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Two-Step Iteration Scheme with Errors for Asymptotically Quasi-Nonexpansive Mappings in Convex Cone Metric Spaces

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Abstract

The object of this paper is to consider a generalized Ishikawa type iteration process with errors, which approximate the fixed point of two asymptotically quasi-nonexpansive mappings in convex cone metric spaces. Our results also extend, improve and generalize many known results of the existing literature.

Key words and phrases. Two step iteration process with errors, cone metric space, asymptotically quasi-nonexpansive mapping, common fixed point, normal and non-normal cone.

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1. INTRODUCTION

In recent years, several convergence results have been proved on iterative methods for approximating fixed points of non-expansive, quasi-non-expansive, asymptotically non-expansive, asymptotically quasi-non-expansive, contractive type and Zamfirescu operators using several iteration schemes, for example see ([3], [4], [8], [15],[26].It is worth mentioning that such iterative type methods are known as Mann iterations and Ishikawa iteration schemes.

In 1967, Diaz and Metcalf [6] introduced the concept of quasi-nonexpansive mappings. In 1972, Goebel and Kirk [2] coined the term asymptotically nonexpansive mappings who proved that every asymptotically nonexpansive self-mapping of nonempty closed bounded and convex subset of a uniformly convex Banach space has fixed point. In this connection many authors have been studied the convergence of Ishikawa iterates of asymptotically quasi-nonexpansive mappings on convex metric spaces. The convergence theorems for some iterates of nonexpansive mappings, quasi-nonexpansive mappings and their generalized types have been proved in metric and Banach spaces (see, e.g., ([7, 17, 28, 29, 30]).

Recently, a class of two-step approximation schemes, which includes Mann and Ishikawa iterative schemes for solving common fixed points in Banach and Hilbert spaces is obtained. In 1973, Petryshyn and Williamson [25] gave necessary and sufficient conditions for Mann iterative sequence [23] to converge to fixed points of quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [9] extended the results of Petryshyn and Williamson [25] and gave necessary and sufficient conditions for Ishikawa iterative sequence to converge to fixed points for quasi-nonexpansive mappings.

In 2001, Liu [22] extended results of [9, 25] and gave necessary and sufficient conditions for Ishikawa iterative sequence with errors to converge to fixed point of asymptotically quasi-nonexpansive mappings.

In 1970, Takahashi [33] first introduced a notion of convex metric space which is more general space. It should be pointed out that each linear normed space is a special example of convex metric space, but there exist some convex metric spaces which cannot be embedded into normed space [33]. In recent years, asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings have been studied extensively in the setting of convex metric spaces ([7, 14, 35, 22, 28]).

Recently, a cone metric space [12] in Banach spaces, which is a cone version of the usual metric in R is very applicable in applied mathematics including nonlinear analysis by joining it with convex structures. In 2013, Lee [21] extended an Ishikawa type iterative scheme with errors to approximate a common fixed point of two sequences of uniformly quasi-Lipschitzian mappings on convex cone metric spaces and many other authors are also proved some fixed point theorems for contractive-type mappings in cone metric spaces.

1.1. Brief Introduction of Various Iteration Schemes: Let K be a nonempty convex subset of a linear space E and $T:K\to K$ be a mapping. The Mann iteration schemes(see [23]) for a mapping $T:K\to K$ are defined by

$$\begin{cases} x_1 = x_0 \in K \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, n \in N \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ is in (0,1). It is well-known that Picard iteration scheme converges for contractions but do not converges for nonexpansive mapping whereas Mann iteration scheme converges for nonexpansive.

The Ishikawa's iteration process [13] which is defined recursively by

$$\begin{cases} x_1 = x_0 \in K \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, n \in N, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval [0,1].

In 2007, Agrawal et al. [1] introduced the following iteration process:

$$\begin{cases} x_1 = x_0 \in K \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, n \in N, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in (0,1). They showed that this process converges at the rate same as that of Picard iteration and faster than Mann iteration for contractions.

In 2001, Khan and Takahashi [19] approximated the fixed points of two asymptotically nonexpansive mappings $S, T: K \to K$ through the sequence $\{x_n\}$ given by

$$\begin{cases} x_1 = x_0 \in K \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, n \in N, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence in (0,1).

Recently, Khan et al. [16] modified the iteration process (1.4) to the case of two mappings as follows:

$$\begin{cases} x_1 = x_0 \in K \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n S^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, n \in N, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0,1).

The above iterative schemes (1.1 -1.5)were extend to a convex structure and studied fixed point theory for nonexpansive mappings in a uniformly convex metric space by the various authors(see [2, 11, 20, 18, 27]).

1.2. Our Proposed Scheme in Convex Cone Metric Spaces:

Inspired and motivated by the above facts, we construct an Ishikawa type iteration process which converges strongly to a common fixed point of a asymptotically quasi-nonexpansive mappings in convex cone metric spaces as below:

Definition 1.1. Let (X,d,W) be a cone metric space with a convex structure $W: X^3 \times I^3 \to X$, where I = [0,1] and $S, T: X \to X$ be asymptotically quasinonexpansive mappings with six sequences of real numbers $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ in [0,1] with $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$, n = 1,2,... . For any given $x_1 \in X$, define a sequence $\{x_n\}$ as follows:

$$\begin{cases} x_0 = x \in K \\ x_{n+1} = W(x_n, T^n y_n, u_n, a_n, b_n, c_n) \\ y_n = W(S^n x_n, T^n x_n, v_n, a'_n, b'_n, c'_n), \end{cases} \tag{1.6}$$

where $\{u_n\}$, $\{v_n\}$ are two sequences in X satisfying the following condition: for any nonnegative integers $n, m, 1 \le n < m$, if $\delta(A_{nm}) > 0$, then

$$\max_{n\le i,j\le m} \{\| d(x,y) \|\} : x \in \{u_i, v_i\}, y \in \{x_j, y_j, Tx_j, Sy_j, u_j, v_j\}\} < \delta(A_{nm}), \tag{1.7}$$

where $A_{nm} = x_i, y_i, Ty_i, Sx_i, u_i, v_i : n \le i \le m, \delta(A_{nm}) = \sup_{x,y\in A_{nm}} \| d(x,y) \|$.

Then a sequence $\{x_n\}$ is called a generalized Ishikawa type iteration process with errors for two asymptotically quasi-nonexpansive mappings S and T in convex cone metric space (X,d).

Putting $S^n = I$ in the inequality **1.6**, where I is identity mapping, we get well known Ishikawa iteration scheme with errors [5] in convex metric spaces.

Obviously, the Ishikawa iterative sequence is a special case of (1.6) with $c_n = c'_n = 0$, $u_n = v_n = 0$ and $S^n = I$, where I is identity mapping.

1.2. Convexity of Various Contractive Mappings in Cone Metric Space: Let (X, d) be a cone metric space with solid one P and $T: X \to X$ a given mapping. Let

F(T) denote the set of fixed points of T (see [10, 32, 34].

Definition 1.2. (i) T is called Nonexpansive if

$d(Tx,Ty) \preceq d(x,y),$

for all $x,y \in X$.

(ii) T is called Quasi-nonexpansive if $F(T) = \phi$ and

$d(Tx,p) \preceq d(x,p),$

for all $x \in X$, and $p \in F(T)$.

(iii) T is called asymptotically nonexpansive mapping if there exist a sequence $k_n \in [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$\| T^n x - T^n y) \preceq (1+k_n) \| x - y \|,$

or all $x,y \in X$ and $n \geq 1$.

(iv) T is called quasi-nonexpansive mapping provided

$\| T^n x - p) \preceq (1+k_n) \| x - p \|,$

for all $x \in X$, $p \in F(T)$. and $n \geq 1$.

.

From the above Definition , if F (T)= φ, it follows that an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive, and an asymptotically quasi-nonexpansive must be uniformly quasi-Lipschitzian where $L = \sup\{k_n, n \in N\} < \infty.$

2. PRELIMINARIES:

Throughout this paper, E is a normed vector space with a normal solid cone P . A nonempty subset P of E is called a cone if P is closed, $P \neq \{0\}$, for a, b $R^+ = [0,\infty)$ and $x, y \in P$, $ax + by \in P$ and $P \cap (-P) \neq \{0\}$. We define a partial ordering $\preceq$ in E as $x \preceq y$ if y-x $\in$ P, x << y indicates that y-x $\in$ intP and $x \prec y$ means that $x \preceq y$ but $x \neq y$. A cone P is said to be solid if its interior intP is nonempty.

There exist two kinds of cones-normal (with the normal constant k) and non-normal ones [6]). Let X be a real Banach space, $P \subset E$ a cone and $\leq$ partial ordering defined by P . A cone P is called normal if there exists a constant K > 0 such that $0 \leq x \leq y$ implies

$$\| x \| \leq k \| y \| \tag{2.1}$$

for all $x, y \in P$. or equivalently, if (for all n) $x_n \leq y_n \leq z_n$ and

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} z_n = x \text{ implies } \lim_{n \to \infty} y_n = x \tag{2.2}$$

The least positive number k satisfying (2.1) is called the normal constant of P .

Definition 2.1. Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satisfies:

(d1) 0 < d(x, y) for all $x, y \in X$ and d(x, y) = 0 if and only if x = y;

(d2) d(x, y) = d(y, x) for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space. It is clear that the concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where E = R and $P = [0, +\infty)$.

Definition 2.2. Suppose $\{x_n\}$ be a sequence in a cone metric space (X, d) and P be a normal cone with a normal constant K (see [12]). Then

(i) a Cauchy sequence if for every ε in E with $0 << \varepsilon$, then there is an nature number N such that for all n, m > N, $d(x_{n,} x_m) << \varepsilon$;

(ii) a convergent sequence if for every ε in E with $0 << \varepsilon$, then there is a natural number N such that for all $n \geq N$, $d(x_{n,} x) << \varepsilon$ for some fixed x in X. A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

We recall [12] that if P is a normal solid cone, then $\{x_n\} \in X$ is a Cauchy sequence if and only if $\| d(x_n, x_m) \| \to 0$, as $n,m \to \infty$ Moreover, $\{x_n\} \in X$ converges to $x \in X$ if and only if $\| d(x_n, x) \| \to 0$ as $n \to \infty$.

In 1970, Takahashi[33] introduced the concept of convexity in a metric space and the properties of the space.

Definition 2.3. Let (X,d) be a metric space, and I = [0,1]. A mapping $W: X \times X \times I \to X$ is said to be convex structure on X, if for any $(x,y,\lambda) \in X \times X \times I$ and $u \in X$, the following inequality holds:

$d(W(x,y,\lambda),u) \leq \lambda d(x,u) + (1-\lambda)d(y,u)$.

X together with a convex structure W is called a convex metric space, denoted by (X,d,W). A nonempty subset K of X is said to be convex if $W(x,y,\lambda) \in K$ for all $(x,y,\lambda) \in K \times K \times I$.

Remark. Every normed space is a convex metric space, where a convex structure $W(x,y,z,\alpha,\beta,\gamma) = \alpha x+\beta y+\gamma z$, for all $x,y,z \in E$ and $\alpha,\beta,\gamma \in I$ with $\alpha+\beta+\gamma = 1$. In fact,

$$\begin{aligned} d(u, W(x,y,z;\alpha,\beta,\gamma)) &= \| u - (\alpha x + \beta y + \gamma z) \| \\ &\leq \alpha \| u - x \| + \beta \| u - y \| + \gamma \| u - z \| \\ &= \alpha d(u,x) + \beta d(u,y) + \gamma d(u,z), \end{aligned}$$

for all $u \in X$. But there exists some convex metric spaces which can not be embedded into normed space.

We recall some Definitions and lemmas in a convex cone metric space (X,d,W) which will be used in our main results:

Definition 2.4. Let (X,d) be a cone metric space, and I = [0,1]. A mapping $W: X^2 \times I \to X$ is said to be convex structure on X, if for any $(x,y,\lambda) \in X^2 \times I$ and $u \in X$, the following inequality holds:

$d(W(x,y,\lambda),u) \preceq \lambda d(x,u) + (1 - \lambda)d(y,u)$.

If (X,d) is a cone metric space with a convex structure W, then (X,d) is called a convex abstract metric space or convex cone metric space (see also [14], [23]). Moreover, a nonempty subset K of X is said to be convex if $W(x,y,\lambda) \in K$, for all $(x,y,\lambda) \in K^2 x I$.

Definition 2.5. Let (X,d) be a cone metric space, I = [0,1], and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ real sequences in [0,1] with $a_n+b_n+c_n=1$. A mapping $W: X^3 \times I^3 \to X$ is said to be convex structure on X, if for any $(x, y,z,a_n,b_n,c_n) \in X_3 \times I_3$ and $u \in X$, the following inequality holds:

$d(W(x, y,z,a_n,b_n,c_n),u) \preceq a_n d(x, u) + b_n d(y,u) + c_n d(z,u)$.

If (X,d) is a cone metric space with a convex structure W, then (X,d) is called a generalized convex cone metric space. Furthermore, a nonempty subset K of X is said to be convex if $W(x, y,z,a_n,b_n,c_n) \in K$, for all $(x, y,z,a_n,b_n,c_n) \to K^3 \times I^3$

.

Remark. If E = R, $P=[0,+\infty)$, $\|.\|=|.|$ then (X,d) is a convex metric space, i.e., generalized convex metric space as in [36].

Example 2.6.

(a) [12] Let E = R^2, $P=\{(x,y)\in R^2 : x\geq 0, y\geq 0\}$, X = R and $d: X\times X\rightarrow E$ defined by $d(x,y)=(|x-y|,\alpha|x-y|)$, where $\alpha\geq 0$ is a constant. Then (X,d) is a cone metric space [11] with normal cone P where k = 1.

(b) For other examples of a cone metric spaces, i.e., P-metric spaces one can see [37].

Example 2.7. Let (X,d) be a cone metric space as in Example 2.6(a). If W(x,y,λ) = λx+ (1-λ)y, then (X,d) is a cone metric space. Therefore, this notion is more general than that of a convex metric space.

Lemma 2.8. (see [22].) Let $\{a_n\}$, $\{b_n\}$ and $\{\alpha_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} = (1+\alpha_n)a_n + b_n, n\geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty}\alpha_n < \infty$. Then

(a) $\lim_{n\to\infty} a_n$ exists.

(b) If $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9. Let K be a nonempty, closed, convex subset of a complete convex cone metric space (X,d,W). Let $S,T: K\rightarrow K$ be an asymptotically quasi nonexpansive mapping with a sequence $\{k_n\}\in[0,\infty)$ Assume that $F=F(S)\cup F(T)\neq\phi$. Let $\{x_n\}$ be the generalized Ishikawa type iteration process with errors defined by (1.6) and $\{u_n\}$, $\{v_n\}$ satisfying (1.7) and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n'\}$, $\{b_n'\}$ and $\{c_n'\}$ be six sequences in [0,1] with restriction $a_n + b_n + c_n = a_n' + b_n' + c_n' = 1$ and $\sum_{n=1}^{\infty} c_n' + c_n < \infty$ and if $F=F(S)\cup F(T)\neq\phi$ then:

(i) there exists a constant vector $\nu \in P\{0\}$ such that $\| d(x_{n+1}, p) \leq k\cdot(1+A_n).\| d(x_{n,} p)\| + k.\| \nu \| ..\lambda_n$

for all n $\in$ N and for all p $\in$ F, where k is the normal constant of a cone P;

(ii) there exists a real constant M > 0 such that

$$\| d(x_{n+m}, p)\| \leq k.M.\| d(x_n, p)\| + k.M.\nu.\sum_{i=n}^{n+m-1}\lambda_n,$$

for all n,m $\in$ N and for all p $\in$ F, where k is the normal constant of a cone P.

Proof. (i) We suppose that p $\in$ F. Then, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(x_n, T^n y_n, u_n, a_n, b_n, c_n), p) \\ &\preceq a_n d(x_n, p) + b_n d(T^n y_n, p) + c_n d(u_n, p) \\ &\preceq d(x_n, p) + b_n(1+k_n) d(y_n, p) + c_n d(u_n, p), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} d(y_n, p) &= d(W(S^n x_n, T^n x_n, v_n, a_n^{'}, b_n^{'}, c_n^{'}), p) \\ &\preceq a_n^{'} d(S^n x_n, p) + b_n^{'} d(T^n x_n, p) + c_n^{'} d(v_n, p), p) \\ &\preceq a_n^{'}(1+k_n) d(x_n, p) + b_n^{'}(1+k_n) d(x_n, p) + c_n^{'} d(v_n, p) \\ &\preceq (1+k_n)(a_n^{'} + b_n^{'}) d(x_n, p) + c_n^{'} d(v_n, p) \end{aligned}$$
(2.4)

Substituting (2.3) into (2.4), it can be obtained that

$$\begin{aligned} d(x_{n+1}, p) &\preceq a_n d(x_n, p) + b_n(1+k_n)[(1+k_n)(a_n^{'} + b_n^{'}) d(x_n, p) \\ &\quad + c_n^{'} d(v_n, p)] + c_n d(u_n, p) \\ &= a_n d(x_n, p) + b_n(1+k_n)[(1+k_n)(a_n^{'} + b_n^{'}) d(x_n, p)] \\ &\quad + b_n(1+k_n) c_n^{'} d(v_n, p) + c_n d(u_n, p) \\ &\preceq [a_n + b_n \delta_n (1+k_n)^2] d(x_n, p) + b_n(1+k_n) c_n^{'} d(v_n, p) + c_n d(u_n, p) \\ &\preceq [1 + \delta_n(1+2k_n + k_n^2)] d(x_n, p) + b_n(1+k_n) c_n^{'} d(v_n, p) + c_n d(u_n, p) \\ &\preceq (1+A_n) d(x_n, p) + \lambda_n \nu; \end{aligned} \tag{2.5}$$

where $A_n = \delta_n(1+2k_n + k_n^2)$, $\nu = \sup_{p\in F,\, n\geq 1}\{b_n(1+k_n) d(v_n, p) + d(u_n, p)\}$

and $\lambda_n = c_n^{'} + c_n \,(n \in N \cup \{0\})$.

Thus, by the normality of P, for the normal constant k > 0, we have

$\| d(x_{n+1}, p) \| \leq k\,.(1 + A_n) \| d(x_n, p) \| + k. \| \nu \| .\lambda_n$

for all $n \in N$ and for all $p \in F$.

(ii) We know that $1 + x \leq e^x$ for all $x \geq 0$. Considering it for the **condition (i) of Lemma 2.9**, we get

$$
\begin{aligned}
d(x_{n+m}, p) &\preceq (1 + A_{n+m-1}) d(x_{n+m-1}, p) + \lambda_{n+m-1} \cdot \nu \\
&\preceq e^{A_{n+m-1}}) d(x_{n+m-1}, p) + \nu.\lambda_{n+m-1} \\
&\preceq e^{A_{n+m-1}} [(1 + A_{n+m-2}\} d(x_{n+m-2}, p) \\
&\quad + \nu..\lambda_{n+m-2}] + \nu..\lambda_{n+m-1} \\
&\preceq e^{A_{n+m-1} + A_{n+m-2}} d(x_{n+m-2}, p) \\
&\quad + e^{A_{n+m-1} + A_{n+m-2}} [\lambda_{n+m-1} + \lambda_{n+m-2}].\nu \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&\preceq M.d(x_n, p) + \left(M \sum_{i=n}^{n+m-1} \lambda_n \right).\nu,
\end{aligned}
$$

where $M = e^{\sum_{i=1}^{\infty} A_i}$. Hence, (ii) follows from **(1.1),** since P is a normal cone with the normal constant k. This completes the proof.

3. MAIN RESULTS

In this section, we propose more generalized convergence theorem regarding Ishikawa type iteration schemes with errors for approximating a common fixed point of two sequences of asymptotically quasi-nonexpansive mappings in convex cone metric spaces.

Theorem 3.1. Let K be a nonempty, closed, convex subset of a complete convex cone metric space (X,d,W). Let $S, T : K \to K$ be an asymptotically quasi nonexpansive mapping with a sequence $\{k_n\} \in [0, \infty)$. Assume that $F = F(S) \cup F(T) \neq \phi$. Let $\{x_n\}$ be the generalized Ishikawa type iteration process with errors defined by (1.6) and $\{u_n\}$, $\{v_n\}$ satisfying (1.7) and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n^{'}\}$, $\{b_n^{'}\}$, and $\{c_n^{'}\}$ be six sequences in [0,1] with restriction $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = 1$ and

$\sum_{n=1}^{\infty}(c_n' + c_n) < \infty$. Then, $\{x_n\}$ converges to a common fixed point of S and T if and only if $\liminf_{n\to\infty} \| d(x_n, F)\| = 0$ where $\| d(x,F)\| = \inf\{\| d(x,q) : q \in F\}$.

Proof. The necessity of condition is obvious. Thus, we will only prove the sufficiency. Then from Lemma 2.9(i), we have

$$\| d(x_{n+1}, p) \le k\cdot(1 + A_n).\| d(x_{n,} p)\| + k.\| \nu \| ..\lambda_n$$

where

$$A_n = \delta_n(1 + 2k_n + k_n^2) \ \nu = \sup_{p\in F,\ n\ge 1} \{b_n(1 + k_n)d(v_n, p) + d(u_n, p)\}$$

and $\lambda_n = c_n' + c_n (n \in N \cup \{0\})$ with $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty}\lambda_n < \infty$. Because $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty}\lambda_n < \infty$, it follows from Lemma 2.8 that $\lim_{n\to\infty} \| d(x_n, F)\|$ exists. Now, $\liminf_{n\to\infty} \| d(x_n, F)\| = 0$, therefore, implies that $\lim_{n\to\infty} \| d(x_n, F)\| = 0$. Secondly, we show that $\{x_n\}$ is a Cauchy sequence in K, for any positive real number ε, there exists a natural number $N_0 \in N$ such that $n > N_0$, we get

$$\| d(x_n, F)\| < \frac{\varepsilon}{4K^2M} \text{ and } \sum_{n=N_0+1}^{\infty}\lambda_n < \frac{\varepsilon}{4K^2 \| \nu \| M}.$$

In particular, there exists a $q \in F$ and an integer $n_1 > N_1$ such that

$$\| d(x_1, p_1)\| < \frac{\varepsilon}{4K^2M}.$$

It follows from Lemma 2.9 (ii), that when $n > N_1$, we have

$$\begin{aligned} \| d(x_{n+m}, p_1)\| &= \| d(x_{n_1+(n+m-n_1)}, p_1)\| \\ &\le \| d(x_{n+m}, p)\| \le k.M.\| d(x_n, p)\| + k.M.v.\sum_{i=n_1}^{n+m-1}\lambda_n, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \| d(x_n, p_1)\| &= \| d(x_{n_1+(n-n_1)}, p_1)\| \\ &\le \| d(x_{n+m}, p)\| \le k.M.\| d(x_n, p)\| + k.M.v.\sum_{i=n_1}^{n-1}\lambda_n, \end{aligned} \tag{3.2}$$

Hence, from (2.1), (3.1) and (3.3), we have

$$\begin{aligned}\| d(x_{n+m}, x_n) \| &\le k.\| d(x_{n+m}, p_1) + d(p_1, x_n) \| \\ &\le k.\| d(x_{n+m}, p_1) \| + k.\| d(p_1, x_n) \| \\ &\le 2k^2.M.\| d(x_{n_1}, p_1) \| + k^2.\| v \|.M\left(\sum_{i=n_i}^{n+m-1}\lambda_i + \sum_{i=n_i}^{n-1}\lambda_i\right) \\ &\le 2k^2.M.\| d(x_{n_1}, p_1) \| + 2k^2.\| v \|.M.\sum_{i=n_i}^{n+m-1}\lambda_i \\ &\le 2k^2.M.\frac{\varepsilon}{4K^2M} + 2k^2.\| v \|.M.\frac{\varepsilon}{4K^2 \| v \| M} = \varepsilon.\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in closed convex subset K of a complete cone metric space X. So that it must be convergent to a point in K. Let $\lim_{n\to\infty} x_n = p$. We will prove that $p \in F$. For given $\varepsilon > 0$, there exists an integer n_2 such that for all $n \ge n_2$, we define

$$\begin{cases} \| d(x_n, p) \| < \dfrac{\varepsilon}{2k(2+k_1)}, \\ \| d(x_n, F) \| < \dfrac{\varepsilon}{2k(2+k_1)} \end{cases} \tag{3.3}$$

In particular, there exists a $p_1 \in F$ and an integer $n_3 > n_2$ such that

$$\| d(x_{n_3}, p) \| < \frac{\varepsilon}{2k(2+k_1)}. \tag{3.4}$$

Then, we obtained

$$\begin{aligned} d(Sp, p) &\preceq d(Sp, p_1) + d(p_1, x_{n_3}) + d(x_{n_3}, p) \\ &\preceq (1+k_1)d(p, x_{n_3}) + (1+k_1)d(x_{n_3}, p_1) + d(p_1, x_{n_3}) + d(x_{n_3}, p) \\ &= (2+k_1)d(x_{n_3}, p) + (2+k_1)d(x_{n_3}, p_1)\end{aligned}$$

Now using (2.1), (3.3) and (3.4), we have

$$\begin{aligned} d(Sp, p) &\le k(2+k_1)d(x_{n_3}, p) + k(2+k_1)d(x_{n_3}, p_1) \\ &< k(2+k_1)\cdot\frac{\varepsilon}{2k(2+k_1)} + k(2+k_1).\frac{\varepsilon}{2k(2+k_1)}. \\ &= \varepsilon.\end{aligned}$$

Similarly, we also have $\| d(Tp, p) \| < \varepsilon$. Since ε is arbitrary, it follows that $d(Sp, p) = d(Tp, p) = 0$, that is, p is a common fixed point of S and T. This completes the proof.

Taking S^n = I in **Theorem 3.1**, where I is identity mapping, we can obtain the following result immediately.

Corollary 3.2. Let K be a nonempty, closed, convex subset of a complete convex cone metric space (X,d,W). Let $T: K \to K$ be an asymptotically quasi nonexpansive mapping with a sequence $\{k_n\} \in [0,\infty)$. Assume that $F(T) = \phi$. Let $\{x_n\}$ be the generalized Ishikawa type iteration process with errors defined by (1.6) and $\{u_n\}$, $\{v_n\}$ satisfying (1.7) and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n^{'}\}$, $\{b_n^{'}\}$, and $\{c_n^{'}\}$ be six sequences in [0,1] with restriction $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = 1$ and $\sum_{n=1}^{\infty}(c_n^{'} + c_n) < \infty$. Then, $\{x_n\}$ converges to a common fixed point of T if and only if $\liminf_{n\to\infty} \| d(x_n, F) \| = 0$ where $\| d(x, F) \| = \inf\{\| d(x, q) \| : q \in F\}$.

Proof. It follows from Lemma 2.9 and Theorem 3.1 with S^n = I for all $n \geq 1$.

Corollary 3.3. Let K be a nonempty, closed, convex subset of a complete convex cone metric space (X,d,W). Let $T: K \to K$ be uniformly quasi-Lipschitzian mapping with $L > 0$ and $\{k_n\} \in [0,\infty)$. Assume that $F(T) = \phi$. Let $\{x_n\}$ be the generalized Ishikawa type iteration process with errors defined by (1.6) and $\{u_n\}$, $\{v_n\}$ satisfying (1.7) and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n^{'}\}$, $\{b_n^{'}\}$, and $\{c_n^{'}\}$ be six sequences in [0,1] with restriction $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = 1$ and $\sum_{n=1}^{\infty}(c_n^{'} + c_n) < \infty$. Then, $\{x_n\}$ converges to a common fixed point of T if and only if $\liminf_{n\to\infty} \| d(x_n, F) \| = 0$ where $\| d(x, F) \| = \inf\{\| d(x, q) : q \in F\}$.

Proof. Since $\{k_n\} \in [0,\infty)$ with $k_n \to \infty$ as $n \to \infty$ n, then there exists L > 0 such that $L = \sup\{k_n : n \geq 1\}$. In this case S and T are uniformly quasi-Lipschitzian mappings with L > 0. Put $S^n = I$ in Theorem 3.1, where I is identity mapping. Hence, Corollary 3.3 can be proved by Corollary 3.2.

Put $c_n = c_n^{'} = 0$, $u_n = v_n = 0$ and $S^n = I$, where I is identity mapping in **Definition 1.6**. We can be found the following result:

Corollary 3.4. Let K be a nonempty, closed, convex subset of a complete convex cone metric space (X, d, W). Let $T: K \to K$ an asymptotically quazi-nonexpansive mapping of K (T need not be continuous) with $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \phi$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is Ishikawa type iterative scheme defined by:

$$\begin{cases} x_0 = x \in K \\ x_{n+1} = W(x_n, T^n y_n, a_n) \\ y_n = W(S^n x_n, T^n x_n, b_n), \end{cases}$$

where $\{a_n\}, \{b_n\} \in [0,1]$; and satisfying all the condition of Lemma 2.9. Then, $\{x_n\}$ converges to a fixed point of T if and only if $\liminf_{n \to \infty} d(x_n, F(T)) = 0$, where $d(y, X)$ denotes the distance of y to set X, i.e., $d(y, X) = \inf d(y, x)$, for all $x \in X$.

Proof. Corollary 3.4 can be proven by Theorem 3.1. �

Remark. Our results extend the corresponding results in convex metric spaces [7,36].

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