

Pricing formula for power quanto options with each type of payoffs at maturity

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Abstract

Nowadays, power options become widely used in derivative markets on foreign exchange, which means the combination of quanto option and power option can be seriously considered on its optimal pricing. Here, we drive full closed-form expressions for the price of a European power quanto call options with four different forms of terminal payoff under the assumption of the classical log-normal asset price and exchange rate model.

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1. Introduction

Power options are a class of exotic options in which the payoff at maturity is related to the certain positive power of the underlying asset price, which gives the buyer of the option the potential to receiving a much higher payoff than that from a vanilla option. Since a small change in the value of the underlying of a power option may lead to a significant change in the option price, power options are widely used in markets to provide high leverage strategy.

A quanto is a type of financial derivative whose pay-out currency differs from the natural denomination of its underlying financial variable. A quanto option is a cross-currency option which has a payoff defined with respect to an underlying in one country, but the payoff is converted to another currency for payment. For example, an investor trades a quanto option on an asset quoted in foreign currency, at maturity, the option's value in foreign currency will be converted at a fixed rate into domestic currency. Indeed, quanto options may have four different forms of terminal payoffs. To mention each type of payoffs at maturity, it is a foreign equity option converted to domestic currency, a foreign equity option struck in domestic currency, a foreign equity option struck in pre-determined domestic currency or an FX option denoted in domestic currency, respectively.

There have been many researches on pricing power options or quanto options, both of which are relatively easy under the classical Black-Scholes [1] model. To mention some references on those subjects, Heynen and Kat [3] and Tompkins [7] focused on power options both theoretically and from the market point of view. Wystup [9] obtained various quanto option price formula and introduced three vega positions on hedging of quanto options, and textbooks of Wystup [8] and Kwok [4] contain general theory of quanto options.

Since power options become widely used in derivative markets on foreign exchange, the combination of quanto option and power option can be seriously considered on its pricing. Although we can not find any previous research on power quanto option pricing, there is a research paper on pricing power exchange option by Blenman and Clark [2], on which the authors combine power option and Margrabe [6] type exchange option.

In this paper, we derive full closed-form expressions for the price of power quanto call options with four different forms of terminal payoff under the assumption of the classical log-normal asset price and exchange rate model. In section 2, we specify the dynamics of the processes of underlying asset price and exchange rate in the risk-neutral world. In section 3, we specify four different types of payoff at maturity and obtain the analytic expressions for the price of a power quanto call option in each case.

2. Risk-neutral dynamics on currencies

For a dividend paying asset with the dividend yield rate q , let S_t be the asset price in foreign currency X , and let V_t be the foreign exchange rate in foreign currency per unit of the domestic currency with the constant volatilities σ_S and σ_V , respectively. which

have the following risk-neutral dynamics:

$$\begin{cases} dS_t = (r_f - q) S_t dt + \sigma_S S_t dB_t^{\mathbb{Q}^f}, \\ dV_t = (r_f - r_d) V_t dt + \sigma_V V_t dW_t^{\mathbb{Q}^f} \end{cases} \quad (2.1)$$

under the risk-neutral probability measure \mathbb{Q}^f , where $B_t^{\mathbb{Q}^f}$ and $W_t^{\mathbb{Q}^f}$ are two standard Brownian motions in foreign currency with the correlation ρ . Also, r_f and r_d are the constant foreign and domestic riskless rates, respectively.

Then from well-known standard procedure (see page 95 of [8] or section 2 of [5]), we get the risk-neutral dynamics of (2.1) in domestic currency are given by

$$\begin{cases} dS_t = (r_f - q - \rho\sigma_S\sigma_V) S_t dt + \sigma_S S_t dB_t^{\mathbb{Q}^d}, \\ dV_t = (r_d - r_f) V_t dt + \sigma_V V_t dW_t^{\mathbb{Q}^d}, \end{cases} \quad (2.2)$$

where $B_t^{\mathbb{Q}^d}$ and $W_t^{\mathbb{Q}^d}$ are two correlated standard Brownian motions in domestic currency.

3. Quanto option prices for various quanto payoffs

The following theorems in next subsections give the explicit formulas for the prices of European power quanto call options with constant foreign and domestic riskless rates according to four different forms of terminal payoffs mentioned in section 1.

3.1. Type I

The foreign equity power- α quanto call option (struck in foreign currency) converted to domestic currency has a maturity payoff given by

$$V_T \max(S_T^\alpha - K_f, 0), \quad (3.1)$$

where K_f is the foreign currency strike price.

Theorem 3.1. Under the assumptions of (2.2) with $\alpha > 0$, the price of a European power- α quanto call option at time t in domestic currency with the payoff (3.1) is given by

$$C_q^{(1)}(t, S_t^\alpha, V_t) = V_t e^{-r_f(T-t)} \left[S_t^\alpha e^{\alpha \left\{ r_f - q + \frac{(\alpha-1)\sigma_S^2}{2} \right\} (T-t)} N(d_1) - K_f N(d_2) \right],$$

where

$$d_1 = \frac{\ln \frac{S_t^\alpha}{K_f} + \alpha \left\{ r_f - q + \frac{(2\alpha-1)\sigma_S^2}{2} \right\} (T-t)}{\alpha\sigma_S\sqrt{T-t}}, \quad d_2 = d_1 - \alpha\sigma_S\sqrt{T-t}$$

and $N(\cdot)$ denotes the cumulative distribution function for the standard normal distribution.

Proof. We may write $C_q^{(1)}$ as

$$\begin{aligned} C_q^{(1)}(t, S_t^\alpha, V_t) &= e^{-rd(T-t)} \mathbb{E}_{\mathbb{Q}^d} [V_T \max(S_T^\alpha - K_f, 0) | \mathcal{F}_t] \\ &= V_t e^{-r_f(T-t)} \mathbb{E}_{\mathbb{Q}^d} \left[e^{-\frac{\sigma_V^2}{2}(T-t) + \sigma_V(W_T^{\mathbb{Q}^d} - W_t^{\mathbb{Q}^d})} \max(S_T^\alpha - K_f, 0) \middle| \mathcal{F}_t \right]. \end{aligned} \quad (3.2)$$

For a new risk-neutral probability measure $\tilde{\mathbb{Q}}^d$, the Radon-Nykodým derivative of $\tilde{\mathbb{Q}}^d$ with respect to \mathbb{Q}^d is defined by

$$\left. \frac{d\tilde{\mathbb{Q}}^d}{d\mathbb{Q}^d} \right|_{\mathcal{F}_t} = e^{-\frac{\sigma_V^2}{2}t + \sigma_V W_t^{\mathbb{Q}^d}}.$$

Then the Girsanov's theorem implies that $B_t^{\tilde{\mathbb{Q}}^d} = B_t^{\mathbb{Q}^d} - \rho\sigma_V t$ is again a standard Brownian motion under the domestic risk-neutral probability measure $\tilde{\mathbb{Q}}^d$. Moreover, the dynamics of S_t^α under the measure $\tilde{\mathbb{Q}}^d$ is given by

$$dS_t^\alpha = \alpha \left\{ r_f - q + \frac{(\alpha-1)\sigma_S^2}{2} \right\} S_t^\alpha dt + \alpha\sigma_S S_t^\alpha dB_t^{\tilde{\mathbb{Q}}^d}. \quad (3.3)$$

Thus, (3.2) becomes

$$\begin{aligned} C_q^{(1)}(t, S_t^\alpha, V_t) &= V_t e^{-r_f(T-t)} \mathbb{E}_{\tilde{\mathbb{Q}}^d} [\max(S_T^\alpha - K_f, 0) | \mathcal{F}_t] \\ &= V_t e^{-r_f(T-t)} \left[S_t^\alpha e^{\alpha \left\{ r_f - q + \frac{(\alpha-1)\sigma_S^2}{2} \right\} (T-t)} N(d_1) - K_f N(d_2) \right], \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln \frac{S_t^\alpha}{K_f} + \alpha \left\{ r_f - q + \frac{(2\alpha-1)\sigma_S^2}{2} \right\} (T-t)}{\alpha\sigma_S \sqrt{T-t}}, \\ d_2 &= d_1 - \alpha\sigma_S \sqrt{T-t}. \end{aligned}$$

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3.2. Type II

The foreign equity power- α quanto call option struck in domestic currency has a maturity payoff given by

$$\max(V_T S_T^\alpha - K_d, 0), \quad (3.4)$$

where K_d is the domestic currency strike price.

Theorem 3.2. Under the assumptions of (2.2) with $\alpha > 0$, the price of a European power- α quanto call option at time t in domestic currency with the payoff (3.4) is given by

$$C_q^{(2)}(t, S_t^\alpha, V_t) = V_t S_t^\alpha e^{\left[-r_f + \alpha \left\{ r_f - q + \frac{(\alpha-1)\sigma_S^2}{2} \right\}\right](T-t)} N(d_1) - K_d e^{-r_d(T-t)} N(d_2),$$

where

$$d_1 = \frac{\ln \frac{V_t S_t^\alpha}{K_d} + \left[r_d - r_f + \alpha \left\{ r_f - q + \rho \sigma_S \sigma_V + \frac{(2\alpha-1)\sigma_S^2}{2} \right\} + \frac{\sigma_V^2}{2} \right] (T-t)}{\sqrt{(\alpha^2 \sigma_S^2 + \sigma_V^2 + 2\rho \alpha \sigma_S \sigma_V) (T-t)}},$$

$$d_2 = d_1 - \sqrt{(\alpha^2 \sigma_S^2 + \sigma_V^2 + 2\rho \alpha \sigma_S \sigma_V) (T-t)}.$$

Proof. We may write $C_q^{(2)}$ as

$$C_q^{(2)}(t, S_t^\alpha, V_t) = e^{-r_d(T-t)} \mathbb{E}_{\mathbb{Q}^d} \left[\max(\hat{S}_T - K_d, 0) \middle| \mathcal{F}_t \right], \tag{3.5}$$

where $\hat{S}_T = V_T S_T^\alpha$. We note that the risk-neutral dynamic of \hat{S}_t in domestic currency is given by

$$d\hat{S}_t = \left[r_d - r_f + \alpha \left\{ r_f - q + \frac{(\alpha-1)\sigma_S^2}{2} \right\} \right] \hat{S}_t dt + \alpha \sigma_S \hat{S}_t dB_t^{\mathbb{Q}^d} + \sigma_V \hat{S}_t dW_t^{\mathbb{Q}^d}$$

from (2.2) and (3.3). Thus, (3.5) becomes

$$C_q^{(2)}(t, S_t^\alpha, V_t) = V_t S_t^\alpha e^{\left[-r_f + \alpha \left\{ r_f - q + \frac{(\alpha-1)\sigma_S^2}{2} \right\}\right](T-t)} N(d_1) - K_d e^{-r_d(T-t)} N(d_2),$$

where

$$d_1 = \frac{\ln \frac{V_t S_t^\alpha}{K_d} + \left[r_d - r_f + \alpha \left\{ r_f - q + \rho \sigma_S \sigma_V + \frac{(2\alpha-1)\sigma_S^2}{2} \right\} + \frac{\sigma_V^2}{2} \right] (T-t)}{\sqrt{(\alpha^2 \sigma_S^2 + \sigma_V^2 + 2\rho \alpha \sigma_S \sigma_V) (T-t)}},$$

$$d_2 = d_1 - \sqrt{(\alpha^2 \sigma_S^2 + \sigma_V^2 + 2\rho \alpha \sigma_S \sigma_V) (T-t)}.$$

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3.3. Type III

A foreign equity power- α quanto call option struck in pre-determined domestic currency has a maturity payoff given by

$$V_0 \max (S_T^\alpha - K_f, 0), \quad (3.6)$$

where V_0 is the some fixed exchange rate and K_f is the foreign currency strike price.

Theorem 3.3. Under the assumptions of (2.2) with $\alpha > 0$, the price of a European power- α quanto call option at time t in domestic currency with the payoff (3.6) is given by

$$C_q^{(3)}(t, S_t^\alpha) = V_0 e^{-r_d(T-t)} \left[S_t^\alpha e^{\alpha \left\{ r_f - q - \rho \sigma_S \sigma_V + \frac{(\alpha-1)\sigma_S^2}{2} \right\} (T-t)} N(d_1) - K_f N(d_2) \right],$$

where

$$d_1 = \frac{\ln \frac{S_t^\alpha}{K_f} + \alpha \left\{ r_f - q - \rho \sigma_S \sigma_V + \frac{(2\alpha-1)\sigma_S^2}{2} \right\} (T-t)}{\alpha \sigma_S \sqrt{T-t}},$$

$$d_2 = d_1 - \alpha \sigma_S \sqrt{T-t}.$$

Proof. We may write $C_q^{(3)}$ as

$$\begin{aligned} C_q^{(3)}(t, S_t^\alpha) &= V_0 e^{-r_d(T-t)} \mathbb{E}_{\mathbb{Q}^d} \left[\max (S_T^\alpha - K_f, 0) \mid \mathcal{F}_t \right] \\ &= V_0 e^{-r_d(T-t)} \mathbb{E}_{\mathbb{Q}^d} \left[S_t^\alpha e^{\alpha \left\{ r_f - q - \rho \sigma_S \sigma_V + \frac{(\alpha-1)\sigma_S^2}{2} \right\} (T-t)} N(d_1) - K_f N(d_2) \right], \end{aligned}$$

where

$$d_1 = \frac{\ln \frac{S_t^\alpha}{K_f} + \alpha \left\{ r_f - q - \rho \sigma_S \sigma_V + \frac{(2\alpha-1)\sigma_S^2}{2} \right\} (T-t)}{\alpha \sigma_S \sqrt{T-t}},$$

$$d_2 = d_1 - \alpha \sigma_S \sqrt{T-t}.$$

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3.4. Type IV

An FX power- α call option denoted in domestic currency is an equity-linked foreign exchange option which has a maturity payoff given by

$$S_T^\alpha \max (V_T - K_e, 0), \quad (3.7)$$

where K_e is the strike price on the exchange rate.

Theorem 3.4. Under the assumptions of (2.2) with $\alpha > 0$, the price of a European power- α quanto call option at time t in domestic currency with the payoff (3.7) is given by

$$C_q^{(4)}(t, S_t^\alpha, V_t) = S_t^\alpha e^{\left[\alpha \left\{ r_f - q - \rho \sigma_S \sigma_V + \frac{\alpha(\alpha-1)\sigma_S^2}{2} \right\} - r_d\right](T-t)} \times \left\{ V_t e^{(r_d - r_f + \rho \sigma_S \sigma_V)(T-t)} N(d_1) - K_e N(d_2) \right\},$$

where

$$d_1 = \frac{\ln \frac{V_t}{K_e} + \left(r_d - r_f + \rho \sigma_S \sigma_V + \frac{\sigma_V^2}{2} \right) (T-t)}{\sigma_V \sqrt{T-t}},$$

$$d_2 = d_1 - \sigma_V \sqrt{T-t}.$$

Proof. We may write $C_q^{(4)}$ as

$$\begin{aligned} C_q^{(4)}(t, S_t^\alpha, V_t) &= e^{-r_d(T-t)} \mathbb{E}_{\mathbb{Q}^d} \left[S_T^\alpha \max(V_T - K_e, 0) \mid \mathcal{F}_t \right] \\ &= S_t^\alpha e^{\left[\alpha \left\{ r_f - q - \rho \sigma_S \sigma_V + \frac{\alpha(\alpha-1)\sigma_S^2}{2} \right\} - r_d\right](T-t)} \\ &\quad \times \mathbb{E}_{\mathbb{Q}^d} \left[e^{-\frac{\alpha^2 \sigma_S^2}{2}(T-t) + \alpha \sigma_S (B_T^{\mathbb{Q}^d} - B_t^{\mathbb{Q}^d})} \max(V_T - K_e, 0) \mid \mathcal{F}_t \right]. \end{aligned} \tag{3.8}$$

For a new risk-neutral probability measure $\tilde{\mathbb{Q}}^d$, the Radon-Nikoým derivative $\tilde{\mathbb{Q}}^d$ with respect to \mathbb{Q}^d is defined by

$$\left. \frac{d\tilde{\mathbb{Q}}^d}{d\mathbb{Q}^d} \right|_{\mathcal{F}_t} = e^{-\frac{\alpha^2 \sigma_S^2}{2} t + \alpha \sigma_S B_t^{\mathbb{Q}^d}}.$$

Then the Girsanov's theorem implies that $B_t^{\tilde{\mathbb{Q}}^d} = B_t^{\mathbb{Q}^d} - \alpha \sigma_S t$ and $W_t^{\tilde{\mathbb{Q}}^d} = W_t^{\mathbb{Q}^d} - \rho \alpha \sigma_S t$ are again standard Brownian motions under the domestic risk-neutral probability measure $\tilde{\mathbb{Q}}^d$. Moreover, the dynamics of V_t under the measure $\tilde{\mathbb{Q}}^d$ is given by

$$dV_t = (r_d - r_f + \rho \sigma_S \sigma_V) V_t dt + \sigma_V V_t d\tilde{W}_t^{\tilde{\mathbb{Q}}^d}.$$

Thus, (3.8) becomes

$$\begin{aligned}
 C_q^{(4)}(t, S_t^\alpha, V_t) &= e^{-r_d(T-t)} \mathbb{E}_{\mathbb{Q}^d} [S_T^\alpha \max(V_T - K_e, 0) | \mathcal{F}_t] \\
 &= S_t^\alpha e^{\left[\alpha \left\{ r_f - q - \rho \sigma_S \sigma_V + \frac{(\alpha-1)\sigma_S^2}{2} \right\} - r_d \right] (T-t)} \mathbb{E}_{\mathbb{Q}^d} [\max(V_T - K_e, 0) | \mathcal{F}_t] \\
 &= S_t^\alpha e^{\left[\alpha \left\{ r_f - q - \rho \sigma_S \sigma_V + \frac{\alpha(\alpha-1)\sigma_S^2}{2} \right\} - r_d \right] (T-t)} \\
 &\quad \times \left\{ V_t e^{(r_d - r_f + \rho \sigma_S \sigma_V)(T-t)} N(d_1) - K_e N(d_2) \right\},
 \end{aligned}$$

where

$$\begin{aligned}
 d_1 &= \frac{\ln \frac{V_t}{K_e} + \left(r_d - r_f + \rho \sigma_S \sigma_V + \frac{\sigma_V^2}{2} \right) (T-t)}{\sigma_V \sqrt{T-t}}, \\
 d_2 &= d_1 - \sigma_V \sqrt{T-t}.
 \end{aligned}$$

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