22} \text{ and there exists an edge } (v_{11}, v_{21}) \text{ in } G_1 \end{aligned}$$

Now $(f_V(v_{11}, v_{12}), f_V(v_{21}, v_{22})) = ((v_{12}, v_{11}), (v_{22}, v_{21}))$ is an edge in $G_2 \times G_1$ if and only if

$$\begin{aligned} & v_{12} = v_{22} \text{ and there exists an edge } (v_{11}, v_{21}) \text{ in } G_1 \\ \text{or } & v_{11} = v_{21} \text{ and there exists an edge } (v_{12}, v_{22}) \text{ in } G_2 \end{aligned}$$

Which is obvious for any edge E in a disemigraph.
If $E = ((v_{11}, v_{12}), (v_{21}, v_{22}), \dots, (v_{n1}, v_{n2}))$.

$$f((v_{11}, v_{12}), (v_{21}, v_{22}), \dots, (v_{n1}, v_{n2})) = (f(v_{11}, v_{12}), f(v_{21}, v_{22}), \dots, f(v_{n1}, v_{n2}))$$

Hence $G_1 \times G_2$ is isomorphic to $G_2 \times G_1$. ■

Remark 2.1.7. If n_1 is the number of middle vertices in G_1 and n_2 the number of middle vertices in G_2 then the number of middle vertices in $G_1 \times G_2$ is $n_1 \cdot n_2$.

2.2. The Vertex Product of two disemigraphs

Definition 2.2.1. If G_1 is a disemigraph $(V(G_1), E(G_1))$ and G_2 is a disemigraph $(V(G_2), E(G_2))$ then the vertex product of G_1 and G_2 denoted by $G_1 \times_V G_2$ is defined as the disemigraph $G = G_1 \times_V G_2$ where $G = (V(G), E(G))$ with vertex set

$$\begin{aligned} V(G) &= V(G_1) \times V(G_2) \\ &= \{(v_1, v_2)/v_1 \in G_1, v_2 \in G_2\} \text{ and} \\ \text{edge set } E(G) &= \{e/e \text{ is an edge } (e_1, e_2) \text{ between } (v_1, v_1') \text{ and} \\ & \quad (v_2, v_2') \in V(G) \text{ if there exists partial edges or edges} \\ & \quad e_1 = (v_1, v_2) \in E(G_1) \text{ and} \\ & \quad e_2 = (v_1', v_2') \in E(G_2)\} \end{aligned}$$

and (v_1, v_2) is a middle edge if and only if v_1 and v_2 are middle vertices in G_1 and G_2 respectively all the other vertices are end vertices.

Example 2.2.2. For the disemigraph G_1 and G_2 given in Fig. 8 the vertex product $G_1 \times_V G_2$ is given in Fig. 9.

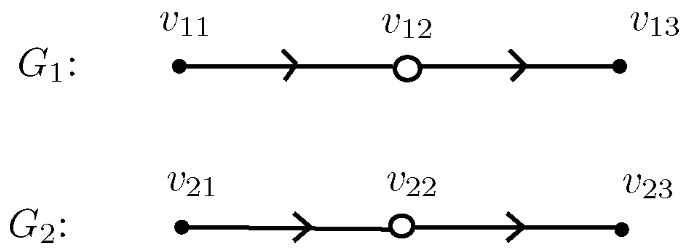

Figure 8:

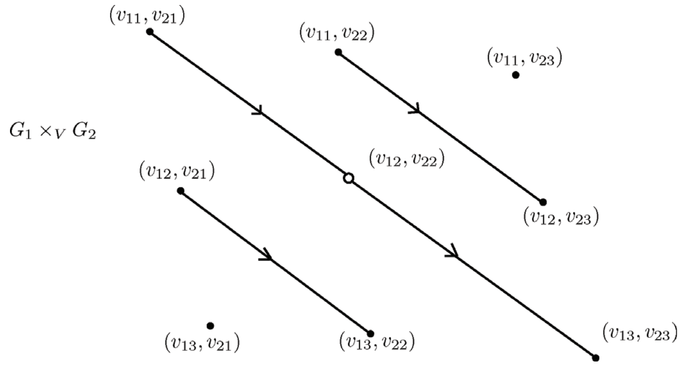

Figure 9:

Theorem 2.2.3. If G_1 and G_2 are two disemigraphs then their vertex product $G_1 \times_V G_2$ is isomorphic to $G_2 \times_V G_1$.

The proof is similar to the proof of Theorem 2.1.6.

2.3. The Edge Product of two disemigraphs

Definition 2.3.1. If G_1 is the disemigraph $(V(G_1), E(G_1))$ and G_2 is the disemigraph $(V(G_2), E(G_2))$ then the edge product of G_1 and G_2 denoted by $G_1 \times_E G_2$ is defined as $G = G_1 \times_E G_2$ where $G = (V(G), E(G))$ with edge set

$$\begin{aligned} E(G) &= E(G_1) \times E(G_2) \\ &= \{e/e \text{ is an edge } (e_1, e_2) \text{ where } e_1 \in E(G_1), e_2 \in E(G_2)\} \\ \text{vertex set } V(G) &= \{(v_1, v_2) \in V(G_1) \times V(G_2)/ \text{ there exists } (e_1, e_2) \in E(G) \text{ with} \\ &\quad (e_1, e_2) = (v_1, v_2)(e_1, e_2) = (v_1 e_1, v_2 e_2) \\ &\quad \text{or} \\ &\quad (e_1, e_2) = (e_1, e_2)(v_1, v_2) = (e_1 v_1, e_2 v_2)\} \end{aligned}$$

and (v_1, v_2) is a middle vertex if and only if v_1 and v_2 are middle vertices in G_1 and G_2 respectively all the other vertices are end vertices.

Example 2.3.2. For the disemigraph G_1 and G_2 given in fig. 10 the edge product $G_1 \times_E G_2$ is given in fig. 11.

Theorem 2.3.3. If G_1 and G_2 are two disemigraphs then the edge product disemigraph $G_1 \times_E G_2$ is isomorphic to $G_2 \times_E G_1$.

Proof. Let $G = G_1 \times_E G_2$ and $G' = G_2 \times_E G_1$. We have to prove that G_1 is isomorphic to G_2 for that define $f_V : V(G) \to V(G')$ as $f_V(v_1, v_2) = (v_2, v_1)$ then f_V is a bijection.

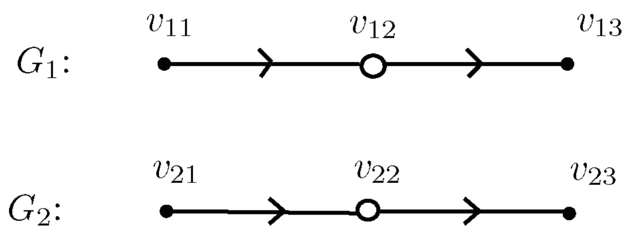

Figure 10:

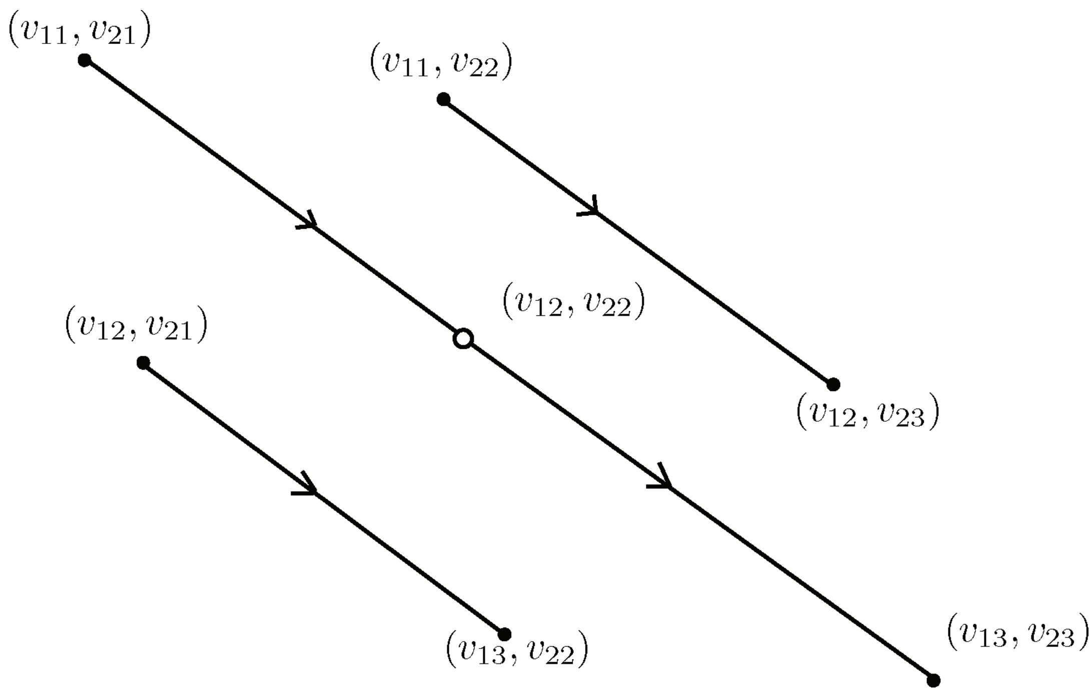

Figure 11:

Since

$$\begin{aligned} f_V(v_1, v_2) &= f_V(v_1', v_2') \\ &\Rightarrow (v_2, v_1) = (v_2', v_1') \\ &\Rightarrow v_2 = v_2' \text{ and } v_1 = v_1'. \end{aligned}$$

Hence f_V is one to one. Also for every $(v_2, v_1) \in V(G')$ there exists $(v_1, v_2) \in V(G)$. Hence f_V is onto. Hence f_V is a bijection. Now define $f_E : E(G) \to E(G')$ by

$$f_E(e_1, e_2) = (e_2, e_1). \text{ Then } f_E \text{ is a bijection. Since,}$$

$$f_E(e_1, e_2) = f_E(e_1', e_2')$$

$$\Rightarrow (e_2, e_1) = (e_2', e_1')$$

$$\Rightarrow \; e_2 = e_2', e_1 = e_1' \text{ hence } f_E \text{ is one to one.}$$

Also for every $(e_2, e_1) \in E(G')$ there exists $(e_1, e_2) \in E(G)$. Hence f_E is onto. Hence f_E is a bijection.

Now define the mapping $f : G \to G'$ by $f_E \cup f_V$

$$\begin{aligned} f(w_1, w_2) &= f_V(w_1, w_2) \text{ if } (w_1, w_2) \in V(G) \\ &= f_E(w_1, w_2) \text{ if } (w_1, w_2) \in E(G). \end{aligned}$$

In G' for all $(w_1, w_2) \in V(G) \cup E(G)$. This map is bijective map from $V(G) \cup E(G)$ to $V(G') \cup E(G')$ since f_E and f_V are bijections. Also for any $(e_1, e_2) \in E(G)$.

$$\begin{aligned} f(e_1, e_2) &= f_E((v_1, v_2)(e_1, e_2)(v_1', v_2')) \\ &= (e_2, e_1) \\ &= (v_2, v_1)(e_2, e_1)(v_2', v_1') \\ &= f_V(v_1, v_2) f_E(e_1, e_2) f_V(v_1', v_2') \\ &= f(v_1, v_2) f(e_1, e_2) f(v_1', v_2') \end{aligned}$$

Hence $G_1 \times_E G_2$ is isomorphic to $G_2 \times_E G_1$. ∎

3. Conclusion

In this paper we introduced product operations on semigraphs . These results and examples show a wider applicability of semigraphs due to the speciality of its structure.The different products give different resultant disemigraphs. Furthur attempts will be to find more applications of product of disemigraphs.

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