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Best Approximation Results via Common Fixed Point for C_q^* - commuting mapping Mappings in Hyperbolic Ordered Metric Spaces

Savita Rathee [a] and Reetu [b]

[a] Department of Mathematics, M.D. University, Rohtak,
[b] Vaish College, Rohtak (Haryana)-124001, India.

Abstract

In the present paper we give the notion of C_q^* - commuting mapping as a generalization of various existing noncommuting mappings existing in the literature and apply it to find common fixed point and best approximation results in the hyperbolic ordered metric space. Our results unify, generalize and complement various known results. Further, some suitable examples are also given which shows the generality of our proved results over the existing ones.

Keywords: Hyperbolic ordered metric space, Common fixed point, ordered C_q - commuting mapping, ordered C_q^* - commuting mappings, ordered asymptotically S-nonexpansive mapping, ordered best approximation.

Mathematics Subject Classification: 54H25, 41A50, 47H09, 47H10, 47H19.

1. INTRODUCTION:

The Banach fixed point theorem for contraction mappings has been generalized and extended in many directions. Existence of fixed points in ordered metric spaces was first investigated in 2004 by Ran and Reurings [18] and then by Nieto and Lopez [15]. Afterwards, Dorić [7], Radenović and Kadelburg [17] proved some fixed point theorems for generalized weak contractive mappings in ordered metric spaces (see also [20, 21] and references mentioned therein). In 2011, Khamsi and Khan [12] studied some inequalities in hyperbolic metric spaces which lay foundation for a new mathematical field: the application of geometric theory of Banach spaces to fixed point theory.

Since the existence of common fixed point theory, various extensions of commuting maps viz. weakly commuting maps, compatible maps, weakly compatible maps, R-subweakly maps have been appeared in the literature. In 2006, proceeding in the same direction, Al-thagafi and Shahzad [4] defined the notion of C_q - commuting mappings in the following sense:

Definition 1.1: Let (X, d) be a normed linear space, M be a q-starshaped subset of X with $q \in$ Fix(S) (Set of fixed point of S) and $C_q(S, T) = \cup\{C(S,T_k): 0 \leq k \leq 1\}$ where $T_k(x) = kTx + (1-k)\, q$. Then the pair (S, T) of self-mappings on X is said to be C_q – commuting if STx = TSx for all $x \in C_q$ (S, T).

Further, by giving suitable examples they showed that the class of C_q – commuting maps is an extension of the class of R- subweakly maps and is different from the class of compatible as well as weakly compatible maps. They proved the following common fixed point theorem for the pair of C_q – commuting maps.

Theorem 1.2 [4]: Let M be a subset of a normed space X, S and T selfmaps of M and $cl(T(M)) \subseteq S(M)$. Suppose that M is q-starshaped, S and T are C_q – commuting, T is continuous and S-nonexpansive, S is q-affine and cl(T(M)) is compact. Then $Fix(S) \cap Fix(T) \neq \emptyset$.

After that several authors utilized the concept of C_q – commuting maps and extended the above results in several possible ways. In 2011, Abbas, Khamsi and Khan[1] proved the common fixed point and invariant approximation results for C_q - commuting mappings in the setting of hyperbolic ordered metric spaces.

Now, consider the following example:

Example: Let $X = \mathbb{R}$ be endowed with usual ordering and M = [0, 1]. Define S, T: M → M by

$Sx = (1 - x)$ for all $x \in M$.

$$Tx = \begin{cases} 1 - x & \text{if } x \leq \frac{1}{2} \\ x & \text{if } x > \frac{1}{2} \end{cases}$$

Here the mappings S and T have a common fixed point x = $\frac{1}{2}$, but S and T are neither C_q – commuting nor weakly compatible as $C_q(S, T) = [0, \frac{1}{2}]$ but ST(x) ≠ TS(x) for all x ∈ $[0, \frac{1}{2})$. Therefore all of the theorems using the hypotheses of C_q – commuting and weakly compatible maps do not ensure the existence of common fixed point for these type of maps. At this stage a question arises whether it is possible to guarantee the existence of common fixed point for such type of mappings, which are neither C_q - commuting nor weakly compatible.

To answer this problem, we improve the class of C_q - commuting mappings by introducing a new class of noncommuting self- mappings called C_q^* - commuting mappings which contain various commuting as well as noncommuting mappings as a proper subclass. We also prove some common fixed point and best approximation results for this newly introduced mapping in the setting of hyperbolic ordered metric spaces which extend the results of various existing noncommuting mappings.

2. PRELIMINARIES

We now give some known definitions and standard notations which will be needed in the sequel:

Let (X, d) be a metric space. A path joining x ∈ X to y ∈ X is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that c(0) = x, c(l) = y and $d(c(t), c(t')) = |t - t'|$ for all t, $t' \in [0, l]$. In particular, c is an isometry and d(x, y) = l. The image of c is called a metric segment joining x and y. When it is unique, the metric segment is denoted by [x, y]. We shall denote by $(1 - \lambda)\, x \oplus \lambda y$ the unique point z of [x, y] which satisfies $d(x, z) = \lambda d(x, y)$ and $d(z, y) = (1 - \lambda)d(x, y)$.

Such metric spaces are usually called convex metric spaces (see Takahashi [25] and Khan at el. [13]). Moreover, if we have for all p, x, y in X

$$d\left(\frac{1}{2}p \oplus \frac{1}{2}x, \frac{1}{2}p \oplus \frac{1}{2}y\right) \leq \frac{1}{2}d(x, y),$$

then X is called a hyperbolic metric space. It is easy to check that in this case for all x, y, z, w in X and λ ∈ [0, 1]

$$d((1 - \lambda)x \oplus \lambda y, (1 - \lambda)z \oplus \lambda w) \leq (1 - \lambda)d(x, z) + \lambda d(y, w).$$

Obviously, normed linear spaces are hyperbolic spaces [12].

Definition 2.1. A subset Y of a hyperbolic ordered metric space X is said to be an ordered convex if Y includes every metric segment joining any two of its comparable points.

Definition 2.2. A subset Y of a hyperbolic ordered metric space X is said to be an ordered q-starshaped if there exists q in Y such that Y includes every metric segment joining any of its point comparable with q.

Definition 2.3. Let X be a hyperbolic ordered metric space. Then X is said to satisfy property (I) if $(1 - \lambda) x \oplus \lambda y \leq (1 - \lambda)z \oplus \lambda w$ for all x, y, z, w in X with $x \leq z$ and $y \leq w$.

Definition 2.4. A self mapping f on an ordered convex subset Y of a hyperbolic ordered metric space X is said to be affine if $f((1 - \lambda) x \oplus \lambda y) = (1 - \lambda) fx \oplus \lambda fy$ for all comparable elements $x, y \in Y$ and $\lambda \in [0, 1]$.

Definition 2.5. Let X be a hyperbolic ordered metric space and $f, g: X \rightarrow X$. A point $x \in X$ is called:

(1) a fixed point of f if $fx = x$,

(2) a coincidence point of the pair (f, g) if $fx = gx$,

(3) a common fixed point of the pair (f, g) if $x = fx = gx$.

We shall denote by Fix(f) and C(f, g), the set of all fixed points of f and the set of all coincidence points of (f, g) respectively.

Definition 2.6. Let $(X, \leq)$ be an ordered set. A pair (f, g) on X is said to be:

(i) weakly compatible if f and g commute at their coincidence points.

(i) weakly increasing if for all $x \in X$, we have $fx \leq gfx$ and $gx \leq fgx$, ([2])

(ii) partially weakly increasing if $fx \leq gfx$, for all $x \in X$.

Remark 2.7. A pair (f, g) is weakly increasing if and only if the ordered pair (f, g) and (g, f) are partially weakly increasing.

Example 2.8. Let X = [0, 1] be endowed with usual ordering. Let $f, g: X \rightarrow X$ be defined by $fx = x^2$ and $gx = \sqrt{x}$. Then $fx = x^2 \leq x = gfx$ for all $x \in X$. Thus (f, g) is partially weakly increasing. But $gx = \sqrt{x} \nleq x = fgx$ for $x \in (0, 1)$. So (g, f) is not partially weakly increasing.

Definition 2.9. Let $(X, \leq)$ be an ordered set. A mapping f is called weak annihilator of g if $fgx \leq x$ for all $x \in X$.

Example 2.10. Let X = [0, 1] be endowed with usual ordering. Define $f, g: X \rightarrow X$ by $fx = x^2$ and $gx = x^3$. Then $fgx = x^6 \leq x$ for all $x \in X$. Thus f is a weak annihilator of g.

Definition 2.11. Let $(X, \leq)$ be an ordered set. A self mapping f on X is called dominating map if $x \leq fx$ for each x in X.

Example 2.12. Let X = [0, 1] be endowed with usual ordering. Let f: X → X be defined by $fx = x^{\frac{1}{3}}$. Then $x \leq x^{\frac{1}{3}} = fx$ for all x ∈ X. Thus f is a dominating mapping.

Example 2.13. Let X = [0, ∞) be endowed with usual ordering. Define f: X → X by

$$fx = \begin{cases} x^{\frac{1}{n}} & \text{for } x \in [0, 1), \\ x^{n} & \text{for } x \in [1, \infty), \end{cases}$$

n ∈ ℕ. Then x ≤ fx for all x ∈ X, hence f is a dominating mapping.

Definition 2.14. Let (X, ≤) be an ordered set and f, g be self mappings on X. Then the pair (f, g) is said to be order limit preserving if $gx_0 \leq f x_0$, for all sequences $\{x_n\}$ in X with $gx_n \leq fx_n$ and $x_n \to x_0$.

Definition 2.15. Let X be a hyperbolic ordered metric space, Y be an ordered q-starshaped subset of X, S and T be a self mapping on X and q ∈ Fix(S). Then T is said to be:

(1) ordered S-contraction if there exists k ∈ (0, 1) such that

$$d(Tx, Ty) \leq kd(Sx, Sy); \text{ for } x, y \in Y \text{ with } x \leq y.$$

(2) ordered asymptotically S-nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $d(T^n x, T^n y) \leq k_n d(Sx, Sy)$ for each x, y in Y with x ≤ y and each n ∈ ℕ. If S = I (identity map) then T is ordered asymptotically nonexpansive mapping. If $k_n = 1$, for all n ∈ ℕ, then T is known as ordered S-nonexpansive mapping.

(3) ordered R-weakly commuting if there exists a real number R > 0 such that

$$d(TSx, STx) \leq Rd(Tx, Sx) \text{ for all } x \text{ in } Y.$$

(4) ordered R-subweakly commuting [6] if there exists a real number R > 0 such that

$$d(TSx, STx) \leq Rd(Sx, Y_q^{T(x)}) \text{ where}$$

$$Y_q^{T(x)} = \{ y_\lambda : y_\lambda = (1-\lambda)q \oplus \lambda Tx \text{ and } \lambda \in [0, 1], q \leq x \text{ or } x \leq q \text{ for all } x \in Y\}.$$

(5) ordered uniformly R-subweakly commuting [6] if there exists a real number R > 0 such that $d(T^n Sx, T^n Sy) \leq Rd(Sx, Y_q^{T^n(x)})$ for all x ∈ Y.

(6) ordered C_q- commuting [5] if STx = TSx for all $x \in C_q(S, T)$, where

$$C_q(S, T) = \cup\{C(S, T_k): 0 \leq k \leq 1\} \text{ and } T_k(x) = (1-k)q \oplus kTx.$$

(7) ordered uniformly C_q-commuting, if S T^nx = T^n Sx for all x ϵ C_q (S, T) and n ϵ ℕ.

(8) uniformly asymptotically regular on Y if, for each η > 0, there exists ℕ(η) = ℕ such that d(T^nx, T^{n+1}x) < η for all η ≥ ℕ and all x ∈ Y.

Definition 2.16. Let M be a closed subset of an ordered metric space X. Let x ∈ X. Define d(x, M) = inf{d(x, y) : y ∈ M, y ≤ x or x ≤ y}. If there exists an element y_0 in M comparable with x such that d(x, y_0) = d(x, M), then y_0 is called an ordered best approximation to x out of M. We denote by P_M(x), the set of all ordered best approximation to x out of M.

3.1. Common fixed point and best approximation results for C_q^* - commuting mappings:

In this section, we first introduce the notion of C_q^* - commuting mappings.

Definition 3.1.1. Let X be a hyperbolic ordered metric space, M be an ordered q-starshaped subset of X, S and T be a self mapping on X and q ∈ Fix(S). Then the mappings S and T are said to be C_q^* - commuting if STx = TSx for all x ∈ C_q^*(S, T), where C_q^*(S, T) = U{C(S, T_k): 0 < k <1} and T_k(x) = (1 − k)q ⊕ kTx.

In the definition of $\mathbf{C_q}$**- commuting** mappings (see [6]), C_q(S, T) = U{C(S,T_k): 0 ≤ k ≤1}, but here k ∈ (0, 1).The following example reveals the impact of this and shows that R-subweakly commuting mappings and also C_q - commuting mappings form a proper subclass of $\mathbf{C_q^*}$ **- commuting** mappings.

Example 3.1.2. Let X = ℝ be endowed with usual ordering, T_k(x) = (1 − k)q ⊕ kTx and M = [0, ∞). Define S, T: M ⟶ M by

$$Sx = \begin{cases} 2 & x = 1 \\ x & x \neq 1 \end{cases}$$

and

$$Tx = \begin{cases} 2 & x = 1 \\ x^2 & x \neq 1 \end{cases}$$

Then M is q-starshaped for q = 0, q ∈ Fix(S) and C(S, T) = {0, 1}, C(S, T_k) = {$\frac{1}{k}$}, for k ∈ (0, 1). Hence S and T are C_q^*- commuting but not C_q – commuting, since for k = 1, 1 ∈ C_q (S, T), but ST1 ≠ TS1. Note that S and T are neither R – subweakly commuting nor R- subcommuting mappings.

Remark 3.1.3. Weakly compatible self-mappings and C_q^*- commuting mappings are of different classes.

Example 3.1.4. Let X = ℝ be endowed with the usual metric and M = [0, ∞).

Define S, T: M → M by

$$Sx = \begin{cases} 5 & x = 1 \\ x & x \neq 1 \end{cases}$$

and

$$Tx = \begin{cases} 5 & x = 1 \\ x^3 & x \neq 1 \end{cases}$$

Then S and T are C_q^*- commuting but not weakly compatible mappings.

Example 3.1.5. Let X = ℝ be endowed with usual ordering and M = [1, ∞). Define S, T: M → M by S(x) = 2x − 1 and T(x) = x^2 for all x ∈ M. Let q = 1. Then M is 1-starshaped with S1 = 1 and C_q^*(S, T) = (1, ∞). Note that S and T are weakly compatible maps but S and T are not C_q^*– commuting maps.

The following theorem of Abbas, Khamsi and Khan[1] is needed to prove our main result:

Theorem 3.1.6 [1, Theorem 3.1]. . Let (X, ≤, d) be an ordered metric space. Let f, g, S, and T be self mappings on X, (T, f) and (S, g) be partially weakly increasing with f(X) ⊆ T(X), g(X) ⊆ S(X) and dominating maps f and g be weak annihilator of T and S respectively. Also, for every two comparable elements x, y ∈ X,

$$d(fx, gy) \leq hM(x, y), \text{where}$$

$$M(x, y) = \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \tfrac{1}{2}[d(Sx, gy) + d(fx, Ty)]\}$$

for h ∈ [0, 1) is satisfied. If one of f(X), g(X), S(X) or T(X) is complete subspace of X, then {f, S} and {g, T} have unique point of coincidence in X provided that for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $x_n \to u$ implies $x_n \leq u$. Moreover, if {f, S} and {g, T} are weakly compatible, then f, g, S, and T have a common fixed point.

Now we prove our main results for C_q^* - commuting mappings:

Theorem 3.1.7. Let M be a nonempty closed q- starshaped subset of a hyperbolic ordered metric space X satisfying property (I) and T and S be self mappings on M – {q} with q ∈ Fix(S) such that S(M) = M and T(M -{q}) ⊂ S(M –{q}). Let (S, T) be partially weakly increasing, order limit preserving and dominating map T is weakly annihilator of S, T is continuous and asymptotically S-nonexpansive with sequence $\{k_n\}$ and S is an affine mapping. If cl(M - {q}) is compact and S and T are uniformly C_q^* - commuting mappings on M - {q}, then Fix(T) ∩ Fix(S) is nonempty provided that for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies $x_n \leq u$. Then for each n ϵ N, Fix(T_n) ∩ Fix(S) is a singleton.

Proof. For each $n \geq 1$, define a mapping T_n on M by $T_n x = (1-\alpha_n)q \oplus \alpha_n T^n x$, where $\alpha_n = \frac{\lambda_n}{k_n}$ and $\{\lambda_n\}$ is a sequence in (0, 1) with $\lim_{n\to\infty} \lambda_n = 1$.

For all x, y ϵ M, we have

$$d(T_n(x), T_n(y)) = d((1-\alpha_n)q \oplus \alpha_n T^n x, (1-\alpha_n)q \oplus \alpha_n T^n y)$$
$$\leq \alpha_n d(T^n x, T^n y)$$
$$\leq \lambda_n d(Sx, Sy).$$

Moreover, since S and T are uniformly C_q^* - commuting mappings and S is affine on M with Sq = q, for each $x \in C(S, T_n) \subseteq C_q^*(S, T)$, we have

$$ST_n x = S((1-\alpha_n)q \oplus \alpha_n T^n x)$$
$$= (1-\alpha_n)q \oplus \alpha_n ST^n x$$
$$= (1-\alpha_n)q \oplus \alpha_n T^n Sx = T_n Sx.$$

Thus S and T_n are weakly compatible for all n. Therefore, by Theorem 3.1.6, there exists x_n in M such that x_n is a common fixed point of S and T_n for each $n \geq 1$ i.e.,

$$x_n = T_n x_n = Sx_n = (1-\alpha_n)q \oplus \alpha_n T^n x_n,$$

Also,

$$d(x_n, T^n x_n) = d((1-\alpha_n)q \oplus \alpha_n T^n x_n, T^n x_n)$$
$$= (1-\alpha_n)\, d(q, T^n x_n).$$

Since T(M –{q}) is bounded, $d(x_n, T^n x_n) \to 0$ as $n \to \infty$. Also,

$$d(x_n, Tx_n) \leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n)$$
$$\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(ST^n x_n, Sx_n)$$
$$\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(ST^n x_n, S((1-\alpha_n)q \oplus \alpha_n T^n x_n))$$
$$\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(ST^n x_n, (1-\alpha_n)q \oplus \alpha_n ST^n x_n)$$
$$\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 (1-\alpha_n) d(ST^n x_n, Sq),$$

which implies that $d(x_n, Tx_n) \to 0$ as $n \to \infty$. As cl(M −{q}) is compact and M is closed, therefore there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x_0 \in Y$ as $i \to \infty$. By the continuity of T, we have $T(x_0) = x_0$. Since T is dominating map, therefore $Sx_k \leq TSx_k$. As T is weak annihilator of S, T is dominating, so $TSx_k \leq x_k \leq Tx_k$. Thus, $Sx_k \leq Tx_k$ and order limit preserving property of (T, S) imply that $Sx_0 \leq Tx_0 = x_0$. Also, $x_0 \leq Sx_0$. Consequently, $Sx_0 = Tx_0 = x_0$. Hence, the result follows.

Theorem 3.1.8. Let M be a nonempty q-starshaped complete subset of a hyperbolic ordered metric space and S, T be selfmaps on M. Suppose that T is continuous, cl(T(M)) is compact and S is affine as well as continuous and $T(M) \subset S(M)$. Let (S, T) be partially weakly increasing and dominating map T be weak annihilators of S. If the pairs {S, T} be C_q^* - commuting and satisfies for all x, y ϵ M

$$d(Tx, Ty) \leq \max\{d(Sx, Sy), dist(Sx, {Y_q}^{Tx}), dist(Sy, {Y_q}^{Ty}), \tfrac{1}{2}[dist(Sx, {Y_q}^{Ty}) + dist(Sy, {Y_q}^{Tx})]\} \quad (3.1)$$

Then S and T have a common fixed point provided that for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies $x_n \leq u$. .

Proof. Define $T_n : M \to M$ by $T_n x = (1 - \lambda_n)q \oplus \lambda_n Tx$, where $\lambda_n \epsilon (0, 1)$ with $\lim_{n\to\infty} \lambda_n = 1$. Since M is q-starshaped, T_n is the self mapping on M for each $n \geq 1$. Since S is affine on M with Sq = q then for $x \epsilon C(S, T_n) \subseteq C_q(S, T)$, we have

$$ST_n x = S((1 - \lambda_n)q \oplus \lambda_n Tx) = (1 - \lambda_n)q \oplus \lambda_n STx = (1 - \lambda_n)q \oplus \lambda_n TSx = T_n Sx.$$

Thus S and T_n are weakly compatible for all n. Also, we have

$$\begin{aligned} d(T_n x, T_n y) &= d((1 - \lambda_n)q \oplus \lambda_n Tx, (1 - \lambda_n)q \oplus \lambda_n Ty) \\ &\leq \lambda_n d(Tx, Ty) \\ &\leq \lambda_n \max\{d(Sx, Sy), d(Sx, {Y_q}^{Tx}), d(Sy, {Y_q}^{Ty}), \tfrac{1}{2}[d(Sx, {Y_q}^{Ty}) + d(Sy, {Y_q}^{Tx})]\} \\ &\leq \lambda_n \max\{d(Sx, Sy), d(Sx, T_n x), d(Sy, T_n y), \tfrac{1}{2}[d(Sx, T_n y) + d(Sy, T_n x)]\} \end{aligned}$$

By Theorem 3.1.6, for each $n \geq 1$, there exists x_n in M such that x_n is a common fixed point of S and T_n. The compactness of cl(T(M)) implies that there exists a subsequence $\{Tx_k\}$ of $\{Tx_n\}$ such that $Tx_k \to y$ as $k \to \infty$. Now, the definition of $T_k x_k$ gives that $x_k \to y$. Hence, we have $y \in Fix(S) \cap Fix(T)$ by the continuity of S and T.

Example 3.1.9. Let X = ℝ be endowed with the usual ordering and M = [0, 1]. Define S, T: M → M by Sx = 1 - x, for all x ϵ M and

$$Tx = \begin{cases} 1 - x & \text{if } x \le \frac{1}{2} \\ x & \text{if } x > \frac{1}{2} \end{cases}$$

Then S(M) = [0, 1], T(M) = [0, $\frac{1}{2}$] so that cl(T(M)) ⊆ S(M) and cl(T(M) is compact. Also M is q-starshaped for q = $\frac{1}{2}$, q ∈ Fix(S), S is affine on M. Also, T is dominating map and weak annihilators of S, the pair {S, T} is partially weakly increasing and the mappings satisfies the inequality (3.1). It is also to be noted that the mappings S and T are neither C_q – commuting nor weakly compatible but these are C_q^* - commuting with C_q^*(S, T) = {$\frac{1}{2}$}. Therefore, by above theorem S and T have a common fixed point and $\frac{1}{2}$ is such the unique fixed point.

Remark 3.1.9. Theorems 3.1.7 and 3.1.8 extend and improve the results of Abbas et. al.[1], Hussain and Rhoades[11], Al-Thagafi's [3, Theorem 2.2], Habiniak's [7, Theorem 4], Hussain and Jungck [10], O'Regan and Shahzad's [16, Theorem 2.2], Shahzad's [23, Theorem 2.2], Al-Thagafi and Shahzad's, [4, Theorem 2.2] and the main result of Rhoades and Saliga [19].

3.2. Best Approximation Results for C_q^* - commuting mappings:

Now, we prove best approximation results for C_q^* - commuting mappings:

Theorem 3.2.1. Let M be a nonempty subset of a of a hyperbolic ordered metric space X and S, T: X → X be continuous mappings such that T(∂M ∩M) ⊂ M, ∂M stands for boundary of M, u ∈ Fix(S) ∩ Fix(T) for some u ∈ X where u is comparable with all x ∈ X. Let (S, T) be partially weakly increasing, order limit preserving, T is uniformly asymptotically regular, asymptotically S-nonexpansive and S is affine on P_M (u) with S(P_M(u)) = P_M(u), q ∈ Fix(S) and P_M(u) is q-starshaped. If cl(P_M(u)) is compact, P_M(u) is complete and S and T are uniformly C_q^* - commuting on P_M(u)∪{u} satisfying d(Tx, Tu) ≤ d(Sx, Su), then P_M(u)∩Fix(T)∩Fix(S) ≠ ∅ provided that for a nondecreasing sequence {x_n} with $x_n \le y_n$ for all n and $y_n \to u$ implies $x_n \le u$.

Proof. Firstly, we show that T is a self mapping on P_M(u), i.e., T: P_M(u) → P_M(u). To do this, let x ϵ P_M(u). Then d(x, u) = d(u, M). Note that for any λ ϵ (0, 1)

$$d(y_\lambda, u) = d((1 - \lambda)u \oplus \lambda x, u)$$
$$= \lambda d(x, u) < d(x, u) = d(u, M).$$

This shows that M and y_x^λ = {y_λ : y_λ = (1 − λ)u ⊕ λx}are disjoint. So, x ∈ ∂M ∩ M which further implies that Tx ϵ M. Since Sx ϵ P_M(u), u is a common fixed point of S and T, therefore by the given contractive condition, we obtain

$$d(Tx, u) = d(Tx, Tu) \leq d(Sx, Su) = d(Sx, u) = d(u, M),$$

which shows that Tx ϵ P_M(u).Thus, P_M(u) is T -invariant. Hence, T(P_M(u)) ⊂ P_M(u) = S(P_M(u)). Now the result follows from Theorem 3.1.7.

Theorem 3.2.2: Let M be a nonempty subset of a of a hyperbolic ordered metric space X, T and S be two self mappings on X such that T(∂M ∩ M) ⊂ M and u ∈ Fix(S) ∩Fix(T) for some u in X. Suppose T is continuous, S is affine on P_M(u) with S(P_M(u)) = P_M(u) and P_M(u) is q-starshaped with q ϵ Fix(S). Let (S, T) be partially weakly increasing and dominating map T be weak annihilators of S. Assume that the pair (S,T) is C_q^* - commuting mapping on P_M(u)∪{u} and satisfies

$$d(Tx, Ty) \leq \begin{cases} d(Sx, Su)\ , & \text{if } y = u, \\ \max\{d(Sx, Sy), d(Sx, {Y_q}^{Tx}), d(Sy, {Y_q}^{Ty}), & \\ \quad \frac{1}{2}[d(Sx, {Y_q}^{Ty}) + d(Sy, {Y_q}^{Tx})]\} & \text{if } y \in P_M(u) \end{cases} \tag{3.2}$$

for all x ∈ P_M(u) ∪{u}. If cl(P_M(u)) is compact and P_M(u) is complete then P_M(u) ∩ Fix(S) ∩ Fix(T) is nonempty provided that for a nondecreasing sequence {x_n} with $x_n \leq y_n$ for all n and $y_n \to u$ implies $x_n \leq u$.

Proof: Let x ϵ P_M(u). Then d(x, u) = d(u, M). Note that for any λ ϵ (0, 1),

$$d(y_\lambda, u) = d((1-\lambda)u \oplus \lambda x, u)$$
$$= \lambda d(x, u) < d(x, u) = d(u, M).$$

which shows that Y_u^x ={ y_λ : y_λ = (1 − λ)u ⊕ λx } ∩ M is empty so x ϵ (∂M ∩ M) and so Tx ϵ M. Since Sx ϵ P_M(u), u is common fixed point of S and T, therefore from the given contractive condition, we obtain

d(Tx, u) = d(Tx, Tu) ≤ d(Sx, Su) = d(Sx, u) = d(u, M).Thus P_M(u) is T-invariant. Hence, T(P_M(u)) ⊂ P_M(u) = S(P_M(u)). Now the result follows from Theorem 3.1.8.

Example 3.2.3. Let X = ℝ be endowed with the usual ordering metric and M = [0, ∞). Define S, T: X → X by S(x) = x for all x ϵ X and

$$Tx = \begin{cases} -1 & x = -1 \\ 0 & x \neq -1 \end{cases}$$

Then clearly Fix(S) = ℝ and Fix(T) = {0, -1} and T(∂M ∩M) = T(0) = 0 ⊂ M. Take u = -1 ∈ Fix(S) ∩ Fix(T) = {-1, 0}. Then D = P_M(u) = {0}. We also observe that P_M(u) is closed, q-starshaped for q = 0, q ∈ Fix(S) and S(D) = D. Also, S is affine and continuous on $P_M(u)$, (S, T) is partially weakly increasing, T is dominating map and weak annihilator of S. Further cl(T(D)) ={0} is compact, T is continuous on D, S and T are C_q^*- commuting and satisfy the inequality (3.2) for all x, y ∈ D ∪{u}. Thus all the conditions of the theorem 3.2.2 are satisfied and hence $P_M(u) \cap Fix(S) \cap Fix(T) \neq \emptyset$. Here 0 is such a point.

Remark 3.2.4. Theorems 3.2.1 and 3.2.2 extend and improve the results of Abbas et. al. [1], Shahzad [22, Theorem 3], Hussain and Rhoades [11, Theorem 2.6], Anderson et. al. [5, Theorem 3.1], Habiniak [9, Theorem 8], Hussain and Jungck [10] and] Shahzad [24, Theorem 6].

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