Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 9 (2017), pp. 5871-5890

http://www.ripublication.com

Some Fixed Point Results in Banach Spaces

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Abstract

In the present paper some fixed point and common fixed point theorems are established for non- contraction mappings, which satisfies the earlier result of Khan [14], Goebel and Zlatkienwicz [8], Singh and Chatterjee [22].

Keywords: Fixed point, Common Fixed Point, Banach spaces, Altering Distance function,

2. INTRODUCTION

The study of non-contraction mapping concerning the existence of fixed points draws attention of various authors in non- linear analysis. It is well known that the differential and integral equations that arise in physical problems are generally non-linear, therefore the fixed point methods specially Banach' contraction principle provides a powerful tool for obtaining the solutions of these equations which were very difficult to solve by any other methods. Recently Verma [24] described about the application of Banach's contraction principle [2]. Browder [4] was the first mathematician to study non-expansive mappings. Meanwhile Brouwder [4] and Ghode [6] have independently proved a fixed point theorem for non-expansive mapping. Many other mathematicians viz; Datson [5] Goebel [6], Goebel and Zlotkienwicz [8], Goebel, Kirk and Simi [9], Iseki [11], Singh and Chatterjee [22], Sharma and Rajput [21], Rajput and Naroliya [20] Pathak and Maity [18], Qureshi and Singh [19], Sharma and Bhagwan [23], Ahmad and Shakil [1], Shahzad and Udomene [24] have done the generalization of non-expansive mappings as well as non-contraction mappings. Kirk [15, 16 and 17] gave the comprehensive survey concerning fixed point theorems for non-expansive mappings.

Definition 2 A: The function $\psi\colon[0,\infty)\to[0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is continuous and non-decreasing.

(ii) $\psi(t)=0$ if and only if $t=0$.

Lemma 2 B: Let (X,d) be a metric space. Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty}\psi[d(x_n,x_{n+1})]=0$$

If $\{x_n\}$ is not Cauchy sequence in X, then there exist an $\epsilon_0>0$ and sequence of integer positive $\{m(k)\}$ and $\{n(k)\}$ with $m(k)>n(k)>k$ such that

$$d\big(x_{m(k)},x_{n(k)}\big)\geq\epsilon_0,\ d\big(x_{m(k)-1},x_{n(k)}\big)<\epsilon_0.$$

3. MAIN RESULTS

Theorem 3.1: Let f be a mapping of a Banach space X into itself. The function $\psi\colon[0,\infty)\to[0,\infty)$ is an altering distance function, if f satisfies the following conditions;

f^2= I, where I is identity mapping. (3.1.1)

$\psi\,\|f(\mathrm{x})-f(\mathrm{y})\|$ (3.1.2)

$$\leq\alpha\psi\left[\frac{\|x-y\|\|x-f(x)\|+\|x-f(x)\|\|x-f(y)\|+\|x-f(y)\|\|y-f(x)\|}{\|x-y\|+\|y-f(y)\|}\right]$$

$$+\lambda\,\psi\left[\frac{\|x-f(x)\|^3+\|y-f(y)\|^3}{1+\|x-f(x)\|^2+\|y-f(y)\|^2}\right]+\mu\,\psi\left[\frac{\|x-f(y)\|^2+\|y-f(x)\|^2}{1+\|x-f(y)\|+\|y-f(x)\|}\right]$$

$$+\gamma\,\psi[\|x-f(x)\|+\|y-f(y)\|]+\delta\,\psi\,[\|x-f(y)\|+\|y-f(x)\|]+\eta\,\psi\,\|x-y\|$$

For every *x, y* Є X, where α, , λ, μ, γ, δ, η > 0 and 5α+β+4 λ+2 μ +4γ+2δ+η < 2, then *f* has a fixed point.

If α + μ +2δ+η < 1 then *f* has an unique fixed point.

Proof: Suppose *x* is a point in the Banach space X.

Taking $y = \frac{1}{2}(f + I)(x)$, $z = f(y)$ and $u = 2y - z$ we have

$$\|z - x\| = \psi\|f(y) - f^2(x)\| = \psi\|f(y) - f(f(x))\|$$

$$\leq \alpha\psi\left[\frac{\|y-f(x)\|\|y-f(y)\| + \|y-f(y)\|\|y-f^2(x)\| + \|y-f^2(x)\|\|f(x)-f(y)\|}{\|y-f(x)\| + \|f(x)-f^2(x)\|}\right]$$

$$+ \lambda\psi\left[\frac{\|y-f(y)\|^3 + \|f(x)-f^2(x)\|^3}{1+\|y-f(y)\|^2 + \|f(x)-f^2(x)\|^2}\right] + \mu\psi\left[\frac{\|y-f^2(x)\|^2 + \|f(x)-f(y)\|^2}{1+\|y-f^2(x)\| + \|f(x)-f(y)\|}\right]$$

$$+ \gamma\psi[\|y - f(y)\| + \|f(x) - f^2(x)\|] + \delta\psi[\|y - f^2(x)\| + \|f(x) - f(y)\|] + \eta\psi\|y - f(x)\|$$

$$\psi\|z - x\| \leq \alpha\psi\left[\frac{\|y-f(x)\|\|y-f(y)\| + \|y-f(y)\|\|x-y\| + \|x-y\|\|f(x)-f(y)\|}{\|y-f(x)\| + \|x-f(x)\|}\right]$$

$$+ \lambda\psi\left[\frac{\|y-f(y)\|^3 + \|f(x)-x\|^3}{1+\|y-f(y)\|^2 + \|f(x)-x\|^2}\right] + \mu\psi\left[\frac{\|y-x\|^2 + \|f(x)-f(y)\|^2}{1+\|y-x\| + \|f(x)-f(y)\|}\right]$$

$$+ \gamma\psi[\|y - f(y)\| + \|x - f(x)\|] + \delta\psi[\|x - y\| + \|f(x) - f(y)\|] + \eta\psi\|y - f(x)\|$$

$$\psi\|z - x\| \leq \alpha\psi\left[\frac{\|y-f(y)\|(\|y-f(x)\| + \|x-y\|) + \|x-y\|\|f(x)-f(y)\|}{\|x-y\|}\right]$$

$$+ \lambda\psi\left[\frac{(\|y-f(y)\|^2 + \|f(x)-x\|^2)(\|y-f(y)\| + \|f(x)-x\|)}{1+\|y-f(y)\|^2 + \|f(x)-x\|^2}\right]$$

$$+ \mu\psi\left[\frac{(\|x-y\| + \|f(x)-f(y)\|)(\|x-y\| + \|f(x)-f(y)\|)}{1+\|y-x\| + \|f(x)-f(y)\|}\right]$$

$$+ \gamma\psi[\|y - f(y)\| + \|x - f(x)\|] + \delta\psi[\|x - y\| + \|f(x) - f(y)\|] + \eta\psi\|y - f(x)\|$$

$$\|z - x\| \leq \alpha\psi\left[\frac{\|y-f(y)\|\|x-f(x)\| + \|x-y\|\|f(x)-f(y)\|}{\|x-y\|}\right]$$

$$+ \lambda\psi\left[\frac{(\|y-f(y)\|^2 + \|f(x)-x\|^2)(\|y-f(y)\| + \|f(x)-x\|)}{\|y-f(y)\|^2 + \|f(x)-x\|^2}\right] + \mu\psi\left[\frac{(\|x-y\| + \|f(x)-f(y)\|)(\|x-y\| + \|f(x)-f(y)\|)}{\|y-x\| + \|f(x)-f(y)\|}\right]$$

$$+ \gamma\psi[\|y - f(y)\| + \|x - f(x)\|] + \delta\psi[\|x - y\| + \|f(x) - f(y)\|] + \eta\psi\|y - f(x)\|$$

$\psi\|z-x\| \leq \alpha\psi\left[\frac{\|y-f(y)\|\|x-f(x)\|}{\|x-y\|} + \|f(x)-f(y)\|\right]$

$+ \lambda\,\psi\,[\|y-f(y)\| + \|x-f(x)\|] + \mu\,\psi\,[\|x-y\| + \|f(x)-f(y)\|]$

$+ \gamma\,\psi[\|y-f(y)\| + \|x-f(x)\|] + \delta\,\psi\,[\|x-y\| + \|f(x)-f(y)\|] + \eta$
$\|y-f(x)\|$

$\psi\|z-x\| \leq \alpha\psi\left[\frac{\|y-f(y)\|\|x-f(x)\|}{\left\|x-\frac{1}{2}(f+I)(x)\right\|} + \left\|f(x)-f\left(\frac{1}{2}(f+I)(x)\right)\right\|\right]$

$+ \lambda\,\psi\,[\|y-f(y)\| + \|x-f(x)\|]$

$+ \mu\,\psi\,\left[\left\|x-\frac{1}{2}(f+I)(x)\right\| + \left\|f(x)-f\left(\frac{1}{2}(f+I)(x)\right)\right\|\right] + \gamma\,\psi$
$[\|y-f(y)\| + \|x-f(x)\|]$

$+\delta\,\psi\left[\left\|x-\frac{1}{2}(f+I)(x)\right\| + \left\|f(x)-f\left(\frac{1}{2}(f+I)(x)\right)\right\|\right] + \eta\,\psi$
$\left\|\frac{1}{2}(f+I)(x)-f(x)\right\|$

$\psi\|z-x\| \leq \alpha\psi\left[\frac{\|y-f(y)\|\|x-f(x)\|}{\frac{1}{2}\|x-f(x)\|} + \frac{1}{2}\|x-f(x)\|\right]$

$+ \lambda\,\psi\,[\|y-f(y)\| + \|x-f(x)\|] + \mu\,\psi\,\left[\frac{1}{2}\|x-f(x)\| +\right.$
$\left.\frac{1}{2}\|x-f(x)\|\right]$

$+ \gamma\,\psi[\|y-f(y)\| + \|x-f(x)\|] + \delta\,\psi\left[\frac{1}{2}\|x-f(x)\| + \frac{1}{2}\|x-f(x)\|\right]$
$+\eta\,\psi\,\frac{1}{2}\|x-f(x)\|$

$\psi\|z-x\| \leq \alpha\psi\left[2\|y-f(y)\| + \frac{1}{2}\|x-f(x)\|\right]$

$+ \lambda\,\psi\,[\|y-f(y)\| + \|x-f(x)\|] + \mu\,\psi\,\|x-f(x)\|$

$+ \gamma\,\psi\,[\|y-f(y)\| + \|x-f(x)\|] + \delta\,\psi\,[\|x-f(x)\|] + \eta\,\psi$
$\frac{1}{2}\|x-f(x)\|$

$\psi\|z-x\| \leq \left(\frac{\alpha}{2} + \lambda + \mu + \gamma + \delta + \frac{\eta}{2}\right)\psi\,\|x-f(x)\| + (2\alpha + \lambda + \gamma)\,\psi\,\|y-f(y)\|$ (3.1.3)

Also

$\psi\|u-x\| = \|2y-z-x\| = \psi\left\|\frac{1}{2}(f+I)(x)-z-x\right\| = \psi\|f(x)-z\| = \psi\|f(x)-$
$f(y)\|$

$$\psi\|u-x\| \leq \alpha\psi\left[\frac{\|x-y\|\|x-f(x)\|+\|x-f(x)\|\|x-f(y)\|+\|x-f(y)\|\|y-f(x)\|}{\|x-y\|+\|y-f(y)\|}\right]$$

$$+\lambda\,\psi\left[\frac{\|x-f(x)\|^3+\|y-f(y)\|^3}{1+\|x-f(x)\|^2+\|y-f(y)\|^2}\right]+\mu\,\psi\left[\frac{\|x-f(y)\|^2+\|y-f(x)\|^2}{1+\|x-f(y)\|+\|y-f(x)\|}\right]$$

$$+\ \gamma\,\psi\,[\|x-f(x)\|+\|y-f(y)\|]+\delta\,\psi\,[\|x-f(y)\|+\|y-f(x)\|]+\eta\,\psi\,\|x-y\|$$

$$\psi\|u-x\| \leq \alpha\psi\left[\frac{\|x-f(x)\|(\|x-y\|+\|x-f(y)\|)+\|x-f(y)\|\|y-f(x)\|}{\|x-f(y)\|}\right]$$

$$+\lambda\,\psi\left[\frac{(\|x-f(x)\|^2+\|y-f(y)\|^2)(\|x-f(x)\|+\|y-f(y)\|)}{1+\|x-f(x)\|^2+\|y-f(y)\|^2}\right]$$

$$+\mu\,\psi\left[\frac{(\|x-f(y)\|+\|y-f(x)\|)(\|x-f(y)\|+\|y-f(x)\|)}{1+\|x-f(y)\|+\|y-f(x)\|}\right]$$

$$+\ \gamma\,\psi[\|x-f(x)\|+\|y-f(y)\|]+\delta\,\psi\,[\|x-f(y)\|+\|y-f(x)\|]+\eta\,\psi\,\|x-y\|$$

$$\psi\,\|u-x\| \leq \alpha\psi\left[\frac{\|x-f(x)\|\|y-f(y)\|}{\|x-f(y)\|}+\|y-f(x)\|\right]$$

$$+\lambda\,\psi\left[\frac{(\|x-f(x)\|^2+\|y-f(y)\|^2)(\|x-f(x)\|+\|y-f(y)\|)}{\|x-f(x)\|^2+\|y-f(y)\|^2}\right]$$

$$+\mu\,\psi\left[\frac{(\|x-f(y)\|+\|y-f(x)\|)(\|x-f(y)\|+\|y-f(x)\|)}{\|x-f(y)\|+\|y-f(x)\|}\right]$$

$$+\ \gamma\,\psi\,[\|x-f(x)\|+\|y-f(y)\|]+\delta\,\psi\,[\|x-f(y)\|+\|y-f(x)\|]+\eta\,\psi\,\|x-y\|$$

$$\psi\ \|u-x\| \leq \alpha\psi\left[\frac{\|x-f(x)\|\|y-f(y)\|}{\|x-f(y)\|}+\|y-f(x)\|\right]$$

$$+\lambda\,\psi\,[\|x-f(x)\|+\|y-f(y)\|]+\mu\,\psi\ [\|x-f(y)\|+\|y-f(x)\|]$$

$$+\ \gamma\,\psi\,[\|x-f(x)\|+\|y-f(y)\|]+\delta\,\psi\,[\|x-f(y)\|+\|y-f(x)\|]+\eta\,\psi\,\|x-y\|$$

$$\psi\|u-x\| \leq \alpha\,\psi\left[\frac{\|x-f(x)\|\|y-f(y)\|}{\left\|x-f\left(\frac{1}{2}(\mathrm{f}+\mathrm{I})(\mathrm{x})\right)\right\|}+\left\|\frac{1}{2}(\mathrm{f}+\mathrm{I})(\mathrm{x})-f(x)\right\|\right]$$

$$+\lambda\,\psi\,[\|x-f(x)\|+\|y-f(y)\|]+\mu\,\psi\left[\left\|x-f\left(\frac{1}{2}(\mathrm{f}+\mathrm{I})(\mathrm{x})\right)\right\|+\left\|\frac{1}{2}(\mathrm{f}+\mathrm{I})(\mathrm{x})-f(x)\right\|\right]$$

$$+\gamma\,\psi\,[\|x-f(x)\|+\|y-f(y)\|]+\delta\,\psi\left[\left\|x-f\left(\frac{1}{2}(\mathrm{f}+\mathrm{I})(\mathrm{x})\right)\right\|+\right.$$

$\left\|\left\|\frac{1}{2}\text{ (f + I) (x)} - f(x)\right\|\right\|$ +η ψ $\left\|x - \frac{1}{2}\text{ (f + I) (x)}\right\|$ $\|u - x\| \leq$

$\alpha\psi\left[\frac{\|x-f(x)\|\|y-f(y)\|}{\frac{1}{2}\|x-f(x)\|} + \frac{1}{2}\|x - f(x)\|\right]$

$+\lambda\,\psi\,[\|x - f(x)\| + \|y - f(y)\|] + \mu\,\psi\left[\frac{1}{2}\|x - f(x)\| + \frac{1}{2}\|x - f(x)\|\right]$

$+\gamma\,\psi\,[\|x - f(x)\| + \|y - f(y)\|] + \delta\,\psi\left[\frac{1}{2}\|x - f(x)\| + \frac{1}{2}\|x - f(x)\|\right]$

$+\eta\,\psi\,\frac{1}{2}\|x - f(x)\|$

$\|u - x\| \leq \alpha\,\psi\,[2\|y - f(y)\| + \frac{1}{2}\|x - f(x)\|] + \lambda\,\psi\,[\|x - f(x)\| + \|y - f(y)\|] + \mu\,\psi\,\|x - f(x)\|$

$+\gamma\,\psi\,[\|x - f(x)\| + \|y - f(y)\|] + \delta\psi\,\|x - f(x)\| + \eta\psi\,\frac{1}{2}\|x - f(x)\|$

$\psi\|u - x\| \leq (\frac{\alpha}{2} + \lambda + \mu + \gamma + \delta + \frac{\eta}{2})\,\psi\|x - f(x)\| + (2\alpha + \lambda + \gamma)\,\psi\|y - f(y)\|$

(3.1.4)

Now,

$\psi\|z - u\| = \psi\|(z - x) - (x - u)\| \leq \psi\,(\|z - x\| + \psi\|x - u\|)$

$\leq [(\frac{\alpha}{2} + \lambda + \mu + \gamma + \delta + \frac{\eta}{2})\,\psi\|x - f(x)\| + (2\alpha + \lambda + \gamma)\,\psi\|y - f(y)\|]$

$+ [(\frac{\alpha}{2} + \lambda + \mu + \gamma + \delta + \frac{\eta}{2})\,\psi\|x - f(x)\| + (2\alpha + \lambda + \gamma)\,\psi\|y - f(y)\|]$

$\psi\|z - u\| \leq (\alpha + 2\lambda + 2\mu + 2\gamma + 2\delta + \eta)\,\psi\|x - f(x)\| + (4\alpha + 2\lambda + 2\gamma)\psi\,\|y - f(y)\|$

Also

$\psi\|z - u\| = \psi\,\|f(y) - (2y - z)\|$

$= \psi\|f(y) - 2y - f(y))\|$

$= 2\psi\|y - f(y)\|$ (3.1.5)

From (3.1.5)

$2\psi\|y - f(y)\| \leq (\alpha + 2\lambda + 2\mu + 2\gamma + 2\delta + \eta)\,\psi\|x - f(x)\| + (4\alpha + 2\lambda + 2\gamma)\,\psi\,\|y - f(y)\|$

$[2 - (4\alpha + 2\gamma + 2\lambda)]\,\psi\,\|y - f(y)\| \leq (\alpha + 2\lambda + 2\mu + 2\gamma + 2\delta + \eta)\,\psi\|x - f(x)\|$

$\|y - f(y)\| \leq q\,\|x - f(x)\|$

Where $q = \frac{(\alpha+2\lambda+2\mu+2\gamma+2\delta+\eta)}{[2-(4\alpha+2\lambda+2\gamma)]} < 1$

Since $5\alpha + 4\lambda + 2\mu + 4\gamma + 2\delta + \eta < 2$

Let $g = \frac{1}{2}(f + I)$ then for every $x \in X$

$$\psi\|g^2(x) - g(x)\| = \psi\|g(y) - y\|$$

$$= \psi\left\|\frac{1}{2}(f + I)y - y\right\|$$

$$= \frac{1}{2}\psi\|y - f(y)\|$$

$$\leq \frac{q}{2}\psi\|x - f(x)\|$$

By the definition of q, we claim that $\{g^n(x)\}$ is a Cauchy sequence in X.

By the completeness, $\{g^n(x)\}$ converges to some element x_0 in X

i.e. $\lim_{n\to\infty} g^n(x) = x_0$

which implies that $g(x_0) = x_0$

hence $f(x_0) = x_0$

i.e. x_0 is fixed point of f

For the uniqueness:

If possible let y_0 ($\neq x_0$) be another fixed point of f then

$$\psi\|x_0 - y_0\| = \psi\|f(x_0) - f(y_0)\|$$

$$\leq \alpha\,\psi\left[\frac{\|x_0-y_0\|\|x_0-f(x_0)\|+\|x_0-f(x_0)\|\|x_0-f(y_0)\|+\|x_0-f(y_0)\|\|y_0-f(x_0)\|}{\|x_0-y_0\|+\|y_0-f(y_0)\|}\right]$$

$$+ \lambda\,\psi\left[\frac{\|y_0-f(y_0)\|^3+\|f(x_0)-x_0\|^3}{1+\|y_0-f(y_0)\|^2+\|f(x_0)-x_0\|^2}\right] + \mu\,\psi\left[\frac{\|y_0-x_0\|^2+\|f(x_0)-f(y_0)\|^2}{1+\|y_0-x_0\|+\|f(x_0)-f(y_0)\|}\right]$$

$$+ \gamma\,\psi\left[\|x_0 - f(x_0)\| + \|y_0 - f(y_0)\|\right] + \delta\,\psi\left[\|x_0 - f(y_0)\| + \|y_0 - f(x_0)\|\right] + \eta\,\psi\,\|x_0 - y_0\|$$

$$\psi\|x_0 - y_0\| \leq \alpha\psi\frac{\|x_0-y_0\|^2}{\|x_0-y_0\|} + \mu\,\psi\left[\frac{2\|y_0-x_0\|^2}{1+2\|y_0-x_0\|}\right] + 2\delta\,\psi\|x_0 - y_0\| + \eta\,\psi\|x_0 - y_0\|$$

$$\leq \alpha\,\psi\|x_0 - y_0\| + \mu\,\psi\left[\frac{2\|y_0-x_0\|^2}{2\|y_0-x_0\|}\right] + 2\delta\,\psi\|x_0 - y_0\| + \eta\,\psi\|x_0 - y_0\|$$

$$\leq \alpha\,\psi\|x_0 - y_0\| + \mu\,\psi\|x_0 - y_0\| + 2\delta\,\psi\|x_0 - y_0\| + \eta\,\psi\|x_0 - y_0\|$$

$$\psi\|x_0 - y_0\| = (\alpha + \mu + 2\delta + \eta)\,\psi\|x_0 - y_0\|$$

Since $\alpha + \mu + 2\delta + \eta < 1$

$\psi\|x_0 - y_0\|$=0

$x_0 = y_0$

This complete the proof

Theorem 3.2: Let K be closed and convex subset of a Banach space X . The function $\psi: [0,\infty) \rightarrow [0,\infty)$ is an altering distance function, let f : K→K, g : K→K satisfy the following conditions:

f and g commute (3.2.1)

f^2=I and g^2= I, where I denotes identity mapping (3.2.2)

$\psi\|f(x) - f(y)\|$ (3.2.3)

$$\leq \alpha\psi\left[\frac{\|g(x)-g(y)\|\|g(x)-f(x)\|+\|g(x)-f(x)\|\|g(x)-f(y)\|+\|g(x)-f(y)\|\|g(y)-f(x)\|}{\|g(x)-g(y)\|+\|g(y)-f(y)\|}\right]$$

$$+\lambda\,\psi\left[\frac{\|g(x)-f(x)\|^3+\|g(y)-f(y)\|^3}{1+\|g(x)-f(x)\|^2+\|g(y)-f(y)\|^2}\right]+\mu\,\psi\left[\frac{\|g(x)-f(y)\|^2+\|g(y)-f(x)\|^2}{1+\|g(x)-f(y)\|+\|g(y)-f(x)\|}\right]$$

$$+\gamma\,\psi\,[\|g(x)-f(x)\|+\|g(y)-f(y)\|]+\delta\,\psi\,[\|g(x)-f(y)\|+\|g(y)-f(x)\|]$$

$$+\eta\,\psi\,\|g(x)-g(y)\|$$

For every x, y Є X, $0 \leq \alpha, \lambda, \mu\ \gamma, \delta, \eta$ and $5\alpha + 4\lambda + 2\mu + 4\gamma + 2\delta + \eta < 2$, then there exits at least one fixed point, x_0 Є X such that f(x_0) = g(x_0) = x_0. Further if $\alpha + \mu + 2\delta + \eta < 1$ then x_0 is the unique fixed point of f and g.

Proof: From (3.2.1) and (3.2.2) it follows that $(fg)^2$ = I and (3.2.2) and (3.2.3) imply.

$\psi\|fgg(x) - fgg(y)\| = \psi\|fg^2(x) - fg^2(y)\|$

$$\leq \alpha\psi\left[\frac{\|gg^2(x)-gg^2(y)\|\|gg^2(x)-fg^2(x)\|+\|gg^2(x)-fg^2(x)\|\|gg^2(x)-fg^2(y)\|+\|gg^2(x)-fg^2(y)\|\|gg^2(y)-fg^2(x)\|}{\|gg^2(x)-gg^2(y)\|+\|gg^2(y)-fg^2(y)\|}\right]$$

$$+\lambda\,\psi\left[\frac{\|gg^2(x)-fg^2(x)\|^3+\|gg^2(y)-fg^2(y)\|^3}{1+\|gg^2(x)-fg^2(x)\|^2+\|gg^2(y)-fg^2(y)\|^2}\right]$$

$$+\mu\,\psi\left[\frac{\|gg^2(x)-fg^2(y)\|^2+\|gg^2(y)-fg^2(x)\|^2}{1+\|gg^2(x)-fg^2(y)\|+\|gg^2(y)-fg^2(x)\|}\right]$$

$$+\gamma\,\psi\,[\|gg^2(x)-fg^2(x)\|+\|gg^2(y)-fg^2(y)\|]+\delta\,\psi\,[\|gg^2(x)-fg^2(y)\|+\|gg^2(y)-fg^2(x)\|]$$

$$+\eta\,\psi\,\|gg^2(x)-gg^2(y)\|$$

$$\leq \alpha\psi\left[\frac{\|g(x)-g(y)\|\|g(x)-fgg(x)\|+\|g(x)-fgg(x)\|\|g(x)-fgg(y)\|+\|g(x)-fgg(y)\|\|g(y)-fgg(x)\|}{\|g(x)-g(y)\|+\|g(y)-fgg(y)\|}\right]$$

$$+\lambda\,\psi\left[\frac{\|g(x)-fgg(x)\|^3+\|g(y)-fgg(y)\|^3}{1+\|g(x)-fgg(x)\|^2+\|g(y)-fgg(y)\|^2}\right]+\mu\,\psi\left[\frac{\|g(x)-fgg(y)\|^2+\|g(y)-fgg(x)\|^2}{1+\|g(x)-fgg(y)\|+\|g(y)-fgg(x)\|}\right]$$

$$+\gamma\,\psi\,[\|g(x)-fgg(x)\|+\|g(y)-fgg(y)\|]+\delta\,\psi\,[\|g(x)-fgg(y)\|+\|g(y)-fgg(x)\|]$$

$$+\eta\,\psi\|g(x)-g(y)\|$$

Now that *g(x) = z* and g(y) = w, then we get

$$\psi\|fg(z)-fg(w)\|\leq\alpha\psi\left[\frac{\|z-w)\|\|z-fg(z)\|+\|z-fg(z)\|\|z-fg(w)\|+\|z-fg(w)\|\|w-fg(z)\|}{\|z-w\|+\|w-fg(w)\|}\right]$$

$$+\lambda\,\psi\left[\frac{\|z-fg(z)\|^3+\|w-fg(w)\|^3}{1+\|z-fg(z)\|^2+\|w-fg(w)\|^2}\right]+\mu\,\psi\left[\frac{\|z-fg(w)\|^2+\|w-fg(z)\|^2}{1+\|z-fg(w)\|+\|w-fg(z)\|}\right]$$

$$+\gamma\,\psi\,[\|z-fg(z)\|+\|w-fg(w)\|]+\delta\,\psi\,[\|z-fg(w)\|+\|w-fg(z)\|]+\eta\,\psi\,\|z-w\|$$

We have $(fg)^2 = I$ and so by theorem 1, fg has at least one fixed point say x_0 in K i.e.

$$fg(x_0)=x_0$$

$$ffg(x_0)=f(x_0)$$

$$g(x_0)=f(x_0)$$

Now

$$\psi\|f(x_0)-x_0\|=\psi\,\|f(x_0)-f^2x_0\|=\psi\,\|f(x_0)-ff(x_0)\|$$

$$\leq\alpha\,\psi\left[\frac{\|g(x_0)-gf(x_0)\|\|g(x_0)-f(x_0)\|+\|g(x_0)-f(x_0)\|\|g(x_0)-ff(x_0)\|+\|g(x_0)-ff(x_0)\|\|gf(x_0)-f(x_0)\|}{\|g(x_0)-gf(x_0)\|+\|gf(x_0)-ff(x_0)\|}\right]$$

$$+\lambda\,\psi\left[\frac{\|g(x_0)-f(x_0)\|^3+\|gf(x_0)-ff(x_0)\|^3}{1+\|g(x_0)-f(x_0)\|^2+\|gf(x_0)-ff(x_0)\|^2}\right]+\mu\,\psi\left[\frac{\|g(x_0)-ff(x_0)\|^2+\|gf(x_0)-f(x_0)\|^2}{1+\|g(x_0)-ff(x_0)\|+\|gf(x_0)-f(x_0)\|}\right]$$

$$+\gamma\,\psi\,[\|g(x_0)-f(x_0)\|+\|gf(x_0)-ff(x_0)\|]+\delta\,\psi\,[\|g(x_0)-ff(x_0)\|+\|gf(x_0)-f(x_0)\|]$$

$+\eta\,\psi\,\|g(x_0)-gf(x_0)\|$

$$\le \alpha\psi\left[\frac{\|f(x_0)-x_0)\|\|f(x_0)-f(x_0)\|+\|f(x_0)-f(x_0)\|\|f(x_0)-x_0\|+\|f(x_0)-x_0)\|\|x_0-f(x_0)\|}{\|f(x_0)-x_0\|+\|x_0-x_0\|}\right]$$

$$+\lambda\,\psi\left[\frac{\|f(x_0)-f(x_0)\|^3+\|x_0-x_0\|^3}{1+\|f(x_0)-f(x_0)\|^2+\|x_0-x_0\|^2}\right]+\mu\,\psi\left[\frac{\|f(x_0)-x_0\|^2+\|x_0-f(x_0)\|^2}{1+\|f(x_0)-x_0\|+\|x_0-f(x_0)\|}\right]$$

$+\gamma\,\psi\,[\|f(x_0)-f(x_0)\|+\|x_0-x_0\|]+\delta\,\psi\,[\|f(x_0)-x_0\|+\|x_0-f(x_0)\|]$

$+\eta\,\psi\,\|f(x_0)-x_0)\|$

$\le \alpha\,\psi\,\|f(x_0)-x_0)\| +\mu\,\psi\,\|f(x_0)-x_0)\|+2\delta\,\psi\,\|f(x_0)-x_0)\|+\eta\,\psi\,\|f(x_0)-x_0)\|$

$\psi\,\|f(x_0)-x_0)\|\le(\alpha+\mu+2\delta+\eta)\ \psi\,\|f(x_0)-x_0)\|$

This is contradiction

Since $(\alpha+\mu+2\delta+\eta)<1$

$\psi\, f(x_0)=x_0$

i.e. x_0 is fixed point of f ,but $f(x_0)=g(x_0)$ therefore we have $g(x_0)=x_0$

i.e. x_0 is the common fixed point of f and g.

Now, we shall prove that x_0 is the unique common fixed point of f and g . If possible let y_0 be another fixed point of f and g.

Now by (3.2.1),(3.2.2),(3.2.3),(3.2.4) and (3.2.5) we have

$\psi\,\|x_0-y_0\|=\psi\,\|f^2(x_0)-f^2(y_0)\|=\psi\,\|ff(x_0)-ff(y_0)\|$

$$\le\alpha\,\psi\left[\frac{\|gf(x_0)-gf(y_0)\|\|gf(x_0)-ff(x_0)\|+\|gf(x_0)-ff(x_0)\|\|gf(x_0)-ff(y_0)\|+\|gf(x_0)-ff(y_0)\|\|gf(y_0)-ff(x_0)\|}{\|gf(x_0)-gf(y_0)\|+\|gf(y_0)-ff(y_0)\|}\right]$$

$$+\lambda\,\psi\left[\frac{\|gf(x_0)-ff(x_0)\|^3+\|gf(y_0)-ff(y_0)\|^3}{1+\|gf(x_0)-ff(x_0)\|^2+\|gf(y_0)-ff(y_0)\|^2}\right]+\mu\,\psi\left[\frac{\|gf(x_0)-ff(y_0)\|^2+\|gf(y_0)-ff(x_0)\|^2}{1+\|gf(x_0)-ff(y_0)\|+\|gf(y_0)-ff(x_0)\|}\right]$$

$+\gamma\,\psi\,[\|gf(x_0)-ff(x_0)\|+\|gf(y_0)-ff(y_0)\|]+\delta\,\psi\,[\|gf(x_0)-ff(y_0)\|+\|gf(y_0)-ff(x_0)\|]$

$+\eta\ \psi\,\|gf(x_0)-gf(y_0)\|$

$$\le\alpha\,\psi\left[\frac{\|x_0-y_0\|\|x_0-x_0)\|+\|x_0-x_0\|\|x_0-y_0\|+\|x_0-y_0\|\|y_0-x_0\|}{\|x_0-y_0\|+\|y_0-y_0\|}\right]$$

$$+\lambda\,\psi\left[\frac{\|x_0-x_0\|^3+\|y_0-y_0\|^3}{1+\|x_0-x_0\|^2+\|y_0-y_0\|^2}\right]+\mu\,\psi\left[\frac{\|x_0-y_0\|^2+\|y_0-x_0\|^2}{1+\|x_0-y_0\|+\|y_0-x_0\|}\right]$$

$+ \gamma\, \psi\, [\|x_0 - x_0\| + \psi\, \|y_0 - y_0\|] + \delta\, \psi\, [\|x_0 - y_0\| + \psi\, \|y_0 - x_0\|]$
$+\eta\, \psi\, \|x_0 - y_0\|$

$$\leq \alpha\ \psi\, \|x_0 - y_0\| + \mu\, \psi \left[\frac{2\|x_0-y_0\|^2}{1+2\|x_0-y_0\|}\right] + 2\delta\, \psi\, \|x_0 - y_0\| + \eta\ \psi\, \|x_0 - y_0\|$$

$$\leq \alpha\ \psi\, \|x_0 - y_0\| + \mu\ \psi\, \|x_0 - y_0\| + 2\delta\, \psi\, \|x_0 - y_0\| + \eta\, \psi\, \|x_0 - y_0\|$$

$\psi\, \|x_0 - y_0)\| \leq (\alpha + \beta + \mu + 2\delta + \eta)\, \psi\ \|x_0 - y_0)\|$

Since $\alpha + \mu + 2\delta + \eta < 1$, it follows

$x_0 = y_0$

Proving the uniqueness of x_0, the proof of theorem 2 is complete.

Theorem 3.3:

Let K be a closed and convex subset of a Banach space X. Let f, g and h be three mappings of X into itself such that

$f g = g f,\ \ g h = h g\ \ and\ \ f h = h$ f (3.3.1)

$f^2 = I$, $g^2 = I$ $h^2 = I$ where I denotes the identity mapping (3.3.2)

$\psi\, \|f(x) - f(y)\|$ (3.3.3)

$$\leq \alpha\, \psi \left[\frac{\|gh(x)-gh(y)\|\|gh(x)-f(x)\|+\|gh(x)-f(x)\|\|gh(x)-f(y)\|+ \|gh(x)-f(y)\|\|gh(y)-f(x)\|}{\|gh(x)-gh(y)\|+\|gh(y)-f(y)\|}\right]$$

$$+ \lambda\, \psi \left[\frac{\|gh(x)-f(x)\|^3+ \|gh(y)-f(y)\|^3}{1+\|gh(x)-f(x)\|^2+ \|gh(y)-f(y)\|^2}\right] + \mu\, \psi \left[\frac{\|gh(x)-f(y)\|^2+ \|gh(y)-f(x)\|^2}{1+\|gh(x)-f(y)\|+\|gh(y)-f(x)\|}\right]$$

$+ \gamma\ \psi\, [\|gh(x) - f(x)\| + \|gh(y) - f(y)\|] + \delta\, \psi\, [\|gh(x) - f(y)\| +$
$\|gh(y) - f(x)\|]$

$+\eta\ \psi\, \|gh(x) - gh(y)\|$

For every *x, y Є K* and $0 \leq \alpha, , \lambda, \mu\ \ \gamma, \delta, \eta$ such that $5\alpha + 4\lambda + 2\mu + 4\gamma + 2\delta + \eta < 2$, then there exist at least one fixed point $x_0 Є$ X such that

$f(x_0) = g\, h(x_0)$ and $f g(x_0) = h(x_0)$

further if $\alpha + \mu + 2\delta + \eta < 1$ then f has an unique fixed point.

Proof: From (3.3.1) and (3.3.2) it follows that $(f g\, h)^2 = I$, where I is identify mapping, from (3.3.2) and (3.3.3) we have

$\psi\, \|fgh.g(x) - fgh.g(y)\| = \psi\, \|f.ghg(x) - f.ghg(y)\|$

$\leq \alpha\, \psi \left[\frac{\|ghghg(x)-ghghg(y)\|\|ghghg(x)-fghg(x)\|+\|ghghg(x)-fghg(x)\|\|ghghg(x)-fghg(y)\|+\|ghghg(x)-fghg(y)\|\|ghghg(y)-fghg(x)\|}{\|ghghg(x)-ghghg(y)\|+\|ghghg(y)-fghg(y)\|}\right]$

$+\lambda\ \psi\left[\frac{\|ghghg(x)-fghg(x)\|^3+\|ghghg(y)-fghg(y)\|^3}{1+\|ghghg(x)-fghg(x)\|^2+\|ghghg(y)-fghg(y)\|^2}\right]$

$+\mu\ \psi\left[\frac{\|ghghg(x)-fghg(y)\|^2+\|ghghg(y)-fghg(x)\|^2}{1+\|ghghg(x)-fghg(y)\|+\|ghghg(y)-fghg(x)\|}\right]$

$+\gamma\,\psi\,[\|ghghg(x)-fghg(x)\|+\|ghghg(y)-fghg(y)\|]$

$+\delta\,\psi\,[\|ghghg(x)-fghg(y)\|+\|ghghg(y)-fghg(x)\|]$

$+\eta\,\psi\,\|ghghg(x)-ghghg(y)\|$

$\leq \alpha$

$\psi\left[\frac{\|g(x)-g(y)\|\|g(x)-fghg(x)\|+\|g(x)-fghg(x)\|\|g(x)-fghg(y)\|+\|g(x)-fghg(y)\|\|g(y)-fghg(x)\|}{\|g(x)-g(y)\|+\|g(y)-fghg(y)\|}\right]$

$+\lambda\,\psi\left[\frac{\|g(x)-fghg(x)\|^3+\|g(y)-fghg(y)\|^3}{1+\|g(x)-fghg(x)\|^2+\|g(y)-fghg(y)\|^2}\right]+\mu\,\psi$
$\left[\frac{\|g(x)-fghg(y)\|^2+\|g(y)-fghg(x)\|^2}{1+\|g(x)-fghg(y)\|+\|g(y)-fghg(x)\|}\right]$

$+\gamma\,\psi\,[\|g(x)-fghg(x)\|+\|g(y)-fghg(y)\|]+\delta\,\psi\,[\|g(x)-$
$fghg(y)\|+\|g(y)-fghg(x)\|]$

$+\eta\,\psi\,\|g(x)-g(y)\|$

Now, if we put g(x) = z and g(y) = w, we get

$\psi\,\|fgh(z)-fgh(w)\| =$
$\leq\alpha\,\psi\left[\frac{\|z-w\|\|z-fgh(z)\|+\|z-fgh(z)\|\|z-fgh(w)\|+\|z-fgh(w)\|\|w-fgh(z)\|}{\|z-w\|+\|w-fgh(w)\|}\right]$

$+\lambda\,\psi\left[\frac{\|z-fgh(z)\|^3+\|w-fgh(w)\|^3}{1+\|z-fgh(z)\|^2+\|w-fgh(w)\|^2}\right]+\mu\,\psi\left[\frac{\|z-fgh(w)\|^2+\|w-fgh(z)\|^2}{1+\|z-fgh(w)\|+\|w-fgh(z)\|}\right]$

$+\gamma\ \psi\,[\|z-fgh(z)\|+\|w-fgh(w)\|]+\delta\,\psi\,[\|z-fgh(w)\|+$
$\|w-fgh(z)\|]$

$+\eta\,\psi\,\|z-w\|$

Put $fg\,h = N$

$\psi\,\|N(z)-N(w)\| = \leq\alpha\,\psi\left[\frac{\|z-w\|\|z-N(z)\|+\|z-N(z)\|\|z-N(w)\|+\|z-N(w)\|\|V-N(z)\|}{\|z-w\|+\|w-N(w)\|}\right]$

$+\lambda\,\psi\left[\frac{\|z-N(z)\|^3+\|w-N(w)\|^3}{1+\|z-N(z)\|^2+\|w-N(w)\|^2}\right]+\mu\,\psi\left[\frac{\|z-N(w)\|^2+\|w-N(z)\|^2}{1+\|z-N(w)\|+\|w-N(z)\|}\right]$

$+\gamma\ \psi\,[\|z-N(z)\|+\|w-N(w)\|]+\delta\,\psi\,[\|z-N(w)\|+\|w-N(z)\|]$

$+\eta\,\psi\,\|z-w\|$

Put $w = \frac{1}{2}(N+I)(z)$, $s = N(w)$ and $t = 2w - s$ we have

Now from (A) we have

$$\psi\,\|s-z\| = \psi\,\|N(w)-z\| = \psi\,\|N(w)-N^2 z\| = \psi\,\|N(w)-NN(z)\|$$

$$\leq \alpha\,\psi\left[\frac{\|w-N(z)\|\|w-N(w)\|+\|w-N(w)\|\|w-NN(z)\|+\|w-NN(z)\|\|N(z)-N(w)\|}{\|w-N(z)\|+\|N(z)-NN(z)\|}\right]$$

$$+\lambda\,\psi\left[\frac{\|w-N(w)\|^3+\|N(z)-NN(z)\|^3}{1+\|w-N(w)\|^2+\|N(z)-NN(z)\|^2}\right]+\mu\,\psi\left[\frac{\|w-NN(z)\|^2+\|N(z)-N(w)\|^2}{1+\|w-NN(z)\|+\|N(z)-N(w)\|}\right]$$

$$+\gamma\,\psi\,[\|w-N(w)\|+\|N(z)-NN(z)\|]+\delta\,\psi\,[\|w-NN(z)\|+\|N(z)-N(w)\|]$$

$$+\eta\,\psi\,\|w-N(z)\|$$

$$\leq \alpha\,\psi\left[\frac{\|w-N(z)\|\|w-N(w)\|+\|w-N(w)\|\|w-z\|+\|w-z\|\|N(z)-N(w)\|}{\|w-z\|}\right]+\lambda\,\psi\left[\frac{\|w-N(w)\|^3+\|N(z)-z\|^3}{1+\|w-N(w)\|^2+\|N(z)-z\|^2}\right]+\mu\,\psi\left[\frac{\|w-z\|^2+\|N(z)-N(w)\|^2}{1+\|w-z\|+\|N(z)-N(w)\|}\right]+\gamma\,\psi\,[\|w-N(w)\|+\|N(z)-z\|]+\delta\,\psi\,[\|w-z\|+\|N(z)-N(w)\|]$$

$$+\eta\,\psi\,\|w-N(z)\|$$

$$\psi\,\|s-z\| \leq \alpha\,\psi\left[\frac{\|w-N(w)\|(\|w-N(z)\|+\|z-w\|)+\|z-w\|\|N(z)-N(w)\|}{\|z-w\|}\right]$$

$$+\lambda\,\psi\left[\frac{(\|w-N(w)\|^2+\|N(z)-z\|^2)(\|w-N(w)\|+\|N(z)-z\|)}{1+\|w-N(w)\|^2+\|N(z)-z\|^2}\right]$$

$$+\mu\,\psi\left[\frac{(\|z-w\|+\|N(z)-N(w)\|)(\|z-w\|+\|N(z)-N(w)\|)}{1+\|w-z\|+\|N(z)-N(w)\|}\right]$$

$$+\gamma\,\psi\,[\|w-N(w)\|+\|z-N(z)\|]+\delta\,\psi\,[\|z-w\|+\|N(z)-N(w)\|]+\eta\,\psi\,\|w-N(z)\|$$

$$\psi\,\|s-z\| \leq \alpha\psi\left[\frac{\|w-N(w)\|\|z-N(z)\|+\|z-w\|\|N(z)-N(w)\|}{\|z-w\|}\right]$$

$$+\lambda\,\psi\left[\frac{(\|w-N(w)\|^2+\|N(z)-z\|^2)(\|w-N(w)\|+\|N(z)-z\|)}{\|w-N(w)\|^2+\|N(z)-z\|^2}\right]$$

$$+\mu\,\psi\left[\frac{(\|z-w\|+\|N(z)-N(w)\|)(\|z-w\|+\|N(z)-N(w)\|)}{\|w-z\|+\|N(z)-N(w)\|}\right]$$

$$+\gamma\,\psi\,[\|w-N(w)\|+\|z-N(z)\|]+\delta\,\psi\,[\|z-w\|+\|N(z)-N(w)\|]+\eta\,\psi\,\|w-N(z)\|$$

$$\psi\|s-z\| \leq \alpha\psi\left[\frac{\|w-N(w)\|\|z-N(z)\|}{\|z-w\|}+\|N(z)-N(w)\|\right]+\lambda\,\psi\,[\|w-N(w)\|+\|z-N(z)\|]+\mu\,\psi\,[\|z-w\|+\|N(z)-N(w)\|]$$

$+ \gamma\, \psi[\|w - N(w)\| + \|z - N(z)\|] + \delta\, \psi\, [\|z - w\| + \|N(z) - N(w)\|] + \eta\, \psi$
$\|w - N(z)\|$

$$\psi\|s - z\| \leq \alpha\psi \left[\frac{\|w-N(w)\|\|z-N(z)\|}{\left\|z-\frac{1}{2}(N + I)(z)\right\|} + \left\|N(z) - N(\tfrac{1}{2}(N + I)(z)\right\|\right]$$

$+ \lambda\, \psi\, [\|w - N(w)\| + \|z - N(z)\|]$

$+ \mu\, \psi \left[\left\|z - \frac{1}{2}(N + I)(z)\right\| + \left\|N(z) - N(\frac{1}{2}(N + I)(z)\right\|\right] + \gamma\, \psi$
$[\|w - N(w)\| + \|z - N(z)\|]$

$+\delta\, \psi \left[\left\|z - \frac{1}{2}(N + I)(z)\right\| + \left\|N(z) - N(\frac{1}{2}(N + I)(z)\right\|\right] + \eta\, \psi$
$\left\|\frac{1}{2}(N + I)(z) - N(z)\right\|$

$$\psi\|s - z\| \leq \alpha\psi \left[\frac{\|w-N(w)\|\|z-N(z)\|}{\frac{1}{2}\|z-N(z)\|} + \frac{1}{2}\|z - N(z)\|\right]$$

$+ \lambda\, \psi\, [\|w - N(w)\| + \|z - N(z)\|] + \mu\, \psi \left[\frac{1}{2}\|z - N(z)\| + \frac{1}{2}\|z - N(z)\|\right]$

$+ \gamma\, \psi\, [\|w - N(w)\| + \|z - N(z)\|] + \delta\, \psi \left[\frac{1}{2}\|z - N(z)\| + \frac{1}{2}\|z - N(z)\|\right] + \eta\, \psi$
$\frac{1}{2}\|z - N(z)\|$

$\psi\|s - z\| \leq \alpha\psi \left[2\|w - N(w)\| + \frac{1}{2}\|z - N(z)\|\right] + \lambda\, \psi\, [\|w - N(w)\| +$
$\|z - N(z)\|] + \mu\, \psi\|z - N(z)\|$

$+ \gamma\, \psi\, [\|w - N(w)\| + \|z - N(z)\|] + \delta\, \psi\, [\|z - N(z)\|] + \eta\, \psi \frac{1}{2}\|z - N(z)\|$

$$\psi\|s - z\| \leq (\frac{\alpha}{2} + \lambda + \mu + \gamma + \delta + \frac{\eta}{2})\psi\, \|z - N(z)\| + (2\alpha + \lambda + \gamma)\, \psi\|w - N(w)\| \tag{3.1.3}$$

Similarly it can be shown that

$\psi\|t - z\| \leq \psi\, (\frac{\alpha}{2} + \lambda + \mu + \gamma + \delta + \frac{\eta}{2})\, \|z - N(z)\| + (2\alpha + \lambda + \gamma)\, \psi\, \|w - N(w)\|$

$\psi\, \|s - t\| \leq \psi\|s - z\| + \psi\|z - t\|$

$\leq (\alpha + 2\lambda + 2\mu + 2\gamma + 2\delta + \eta)\, \|N(z) - z\| + (4\alpha + 2\lambda + 2\gamma)\, \|w - N(w)\|$ (D)

Also

$\psi\|s-t\| = \psi\,\|N(w)-(2w-s)\|$

$= \psi\|N(w)-2w+N(w))\|$

$= 2\psi\|N(w)-w\|$

Putting the above value in equality (D), we have

$2\psi\|N(w)-w\| \le (\alpha+2\lambda+2\mu+2\gamma+2\delta+\eta)\,\psi\|N(z)-z\| + (4\alpha+2\lambda+2\gamma)\,\psi\,\|w-N(w)\|$

$[2-(4\alpha+2\lambda+2\gamma)]\,\psi\|w-N(w)\| \le (\alpha+2\lambda+2\mu+2\gamma+2\delta+\eta)\,\psi\|N(z)-z\|$

$\psi\|N(w)-w\| \le q\,\psi\|N(z)-z\|$

Where $q = \frac{(\alpha+2\lambda+2\mu+2\gamma+2\delta+\eta)}{[2-(4\alpha+2\lambda+2\gamma)]} < 1$

Since $(5\alpha+4\lambda+2\mu+4\gamma+2\delta+\eta) < 2$

i.e. $\psi\,\|N(w)-w\| \le q\psi\,\|N(z)-z\|$ (E)

Put $g = \frac{1}{2}(f+I)$ then for every $z \in X$

$\psi\|g^2(z)-g(z)\| = \psi\|g(w)-w\|$

$= \psi\,\left\|\frac{1}{2}(N+I)w-w\right\|$

$= \frac{1\psi}{2}\|w-N(w)\|$

$\le \frac{q\psi}{2}\|w-N(w)\|$

By the definition of q, we claim that $\{g^n(x)\}$ is a Cauchy sequence in X.

By the completeness, $\{g^n(x)\}$ converges to some point x_0 in X

$\lim_{n\to\infty} g^n(x) = x_0$

which implies that $g(x_0) = x_0$

Hence $N(x_0) = x_0$

$f\,g\,h(x_0) = x_0$ because $N = f\,g\,h$ *(3.3.4)*

and so

$g\,h(f\,g\,h)(x_0) = g\,h(x_0)$

$f(x_0) = g\,h(x_0)$ *(3.3.5)*

Also

h (fgh)(x₀) = h(x₀)

fg(x₀) =h(x₀)

Now by (3.3.1), (3.3.2), (3.3.3), (3.3.4) and (3.3.5) we have

$$\|h(x_0) - x_0\| = \|fg(x_0) - f^2(x_0)\| = \|fg(x_0) - ff(x_0)\|$$

$$\leq \alpha\psi \left[\frac{\|ghg(x_0)-ghf(x_0)\|\|ghg(x_0)-fg(x_0)\|+\|ghg(x_0)-fg(x_0)\|\|ghg(x_0)-ff(x_0)\|+\|ghg(x_0)-ff(x_0)\|\|ghf(x_0)-fg(x_0)\|}{\|ghg(x_0)-ghf(x_0)\|+\|ghf(x_0)-ff(x_0)\|}\right]$$

$$+\lambda\,\psi\left[\frac{\|ghg(x_0)-fg(x_0)\|^3+\|ghf(x_0)-ff(x_0)\|^3}{1+\|ghg(x_0)-fg(x_0)\|^2+\|ghf(x_0)-ff(x_0)\|^2}\right]$$

$$+\mu\,\psi\left[\frac{\|ghg(x_0)-ff(x_0)\|^2+\|ghf(x_0)-fg(x_0)\|^2}{1+\|ghg(x_0)-ff(x_0)\|+\|ghf(x_0)-fg(x_0)\|}\right]$$

$$+\gamma\,\psi\,[\|ghg(x_0) - fg(x_0)\| + \|ghf(x_0) - ff(x_0)\|]$$

$$+\delta\,\psi\,[\|ghg(x_0) - ff(x_0)\| + \|ghf(x_0) - fg(x_0)\|] + \eta\,\|ghg(x_0) - ghf(x_0)\|$$

$$\leq \alpha\left[\frac{\|h(x_0)-(x_0)\|\|h(x_0)-h(x_0)\|+\|h(x_0)-h(x_0)\|\|h(x_0)-(x_0)\|+\|h(x_0)-(x_0)\|\|(x_0)-h(x_0)\|}{\|h(x_0)-(x_0)\|+\|(x_0)-(x_0)\|}\right]$$

$$+\lambda\,\psi\left[\frac{\|h(x_0)-h(x_0)\|^3+\|(x_0)-(x_0)\|^3}{1+\|h(x_0)-h(x_0)\|^2+\|(x_0)-(x_0)\|^2}\right] + \mu\,\psi\left[\frac{\|h(x_0)-(x_0)\|^2+\|(x_0)-h(x_0)\|^2}{1+\|h(x_0)-(x_0)\|+\|(x_0)-h(x_0)\|}\right]$$

$$+\gamma\,\psi\,[\|h(x_0) - h(x_0)\| + \|(x_0) - x_0\|] + \delta\,\psi\,[\|h(x_0) - x_0\| + \|x_0 - h(x_0)\|]$$

$$+\eta\,\|h(x_0) - x_0\|$$

$$\leq \alpha\psi\,\|h(x_0) - x_0\| + \mu\|h(x_0) - x_0\| + 2\delta\,\|h(x_0) - x_0\| + \eta\|h(x_0) - x_0\|$$

$$\leq (\alpha + \mu + 2\delta + \eta)\psi\|h(x_0) - x_0\|$$

Therefore

$$\psi\|h(x_0) - x_0)\| \leq (\alpha + \mu + 2\delta + \eta)\psi\|h(x_0) - x_0\|$$

This is contradiction

Since $(\alpha + \mu + 2\delta + \eta) < 1$

$h(x_0) = x_0$

i.e. x_0 is the fixed point of h. Thus we have from (3.3.5)

$g(x_0) = f(x_0)$

Again

$$\psi\|f(x_0) - x_0\| = \psi\|f(x_0) - f^2(x_0)\| = \psi\|f(x_0) - ff(x_0)\|$$

$$\leq \alpha\psi\left[\frac{\|gh(x_0)-ghf(x_0)\|\|gh(x_0)-f(x_0)\|+\|gh(x_0)-f(x_0)\|\|gh(x_0)-ff(x_0)\|+\|gh(x_0)-ff(x_0)\|\|ghf(x_0)-f(x_0)\|}{\|gh(x_0)-ghf(x_0)\|+\|ghf(x_0)-ff(x_0)\|}\right]$$

$$+\lambda\psi\left[\frac{\|gh(x_0)-f(x_0)\|^3+\|ghf(x_0)-ff(x_0)\|^3}{1+\|gh(x_0)-f(x_0)\|^2+\|ghf(x_0)-ff(x_0)\|^2}\right]+\mu\psi\left[\frac{\|gh(x_0)-ff(x_0)\|^2+\|ghf(x_0)-f(x_0)\|^2}{1+\|gh(x_0)-ff(x_0)\|+\|ghf(x_0)-f(x_0)\|}\right]$$

$$+\gamma\psi[\|gh(x_0)-f(x_0)\|+\|ghf(x_0)-ff(x_0)\|]$$

$$+\delta\psi[\|gh(x_0)-ff(x_0)\|+\|ghf(x_0)-f(x_0)\|]+\eta\psi\|gh(x_0)-ghf(x_0)\|$$

$$\leq \alpha\psi\left[\frac{\|f(x_0)-(x_0)\|\|f(x_0)-f(x_0)\|+\|f(x_0)-f(x_0)\|\|f(x_0)-(x_0)\|+\|f(x_0)-(x_0)\|\|(x_0)-f(x_0)\|}{\|f(x_0)-(x_0)\|+\|(x_0)-(x_0)\|}\right]$$

$$+\lambda\psi\left[\frac{\|f(x_0)-f(x_0)\|^3+\|(x_0)-(x_0)\|^3}{1+\|f(x_0)-f(x_0)\|^2+\|(x_0)-(x_0)\|^2}\right]+\mu\psi\left[\frac{\|f(x_0)-(x_0)\|^2+\|(x_0)-f(x_0)\|^2}{1+\|f(x_0)-(x_0)\|+\|(x_0)-f(x_0)\|}\right]$$

$$+\gamma\psi[\|f(x_0)-f(x_0)\|+\|(x_0)-(x_0)\|]+\delta\psi$$
$$[\|f(x_0)-(x_0)\|+\|(x_0)-f(x_0)\|]$$

$$+\eta\psi\|f(x_0)-(x_0)\|$$

$$\leq \alpha\psi\|f(x_0)-(x_0)\|+\mu\psi\|f(x_0)-(x_0)\|+2\delta\|f(x_0)-(x_0)\|+$$
$$\eta\psi\|f(x_0)-(x_0)\|$$

$$\leq(\alpha+\mu+2\delta+\eta)\psi\|f(x_0)-(x_0)\|$$

$$\psi\|f(x_0)-(x_0)\|\leq(\alpha+\mu+2\delta+\eta)\psi\|f(x_0)-(x_0)\|$$

which is contradiction

Since $(\alpha+\mu+2\delta+\eta)<1$

$f(x_0)=x_0$

but

$f(x_0)=g(x_0)$

x_0 is the common fixed point of *f, g, h.*

Using (3.3.1), (3.3.2), (3.3.3), (3.3.4) and (3.3.5)

$$\psi\|x_0-y_0\|=\psi\|f^2(x_0)-f^2(y_0)\|=\psi\|ff(x_0)-ff(y_0)\|$$

$$\leq$$
$$\alpha\psi\left[\frac{\|ghf(x_0)-ghf(y_0)\|\|ghf(x_0)-ff(x_0)\|+\|ghf(x_0)-ff(x_0)\|\|ghf(x_0)-ff(y_0)\|+\|ghf(x_0)-ff(y_0)\|\|ghf(y_0)-ff(x_0)\|}{\|ghf(x_0)-ghf(y_0)\|+\|ghf(y_0)-ff(y_0)\|}\right]$$

$$+\lambda\psi\left[\frac{\|ghf(x_0)-ff(x_0)\|^3+\|ghf(y_0)-ff(y_0)\|^3}{1+\|ghf(x_0)-ff(x_0)\|^2+\|ghf(y_0)-ff(y_0)\|^2}\right]+\mu\psi\left[\frac{\|ghf(x_0)-ff(y_0)\|^2+\|ghf(y_0)-ff(x_0)\|^2}{1+\|ghf(x_0)-ff(y_0)\|+\|ghf(y_0)-ff(x_0)\|}\right]$$

$$+\gamma\psi[\|ghf(x_0)-ff(x_0)\|+\|ghf(y_0)-ff(y_0\|]$$

$$+\delta\psi[\|ghf(x_0)-ff(y_0\|+\|ghf(y_0)-ff(x_0)\|]+\eta\psi\|ghf(x_0)-ghf(y_0)\|$$

$$\leq \alpha\psi\left[\frac{\|x_0-y_0)\|\|x_0-x_0\|+\|x_0-x_0\|\|x_0-y_0\|+\|x_0-y_0\|\|y_0-x_0\|}{\|x_0-y_0\|+\|y_0-y_0\|}\right]$$

$$+ \lambda \psi \left[\frac{\|(x_0-x_0\|^3 + \|y_0 - y_0\|^3}{1+\|x_0-x_0\|^2 + \|y_0 - y_0\|^2}\right] + \mu \psi \left[\frac{\|x_0-y_0\|^2 + \|y_0 - x_0\|^2}{1+\|x_0-y_0\| + \|y_0 - (x_0\|}\right]$$

$$+ \gamma \psi [\|x_0 - x_0\| + \|y_0 - y_0\|] + \delta \psi [\|x_0 - y_0\| + \|y_0 - x_0\|]$$

$$+ \eta \psi \|x_0 - y_0\|$$

$$\leq \alpha\psi \|x_0 - y_0\| + \beta\|x_0 - y_0\| + \mu \psi \|x_0 - y_0\| + 2\delta\psi\|x_0 - y_0\| + \eta \psi \|x_0 - y_0\|$$

$$\|x_0 - y_0)\| \leq (\alpha + \mu + 2\delta + \eta) \psi \|x_0 - y_0)\|$$

This is contradiction,

Since $\alpha + \mu + 2\delta + \eta < 1$, it follows

$$x_0 = y_0$$

This completes the proof of theorem.

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