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On the Local-in-time Solution to Primitive Equations for the Ocean with Free Surface

Hirotada Honda1[1]
NTT Network Technology Laboratories
3-9-11 Midori-Cho, Musashino-Shi, Tokyo 180-8585 JAPAN

Abstract

The small time existence and the regularity of a strong solution to the free surface problem of primitive equations for the ocean are discussed in this paper. Following formulations of practical models such as OGCM (Oceanic General Circulation Model), we impose the kinematic condition on the free ocean surface, and apply the equation of state in the general form.

Keywords: Primitive Equations, Sobolev-Slobodetskiĭ space, strong solution.

1. INTRODUCTION

This paper concerns the small time existence of a strong solution to the free surface problem of primitive equations for the ocean, which describes the temporal motion and state of the ocean [3], [23].

The novelties of this paper are following issues:

(i) We show that the local-in-time solvability of the free ocean surface model under some initial and boundary conditions;

(ii) the general form of the equation of state is applied;

Since the memorable contributions to the mathematical argument of the primitive equations by Lions, Temam and Wang [17] [18], there have been a number of works concerning the primitive equations for the ocean, the atmosphere, and the coupled model of the ocean and the atmosphere in the mathematical literature. In the present paper, we investigate the free surface problem of the primitive equation for the ocean with the general form of the equation of state and construct a strong local-in-time

[1]Corresponding author E-mail: honda.hirotada@lab.ntt.co.jp, Phone: +081 422 59 3981, Fax: +081 422 59 6364

solution in the anisotropic Sobolev-Slobodetskiĭ spaces. In addition, the vertical velocity, whose regularity was lower than that of the horizontal ones in the preceding works, is shown to have the same regularity as that of the horizontal ones. We first consider the problem in the transformed coordinate system, which makes the ocean region to be fixed and known. In order to show the improved regularity of the vertical velocity, a new unknown variable η_h with a parameter h is introduced. Then, we consider linear problems for each unknown variables in the fixed region of the transformed coordinate system. In order to estimate η_h, some technical calculations are necessary, including the estimates in the Sobolev-Slobodetskiĭ spaces defined on non-cylindrical domains. We finally consider the successive approximation of the problem, which converges uniformly with respect to h in the desired function space.

This paper is organized as follows: in the next section, we formulate the problem. In section 3, we define the function spaces used throughout this paper. In Section 4, we state the main result of this paper, followed by the proof of the main theorem in section 6.

2. FORMULATION OF PROBLEM

Our problem is formulated in the 3-dimensional strip-like region. By $x = (x', x_3) = (x_1, x_2, x_3)$, we denote an orthogonal Cartesian coordinate system with x_3 being the vertical direction. Let the unknown free ocean surface and the known bottom of the ocean be represented by the equations $x_3 = F(x', t)$ and $x_3 = b(x')$, respectively. The initial value $F_0(x')$ of $F(x', t)$ is assumed to satisfy $F_0(x') - b(x') > c_0$ with a positive constant c_0 for any $x' \in \boldsymbol{R}^2$. Then the domain $\Omega(t)$ of the ocean at time t is represented as

$$\Omega(t) = \{(x', x_3) | x' \in \boldsymbol{R}^2, b(x') < x_3 < F(x', t)\}.$$

The equations that we consider in the present paper are as follows:

$$\begin{cases} \dfrac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + w\dfrac{\partial \boldsymbol{v}}{\partial x_3} - \left[\mu_1 \Delta \boldsymbol{v} + \mu_2 \dfrac{\partial^2 \boldsymbol{v}}{\partial x_3^2}\right] + f_0 \boldsymbol{A}\boldsymbol{v} = -\dfrac{1}{\varrho_0}\nabla p, \\ \dfrac{\partial p}{\partial x_3} = -\varrho g, \\ \nabla \cdot \boldsymbol{v} + \dfrac{\partial w}{\partial x_3} = 0, \\ \dfrac{\partial \theta}{\partial t} + (\boldsymbol{v} \cdot \nabla)\theta + w\dfrac{\partial \theta}{\partial x_3} - \left[\mu_3 \Delta \theta + \mu_4 \dfrac{\partial^2 \theta}{\partial x_3^2}\right] = 0, \\ \dfrac{\partial S}{\partial t} + (\boldsymbol{v} \cdot \nabla)S + w\dfrac{\partial S}{\partial x_3} - \left[\mu_5 \Delta S + \mu_6 \dfrac{\partial^2 S}{\partial x_3^2}\right] = 0, \\ \varrho = \varrho(p, \theta, S) \quad x \in \Omega(t), t > 0. \end{cases} \tag{2.1}$$

Here, $f_0\boldsymbol{A}\boldsymbol{v}$ is a Coriolis force with $\boldsymbol{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the Coriolis parameter f_0 is a positive constant due to the f-approximation; ∇ and Δ are 2-dimensional gradient and Laplacian, respectively. The horizontal component of the velocity is represented by $\boldsymbol{v} = (v_1, v_2)^{\mathrm{T}}$ and the vertical one, by w; p, the pressure; $\varrho = \varrho(z_1, z_2, z_3)$, the density; $\varrho_0 > 0$, a constant of the representative value of the density; g, a positive constant denoting the acceleration due to gravity; θ, the temperature; S, the salinity; μ_1 and μ_2, coefficients of turbulent viscosity; (μ_3, μ_4) and (μ_5, μ_6), scaling sums of turbulent and molecular diffusivities, respectively with μ_i $(i = 1,2, \ldots ,6)$ being positive constants. Note that the equation of state is provided in a general form in (2.1) $_6$. Conditions on the free ocean surface $\Gamma(t) = \{x \in \boldsymbol{R}^3 | x_3 = F(x', t), t > 0\}$ are as follows (ses, for instance, [3], [23]):

$$\begin{cases} \mu_2 \dfrac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}_F} = \boldsymbol{\tau}_1(x,t), \ \mu_4 \dfrac{\partial \theta}{\partial \boldsymbol{n}_F} = \tau_2(x,t), \\ \mu_6 \dfrac{\partial S}{\partial \boldsymbol{n}_F} = g_1(x,t) S(x,t), \ p = p_0(x,t) \quad x \in \Gamma(t), \ t > 0. \end{cases} \tag{2.2}$$

$$\boldsymbol{n}_F = (n_1, n_2, n_3)^{\mathrm{T}} = \frac{1}{\sqrt{1 + |\nabla F|^2}} (\frac{\partial F}{\partial x_1}, \ \frac{\partial F}{\partial x_2}, \ -1)^{\mathrm{T}} \tag{2.3}$$

is the unit normal vector to $\Gamma(t)$ at time t pointing to the ocean region, and $p_0(x,t)$, the atmospheric pressure at the ocean surface. $\tau_1 \in \boldsymbol{R}^2$ is the wind shear due to the movement of the atmosphere over the ocean; $\tau_2 \in \boldsymbol{R}$, the heat flux on the ocean surface; $g_1(x,t)$, a given function representing the difference of the precipitation and evaporation rate. In addition, the kinematic condition is imposed ([3], [15], [23]),

$$\frac{D}{Dt}(x_3 - F(x', t)) = 0, \tag{2.4}$$

where $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla + w \frac{\partial}{\partial x_3}$ is an operator known as the material derivative. Following Washington and Parkinson [23], conditions on the bottom of the ocean $\Gamma_b = \{(x', b(x')) | x' \in \boldsymbol{R}^2\}$ are

$$\boldsymbol{v} \cdot \nabla b + w = 0, \quad \mu_2 \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}_b} = 0, \ \mu_4 \frac{\partial \theta}{\partial \boldsymbol{n}_b} = \mu_6 \frac{\partial S}{\partial \boldsymbol{n}_b} = 0, \quad x \in \Gamma_b, t > 0, \tag{2.5}$$

where

$$\boldsymbol{n}_b = \frac{1}{\sqrt{1 + |\nabla b|^2}} (\nabla b, \ -1)^{\mathrm{T}}.$$

Initial conditions are

$$\begin{cases}(\boldsymbol{v},\theta,S)(x,0)=(\boldsymbol{v}_0,\theta_0,S_0)(x), & x\in\Omega\equiv\Omega(0),\\ F(x',0)=F_0(x'), & x'\in\boldsymbol{R}^2.\end{cases} \tag{2.6}$$

We apply the transform $\Phi:(x,t)\longmapsto(y,t)$ of the coordinate system similar to the one used in the preceding papers [11], [12] to (2.1)–(2.6):

$$y'=x',\quad y_3=(b(x')-F_0(x'))\frac{x_3-F(x',t)}{b(x')-F(x',t)}+F_0(x'), \tag{2.7}$$

which is similar to that used by Beale [1]. By this transformation, and making use of the representation

$$x_3=\frac{(y_3-F_0(y'))(b(y')-F(y',t))}{b(y')-F_0(y')}+F(y',t)\equiv X_3^{(F)}(y,t),$$

it is clear that, for an arbitrary $T>0$, the regions

$$\bigcup_{0<t<T}(\Omega(t)\times\{t\}),\ \bigcup_{0<t<T}(\Gamma_b\times\{t\}),\ \bigcup_{0<t<T}(\Gamma(t)\times\{t\})$$

are transformed onto the regions

$$\widetilde{\Omega}_T\equiv\widetilde{\Omega}\times(0,T),\ \widetilde{\Gamma}_{bT}\equiv\widetilde{\Gamma}_b\times(0,T),\ \widetilde{\Gamma}_T\equiv\widetilde{\Gamma}\times(0,T),$$

respectively, where

$$\widetilde{\Omega}=\{(y',y_3)|y'\in\boldsymbol{R}^2,\ b(y')<y_3<F_0(y')\},\ \widetilde{\Gamma}_b=\{(y',y_3)|y'\in\boldsymbol{R}^2,\ y_3=b(y')\},$$
$$\widetilde{\Gamma}=\{(y',y_3)|y'\in\boldsymbol{R}^2,\ y_3=F_0(y')\}.$$

We denote the inverse of transposed matrix of the Jacobian matrix by

$$(J[(x/y)]^{\mathrm{T}})^{-1}=(a^{ij})=(a^{ij}(F))\quad(i,j=1,2,3).$$

Then one can easily derive

$$\boldsymbol{a}^3(F)\equiv(a^{13}(F),\ a^{23}(F))^{\mathrm{T}}=\frac{(F_0(y')-F(y',t))(y_3-F_0(y'))}{(b(y')-F(y',t))(b(y')-F_0(y'))}\nabla b(y')$$
$$+(b(y')-y_3)\left\{\frac{\nabla F_0(y')}{b(y')-F_0(y')}-\frac{\nabla F(y',t)}{b(y')-F(y',t)}\right\},$$
$$a^{33}(F)=\frac{b(y')-F_0(y')}{b(y')-F(y',t)},\quad a^{ij}=\delta_{ij}\quad(i=1,2,\ j=1,2,3),$$

$$A_1 = A_1(F) \equiv \frac{\partial y_3}{\partial t} = \frac{y_3 - b(y')}{b(y') - F(y',t)} \frac{\partial F}{\partial t}(y',t).$$

Hereafter δ_{ij} stands for the Kronecker's delta. In the following, we use the notation

$$(\nabla_F,\ \nabla_{F,3})^{\mathrm{T}} = (J[(x/y)]^{\mathrm{T}})^{-1}\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}\right)^{\mathrm{T}}.$$

Hereafter, we simply denote $\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right)$ by ∇ in the case it is obvious. Except for unknown variables, a function $f(x,t)$ after the coordinate transform is denoted by $\tilde{f}^{(F)}(y,t)$ in order to represent its dependency on F. For a function defined in the whole space $\boldsymbol{R}^3$ in the original coordinate system, we use the same notation to the one restricted on $\Omega(t)$ at each t and then transfomed into the new coordinate system.

3. FUNCTION SPACES

Let us introduce the function spaces used in this paper. Let $T > 0$ and G be a domain in $\boldsymbol{R}^n$ $(n = 2,3)$. By $W_2^l(G)$ we mean a space of functions $u(x)$, $x \in G$, equipped with the norm $\| u \|^2_{W_2^l(G)} = \sum_{|\alpha|<l} \| D_x^\alpha u \|^2_{L_2(G)} + \| u \|^2_{\dot{W}_2^l(G)}$, where D_x^α denotes the usual multi-index of derivatives with respect to x, and

$$\begin{cases} \| u \|^2_{\dot{W}_2^l(G)} = \sum_{|\alpha|=l} \| D_x^\alpha u \|^2_{L_2(G)} = \sum_{|\alpha|=l} \int_G | D_x^\alpha u(x)|^2 \mathrm{d}x \text{ if } l \text{ is an integer}, \\ \| u \|^2_{\dot{W}_2^l(G)} = \sum_{|\alpha|=[l]} \int_G \int_G \frac{|D_x^\alpha u(x) - D_x^\alpha u(y)|^2}{|x-y|^{n+2\{l\}}} \mathrm{d}x\mathrm{d}y \text{ if } l \text{ is a non-integer}, \\ l = [l] + \{l\},\ 0 < \{l\} < 1. \end{cases}$$

We also define the following function spaces for $m > 1$:

$$\overline{W}_2^m(G) = \left\{ u(x) \left| \begin{array}{c} \| u \|^2_{\overline{W}_2^m(G)} \equiv \sup_{x\in G} | u(x)|^2 + \| u \|^2_{\dot{W}_2^{m-[m]}(G)} \\ + \sum_{|\alpha|=1} \| D_x^\alpha u \|^2_{W_2^{m-1}(G)} < \infty \end{array} \right. \right\}.$$

Next we introduce anisotropic Sobolev–Slobodetskiĭ *spaces* $W_2^{l,\frac{l}{2}}(G_T) \equiv W_2^{l,0}(G_T) \cap W_2^{0,\frac{l}{2}}(G_T)$ $(G_T \equiv G \times (0,T))$, whose norms are defined by

$$\| u \|^2_{W_2^{l,\frac{l}{2}}(G_T)} = \int_0^T \| u(\cdot,t) \|^2_{W_2^l(G)} \mathrm{d}t + \int_G \| u(x,\cdot) \|^2_{W_2^{\frac{l}{2}}(0,T)} \mathrm{d}x$$

$$\equiv \| u \|^2_{W_2^{l,0}(G_T)} + \| u \|^2_{W_2^{0,\frac{l}{2}}(G_T)}.$$

We also define function spaces:

$$\overline{W}_2^{m,\frac{m}{2}}(G_T) = \left\{ u(x,t) \left| \begin{array}{c} \| u \|^2_{\overline{W}_2^{m,\frac{m}{2}}(G_T)} \equiv \sup_{G_T} | u(x,t)|^2 \\ + \sup_{x\in G} \| u(x,\cdot) \|^2_{\dot{W}_2^{\frac{m-[m]}{2}}(0,T)} + \sup_{t\in(0,T)} \| u(\cdot,t) \|^2_{\dot{W}_2^{m-[m]}(G)} \\ + \sum_{|\alpha|=1} \|D_x^\alpha u\|^2_{W_2^{m-1,\frac{m-1}{2}}(G_T)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{W_2^{m-2,\frac{m-2}{2}}(G_T)} < \infty \end{array} \right. \right\},$$

for $m > 2$, and

$$\widetilde{W}_2^{m,\frac{m}{2}}(G_T) = \left\{ u(x,t) \in W_2^{m,\frac{m}{2}}(G_T) \left| \| u \|^2_{\widetilde{W}_2^{m,\frac{m}{2}}(G_T)} \equiv \| u \|^2_{W_2^{m,\frac{m}{2}}(G_T)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{W_2^{m-1,\frac{m-1}{2}}(G_T)} < \infty \right. \right\},$$

for $m > 1$.

For the convenience, k times product of a function space W is denoted by W^k. We also denote the product of function spaces W_1 and W_2 by

$$\prod_{i=1}^{2} W_i \equiv W_1 \times W_2.$$

Norms of the vector and product spaces are defined by the standard vector norm and the sum of the norm of the each space, respectively. Function spaces defined on a non-cylindrical domain $\Omega_{x,T} \equiv \bigcup_{0<t<T}(\Omega(t) \times \{t\})$, which are used to solve the free boundary problems (4.1)–(4.4) in Sections 4, are defined by the local charts (see, [19]).

4. MAIN RESULT

Now, denoting $(\boldsymbol{v}, w, \theta, S)$ after the coordinate transform by $(\boldsymbol{u}, u_3, \tilde{\theta}, \tilde{S})$, and introducing notations $\mathcal{U} \equiv (\boldsymbol{u}, \tilde{\theta}, \tilde{S})^{\mathrm{T}}$ and $\mathcal{U}_0 \equiv (\boldsymbol{v}_0, \theta_0, S_0)^{\mathrm{T}}$, we rewrite (2.1)–(2.6) in y-coordinates. Note that the initial data is invariant under the coordinate transform.

$$\begin{cases}\dfrac{\partial \boldsymbol{u}}{\partial t}-L_{1,F}\boldsymbol{u}=\boldsymbol{G}_{1,F}(\boldsymbol{u},u_3,\tilde{\theta},\tilde{S})\\ \equiv -A_1\dfrac{\partial \boldsymbol{u}}{\partial y_3}-(\boldsymbol{u}\cdot\nabla_F)\boldsymbol{u}-u_3a^{33}\dfrac{\partial \boldsymbol{u}}{\partial y_3}-f_0A\boldsymbol{u}-\dfrac{1}{\varrho_0}\nabla_F\tilde{p}^{(F)},\\ a^{33}\dfrac{\partial \tilde{p}^{(F)}}{\partial y_3}=-\varrho(\tilde{p}^{(F)},\tilde{\theta},\tilde{S})g\equiv\tilde{\varrho}g,\\ \nabla_F\cdot\boldsymbol{u}+a^{33}\dfrac{\partial u_3}{\partial y_3}=0,\\ \dfrac{\partial\tilde{\theta}}{\partial t}-L_{2,F}\tilde{\theta}=G_{2,F}(\boldsymbol{u},u_3,\tilde{\theta})\equiv -A_1\dfrac{\partial\tilde{\theta}}{\partial y_3}-(\boldsymbol{u}\cdot\nabla_F)\tilde{\theta}-u_3a^{33}\dfrac{\partial\tilde{\theta}}{\partial y_3},\\ \dfrac{\partial\tilde{S}}{\partial t}-L_{3,F}\tilde{S}=G_{3,F}(\boldsymbol{u},u_3,\tilde{S})\equiv -A_1\dfrac{\partial\tilde{S}}{\partial y_3}-(\boldsymbol{u}\cdot\nabla_F)\tilde{S}-u_3a^{33}\dfrac{\partial\tilde{S}}{\partial y_3}\quad\text{in }\tilde{\Omega}_T,\\ \dfrac{\partial F}{\partial t}+\boldsymbol{u}\cdot\nabla F-u_3=0\quad\text{in }\boldsymbol{R}_T^2,\end{cases}\tag{4.1}$$

$$\begin{cases}\mathcal{B}_{11,F}\boldsymbol{u}\equiv\mu_2D_F\boldsymbol{u}\cdot\boldsymbol{n}_F=\tilde{\boldsymbol{\tau}}_1^{(F)},\ \mathcal{B}_{12,F}\tilde{\theta}\equiv\mu_4D_F\tilde{\theta}\cdot\boldsymbol{n}_F=\tilde{\tau}_2^{(F)},\\ \mathcal{B}_{13,F}\tilde{S}\equiv\mu_6D_F\tilde{S}\cdot\boldsymbol{n}_F=\tilde{g}_1^{(F)}\tilde{S},\ \tilde{p}^{(F)}=\tilde{p}_0^{(F)}\quad\text{on }\tilde{\Gamma}_T,\end{cases}\tag{4.2}$$

$$\begin{cases}\boldsymbol{u}\cdot\nabla_Fb+u_3=0,\ \mathcal{B}_{21,F}\boldsymbol{u}\equiv\mu_2D_F\boldsymbol{u}\cdot\boldsymbol{n}_b=0,\\ \mathcal{B}_{22,F}\tilde{\theta}\equiv\mu_4D_F\tilde{\theta}\cdot\boldsymbol{n}_b=0,\ \mathcal{B}_{23,F}\tilde{S}\equiv\mu_6D_F\tilde{S}\cdot\boldsymbol{n}_b=0\quad\text{on }\tilde{\Gamma}_{bT},\end{cases}\tag{4.3}$$

$$\begin{cases}\mathcal{U}_0(y)\equiv(\boldsymbol{u},\tilde{\theta},\tilde{S})|_{t=0}=(\boldsymbol{v}_0(y),\theta_0(y),S_0(y))\quad\text{on }\tilde{\Omega},\\ F|_{t=0}=F_0(y')\quad\text{on }\boldsymbol{R}^2,\end{cases}\tag{4.4}$$

where

$$L_{i,F}\equiv\mu_{2i-1}\nabla_F^2+\mu_{2i}\left(a^{33}(F)\frac{\partial}{\partial y_3}\right)^2\quad(i=1,2,3),$$

$$D_F\equiv(\nabla_F,\ a^{33}(F)\frac{\partial}{\partial y_3})^{\mathrm{T}}.$$

Note that F is invariant under the coordinate transform (2.7).

Now it is to be noted that we can extend $(\boldsymbol{v}_0,\theta_0,S_0)(y)$ into the half space $t>0$ preserving the regularity, which is denoted by $(\bar{\boldsymbol{u}}_0,\bar{\tilde{\theta}}_0,\bar{\tilde{S}}_0)(y,t)$ [24]. We also introduce notations $\overline{\mathcal{U}}_0\equiv(\bar{\boldsymbol{u}}_0,\bar{\tilde{\theta}}_0,\bar{\tilde{S}}_0)^{\mathrm{T}}$, $\mathcal{U}'=(\boldsymbol{u}',\tilde{\theta}',\tilde{S}')^{\mathrm{T}}\equiv\mathcal{U}-\overline{\mathcal{U}}_0$ and $F'\equiv F-\overline{F}_0$. Then, after solving $\tilde{p}^{(F)}$ explicitly making use of (4.1) $_2$ and (4.2) $_4$, (4.1)–(4.4) becomes the following one for $\mathcal{U}'$.

$$\begin{cases} \dfrac{\partial \mathcal{U}'}{\partial t} - \mathcal{L}_F \mathcal{U}' = \mathcal{G}_{1,u_3,F}\mathcal{U}' + \mathcal{L}_F \overline{\mathcal{U}}_0 - \dfrac{\partial \overline{\mathcal{U}}_0}{\partial t} \quad \text{in } \widetilde{\Omega}_T, \\ \mathcal{B}_{1,F}\mathcal{U}' = \mathcal{G}_{2,F,\tilde{S}}\overline{\mathcal{U}}_0 \quad \text{on } \tilde{\Gamma}_T, \\ \mathcal{B}_{2,F}\mathcal{U}' = \mathcal{G}_{3,F}\overline{\mathcal{U}}_0 \quad \text{on } \tilde{\Gamma}_{bT}, \\ \mathcal{U}'|_{t=0} = (\mathbf{0}, 0,0)^{\mathrm{T}} \end{cases} \tag{4.5}$$

where $(\boldsymbol{u}, \tilde{\theta}, \tilde{S}) = (\boldsymbol{u}' + \bar{\boldsymbol{u}}_0, \tilde{\theta}' + \bar{\tilde{\theta}}_0, \tilde{S}' + \bar{\tilde{S}}_0)$, and

$$\mathcal{L}_F\mathcal{U}' \equiv [L_{1,F}(\boldsymbol{u} - \bar{\boldsymbol{u}}_0), L_{2,F}(\tilde{\theta} - \bar{\tilde{\theta}}_0), L_{3,F}(\tilde{S} - \bar{\tilde{S}}_0)]^{\mathrm{T}},$$
$$\mathcal{B}_{1,F}\mathcal{U}' \equiv [\mathcal{B}_{11,F}\boldsymbol{u}', \mathcal{B}_{12,F}\tilde{\theta}', \mathcal{B}_{13,F}\tilde{S}']^{\mathrm{T}}, \quad \mathcal{B}_{2,F}\mathcal{U}' \equiv [\mathcal{B}_{21,F}\boldsymbol{u}', \mathcal{B}_{22,F}\tilde{\theta}', \mathcal{B}_{23,F}\tilde{S}']^{\mathrm{T}},$$
$$\mathcal{G}_{1,u_3,F}\mathcal{U}' \equiv [\boldsymbol{G}_{1,F}(\boldsymbol{u}, u_3, \tilde{\theta}, \tilde{S}), G_{2,F}(\boldsymbol{u}, u_3, \tilde{\theta}), G_{3,F}(\boldsymbol{u}, u_3, \tilde{S})]^{\mathrm{T}},$$
$$\mathcal{G}_{2,F,\tilde{S}}\overline{\mathcal{U}}_0 \equiv [\tilde{\boldsymbol{\tau}}_1^{(F)} + \mu_2 D_F \bar{\boldsymbol{u}}_0 \cdot \boldsymbol{n}_F,\ \tilde{\tau}_2^{(F)} + \mu_4 D_F \bar{\tilde{\theta}}_0 \cdot \boldsymbol{n}_F,\ \tilde{g}_1^{(F)}\tilde{S} + \mu_6 D_F \bar{\tilde{S}}_0 \cdot \boldsymbol{n}_F]^{\mathrm{T}},$$
$$\mathcal{G}_{3,F}\overline{\mathcal{U}}_0 \equiv [\mu_2 D_F \bar{\boldsymbol{u}}_0 \cdot \boldsymbol{n}_b, \mu_4 D_F \bar{\tilde{\theta}}_0 \cdot \boldsymbol{n}_b, \mu_6 D_F \bar{\tilde{S}}_0 \cdot \boldsymbol{n}_b]^{\mathrm{T}},$$
$$\mathcal{L}_F\overline{\mathcal{U}}_0 \equiv [L_{1,F}\bar{\boldsymbol{u}}_0, L_{2,F}\bar{\tilde{\theta}}_0, L_{3,F}\bar{\tilde{S}}_0]^{\mathrm{T}},$$

$$u_3 = -\boldsymbol{u} \cdot \nabla_F b + \frac{1}{a^{33}(F)} \int_{b(y')}^{y_3} \nabla_F \cdot \boldsymbol{u}(y', z_3, t)\ \mathrm{d}z_3 \ \text{ in } \widetilde{\Omega}_T, \tag{4.6}$$

$$\begin{cases} \dfrac{\partial F'}{\partial t} + \boldsymbol{u} \cdot \nabla F' = u_3 \quad \text{in } \boldsymbol{R}_T^2, \\ F'|_{t=0} = 0 \quad \text{on } \boldsymbol{R}^2. \end{cases} \tag{4.7}$$

The following is the main result of this paper.

Theorem 4.1 Let $1/2 < l < l_0 < 1$, and $T > 0$ be an arbitrary number. Assume that

(i) $(\boldsymbol{v}_0, \theta_0, S_0) \in W_2^{2+l_0}(\boldsymbol{R}^3) \times (\overline{W}_2^{2+l_0}(\boldsymbol{R}^3))^2 \equiv \mathcal{W}_0,\ F_0 \in W_2^{\frac{7}{2}+l}(\boldsymbol{R}^2)$;

(ii) $\underline{\theta}_0 \leq \theta_0(x) < \infty$, and $0 < \underline{S}_0 \leq S_0(x) < \infty$ with positive constants $\underline{\theta}_0$ and $\underline{S}_0$, respectively;

(iii) $\int_{b(x')}^{x_3} \nabla \cdot \boldsymbol{v}_0(x', z_3)\ \mathrm{d}z_3 \in W_2^{2+l}(\boldsymbol{R}^3)$;

(iv) $b \in \overline{W}_2^{\frac{5}{2}+l}(\boldsymbol{R}^2)$ and $F_0(x') - b(x') > c_0 > 0$ on $\boldsymbol{R}^2$ with a positive constant c_0;

(v) $\boldsymbol{\tau}_1,\ \boldsymbol{\tau}_2,\ g_1 \in W_2^{2+l,\frac{2+l}{2}}(\boldsymbol{R}_T^3),\ p_0 \in \overline{W}_2^{3+l,\frac{3+l}{2}}(\boldsymbol{R}_T^3)$;

(vi) $\varrho \in C^{4+\beta}(\mathcal{G})$ on $\mathcal{G} = \{z = (z_1, z_2, z_3) \in \boldsymbol{R}^3 | z_1 > c_u, z_2 > \underline{\theta}_0, z_3 > \underline{S}_0\}$ with $l < \beta < 1 + l/2$, $\inf_{z\in\mathcal{G}} \varrho(z) \geq c_1 > 0$ with positive constants c_u and c_1, and $\sup_{z\in\mathcal{G}} |D_z^\alpha \varrho(z)| \leq M$ for $|\alpha| \leq 4$;

(vii) $D_x^\alpha p_0$ with $|\alpha| = 3$ satisfy the Hölder condition with exponent $\beta > l/2$ with respect to x_3;

(viii) for any $T > 0$ and $(f_\theta, f_S) \in (\overline{W}_2^{3+l,\frac{3+l}{2}}(\Omega_{x,T}))^2$ provided, the problem

$$\frac{\partial p}{\partial x_3} = -\varrho(p, f_\theta, f_S)g \quad x \in \Omega,\ t \in (0, T),$$
$$p(x', F(x', t), t) = p_0(x', F(x', t), t) \quad x' \in \boldsymbol{R}^2,\ t \in (0, T),$$

has a unique solution $p \in \overline{W}_2^{3+l,\frac{3+l}{2}}(\Omega_{x,T})$;

(ix) For any $T > 0$, the problem

$$\frac{\partial p}{\partial x_3} = -\varrho(p, \theta, S)g \quad x \in \Omega,\ t \in (0, T),$$
$$p(x', F(x', t), t) = p_0(x', F(x', t), t) \quad x' \in \boldsymbol{R}^2,\ t \in (0, T),$$

has a unique solution $(p, \theta, S) \in (\overline{W}_2^{3+l,\frac{3+l}{2}}(\Omega_{x,T}))^3$;

(x) $|\nabla \cdot F_0 - 1| \geq \delta_1 > 0$ on $\boldsymbol{R}^2$ with a positive constant δ_1.

Moreover, the compatibility conditions up to the order 1 for $\boldsymbol{v}$, θ, S are satisfied. Then, there exists $T^* \in (0, T]$ such that (2.1)–(2.6) has a unique solution

$$(\boldsymbol{v}, w, \theta, S, F) \in \mathcal{W}_x(T^*),$$

satisfying

$$0 < \frac{\underline{\theta}_0}{2} \leq \theta < \infty, \quad 0 < \frac{\underline{S}_0}{2} \leq S < \infty.$$

Here

$$\mathcal{W}_x(T^*) \equiv (W_2^{3+l,\frac{3+l}{2}}(\Omega_{x,T^*}))^2 \times (\overline{W}_2^{3+l,\frac{3+l}{2}}(\Omega_{x,T^*}))^2 \times \widetilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_{T^*}^2).$$

5. PROOF OF THEOREM 4.1

In this section, we consider the original nonlinear problem. First, we introduce a new unknown variable in the original coordinate system, and then consider the problem in the transformed one. By virtue of the successive approximation, we show the existence and uniqueness of the desired solution.

5.1. Introduction of new variable

First, we introduce a new variable in the original coordinate system:

$$\begin{aligned}\eta_h(x,t) &\equiv \int_{b(x')}^{x_3} \nabla_h \cdot \boldsymbol{v}(x',z_3,t)\ \mathrm{d}z_3 \\ &\equiv \int_{b(x')}^{x_3} h^{-1}\{(v_1(x_1+h,x_2,z_3,t)-v_1(x_1,x_2,z_3,t)) \\ &\quad +(v_2(x_1,x_2+h,z_3,t)-v_2(x_1,x_2,z_3,t))\}\ \mathrm{d}z_3,\end{aligned}$$

with $h>0$. Note that $\boldsymbol{v}$ is twice differentiable at this time. Then, it is obvious from (2.1) that it satisfies the following problem:

$$\begin{cases}\dfrac{\partial\eta_h}{\partial t}-(\mu_1\nabla^2\eta_h+\mu_2\dfrac{\partial^2\eta_h}{\partial x_3^2}) \\ \qquad =\mathcal{N}_{h,1}(\boldsymbol{v},w,\theta,S)+\mathcal{N}_{h,2}(\boldsymbol{v})+\mathcal{N}_{h,3}(\boldsymbol{v},p) \quad x\in\Omega(t),\ t>0, \\ \dfrac{\partial\eta_h}{\partial x_3}=\nabla_h\cdot\boldsymbol{v} \quad x\in\Gamma(t),\ t>0, \\ \eta_h=0 \quad x\in\Gamma_b,\ t>0, \\ \eta_h(x,0)=\displaystyle\int_{b(x')}^{F_0(x')}\nabla_h\cdot\boldsymbol{v}_0(x',z)\ \mathrm{d}z \quad x\in\Omega.\end{cases} \tag{5.1}$$

Here we have used notations

$$\begin{aligned}\mathcal{N}_{h,1}(\boldsymbol{v},w,\theta,S) &\equiv -\int_{b(x')}^{x_3}\nabla_h\cdot((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}+w\frac{\partial\boldsymbol{v}}{\partial x_3})\ \mathrm{d}z_3 \\ &\qquad -\int_{b(x')}^{x_3}\nabla_h\cdot(f_0\boldsymbol{A}\boldsymbol{v})(x',z_3,t)\ \mathrm{d}z_3, \\ \mathcal{N}_{h,2}(\boldsymbol{v}) &\equiv (\nabla_h\cdot\boldsymbol{v})|_{x_3=b(x')}\nabla^2 b+2\nabla b\cdot(\nabla_h\cdot\boldsymbol{v})|_{x_3=b(x')}, \\ \mathcal{N}_{h,3}(\boldsymbol{v},p) &\equiv -\frac{1}{\varrho_0}\int_{b(x')}^{x_3}\nabla_h\cdot\nabla p(x',z_3,t)\ \mathrm{d}z_3+(\sum_{i=1}^{2}\frac{\partial b}{\partial x_i})^2\nabla_h\frac{\partial\boldsymbol{v}}{\partial x_3}_{x_3=b(x')} \\ &\qquad -\mu_2\nabla_h\cdot\frac{\partial\boldsymbol{v}}{\partial x_3}_{x_3=b(x')}.\end{aligned}$$

Now, we consider the following problem for $(\boldsymbol{v}_h,w_h,\theta_h,S_h,F_h)$.

$$\begin{cases}\dfrac{\partial \boldsymbol{v}_h}{\partial t}+(\boldsymbol{v}_h\cdot\nabla)\boldsymbol{v}_h+w_h\dfrac{\partial \boldsymbol{v}_h}{\partial x_3}-\left[\mu_1\Delta\boldsymbol{v}_h+\mu_2\dfrac{\partial^2\boldsymbol{v}_h}{\partial x_3^2}\right]+f_0\boldsymbol{A}\boldsymbol{v}_h=-\dfrac{1}{\varrho_0}\nabla p_h,\\ \dfrac{\partial p_h}{\partial x_3}=-\varrho(p_h,\theta_h,S_h)g,\\ \dfrac{\partial \theta_h}{\partial t}+(\boldsymbol{v}_h\cdot\nabla)\theta_h+w_h\dfrac{\partial \theta_h}{\partial x_3}-\left[\mu_3\Delta\theta_h+\mu_4\dfrac{\partial^2\theta_h}{\partial x_3^2}\right]=0,\\ \dfrac{\partial S_h}{\partial t}+(\boldsymbol{v}_h\cdot\nabla)S_h+w_h\dfrac{\partial S_h}{\partial x_3}-\left[\mu_5\Delta S_h+\mu_6\dfrac{\partial^2 S_h}{\partial x_3^2}\right]=0,\\ w_h=-\boldsymbol{u}_h|_{x_3=b(x')}\cdot\nabla_{F_h}b-\eta_h,\\ \dfrac{D}{Dt}(x_3-F_h(x',t))=0\quad x\in\Omega(t),t>0.\end{cases}\tag{5.2}$$

$$\begin{cases}\mu_2\dfrac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}_{F_h}}=\boldsymbol{\tau}_1(x,t),\ \mu_4\dfrac{\partial\theta_h}{\partial \boldsymbol{n}_{F_h}}=\tau_2(x,t),\\ \mu_6\dfrac{\partial S_h}{\partial \boldsymbol{n}_{F_h}}=g_1(x,t)S_h(x,t),\ p_h=p_0\quad x\in\Gamma(t),\ t>0,\\ \dfrac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}_b}=\dfrac{\partial \theta_h}{\partial \boldsymbol{n}_b}=\dfrac{\partial S_h}{\partial \boldsymbol{n}_b}=0\quad x\in\Gamma_b,\ t>0.\end{cases}\tag{5.3}$$

$$\begin{cases}(\boldsymbol{v}_h,\theta_h,S_h)(x,0)=(\boldsymbol{v}_0,\theta_0,S_0)(x),\quad x\in\Omega,\\ F_h(x',0)=F_0(x'),\quad x'\in\boldsymbol{R}^2.\end{cases}\tag{5.4}$$

Introducing notations

$$\boldsymbol{u}_h(y_1,y_2,y_3)\equiv\boldsymbol{v}_h(y_1,y_2,X_3^{(F_h)}(y,t),t),\ \tilde\theta_h(y_1,y_2,y_3)\equiv\theta_h(y_1,y_2,X_3^{(F_h)}(y,t),t),$$
$$\tilde S_h(y_1,y_2,y_3)\equiv S_h(y_1,y_2,X_3^{(F_h)}(y,t),t),\ \mathcal{U}_h\equiv(\boldsymbol{u}_h,\tilde\theta_h,\tilde S_h),\ \mathcal{U}_h'\equiv\mathcal{U}_h-\overline{\mathcal{U}}_0,$$

we consider (5.1)–(5.4) after the coordinate transform into the y-coordinate system. We introduce notations $\mathcal{K}_h=(\boldsymbol{u}_h,F_h)$, and

$$\mathcal{P}_{F_h}\boldsymbol{u}_h\equiv\int_0^1\frac{\partial v_{h1}}{\partial x_1}(y'+\sigma h\boldsymbol{e}_1,X_3^{(F_h)}(y,t))\,\mathrm{d}\sigma+\int_0^1\frac{\partial v_{h2}}{\partial x_2}(y'+\sigma' h\boldsymbol{e}_2,X_3^{(F_h)}(y,t))\,\mathrm{d}\sigma',$$

where $\boldsymbol{v}_h(x,t)=(v_{h1},v_{h2})(x,t)=\boldsymbol{u}_h(\Phi(x,t))$ with Φ defined in Section 2, $\boldsymbol{e}_i$ $(i=1,2)$ are the orthogonal unit vectors in the ith direction in $\widetilde\Omega$. Thanks to the assumption (iii) on $\boldsymbol{v}_0$ in Theorem 4.1, we have the extension $\bar{\tilde\eta}_0$ of

$$\tilde{\eta}_h|_{t=0} = \tilde{\eta}_0 \equiv \int_{b(y')}^{F_0(y')} \mathcal{P}_{F_0}\, \boldsymbol{v}_0(y', z_3, t)\ \mathrm{d}z_3$$

into the region $t > 0$ conserving the regularity, and are able to consider the problem for $\tilde{\eta}_h' \equiv \tilde{\eta}_h - \bar{\tilde{\eta}}_0$ with (4.5)–(4.7).

$$\begin{cases} \dfrac{\partial \mathcal{U}_h'}{\partial t} - \mathcal{L}_{F_h}\mathcal{U}_h' = \mathcal{G}_{1,u_{3h},F_h}\mathcal{U}_h' + \mathcal{L}_{F_h}\overline{\mathcal{U}}_0 - \dfrac{\partial \overline{\mathcal{U}}_0}{\partial t} \quad \text{in } \widetilde{\Omega}_T, \\ \mathcal{B}_{1,F_h}\mathcal{U}_h' = \mathcal{G}_{2,F_h,\tilde{S}_h}\overline{\mathcal{U}}_0 \quad \text{on } \tilde{\Gamma}_T, \\ \mathcal{B}_{2,F_h}\mathcal{U}_h' = \mathcal{G}_{3,F_h}\overline{\mathcal{U}}_0 \quad \text{on } \tilde{\Gamma}_{bT}, \\ \mathcal{U}_h'|_{t=0} = (\mathbf{0}, 0, 0) \quad \text{on } \widetilde{\Omega}, \end{cases} \tag{5.5}$$

$$\begin{cases} \dfrac{\partial \tilde{\eta}_h'}{\partial t} - L_{1,F_h}\tilde{\eta}_h' = \tilde{G}_{4,F_h}(\mathcal{K}_h, u_{3h}, \tilde{\eta}_h) + L_{1,F_h}\bar{\tilde{\eta}}_0 - \dfrac{\partial \bar{\tilde{\eta}}_0}{\partial t} \quad \text{in } \widetilde{\Omega}_T, \\ a_h^{33}\dfrac{\partial \tilde{\eta}_h'}{\partial y_3} = -a_h^{33}\dfrac{\partial \bar{\tilde{\eta}}_0}{\partial y_3} + \displaystyle\int_{b(y')}^{F_0(y')} \mathcal{P}_{F_h}\, \boldsymbol{u}_h(y', z_3, t)\ \mathrm{d}z_3 \quad \text{on } \tilde{\Gamma}_T, \\ \tilde{\eta}_h' = 0 \quad \text{on } \tilde{\Gamma}_{bT}, \\ \tilde{\eta}_h'|_{t=0} = 0 \quad \text{on } \widetilde{\Omega}, \end{cases} \tag{5.6}$$

$$u_{3h} = -\tilde{\boldsymbol{u}}_h|_{y_3=b(y')} \cdot \nabla_{F_h} b - \tilde{\eta}_h \quad \text{in } \widetilde{\Omega}_T, \tag{5.7}$$

$$\begin{aligned} F_h'(y', t) = F_0(X_{\boldsymbol{u}_h}^{-1}(y', t)) + \int_0^t u_{3h}\,(y', F_0(y'), \tau)\ \mathrm{d}\tau \\ - \overline{F}_0(y', t) \quad \text{in } \boldsymbol{R}_T^2, \end{aligned} \tag{5.8}$$

where $a_h^{33} = a^{33}(F_h)$,

$$\tilde{G}_{4,F_h}(\mathcal{K}_h, u_{3h}, \tilde{\eta}_h) \equiv -A_1(F_h)\frac{\partial \tilde{\eta}_h}{\partial y_3} + \widetilde{\mathcal{N}}_{h,1}(\mathcal{K}_h, u_{3h}) + \sum_{i=2}^{3} \widetilde{\mathcal{N}}_{h,i}\,(\mathcal{K}_h),$$

and $\widetilde{\mathcal{N}}_{h,i}$ $(i = 1,2,3,4)$ represent $\mathcal{N}_i$ after the coordinate transform:

$$\begin{aligned} \widetilde{\mathcal{N}}_{h,1}(\mathcal{K}_h, u_{3h}) = -\int_{b(y')}^{y_3} \mathcal{P}_{F_h}\left\{(\boldsymbol{u}_h \cdot \nabla_{F_h})\boldsymbol{u}_h + u_{3h}a^{33}(F_h)\frac{\partial \boldsymbol{u}_h}{\partial y_3}\right\}(y', z_3, t)\ \mathrm{d}z_3 \\ - \int_{b(y')}^{y_3} f_0\, \boldsymbol{A}\boldsymbol{u}_h(y', z_3, t)\ \mathrm{d}z_3, \end{aligned}$$

$$\widetilde{\mathcal{N}}_{h,2}(\mathcal{K}_h) \equiv \mathcal{P}_{F_h}(\boldsymbol{u}_h)|_{y_3=b(y')}\nabla^2 b + 2\nabla b \cdot \mathcal{P}_{F_h}(\boldsymbol{u}_h)|_{y_3=b(y')}$$

$$\widetilde{\mathcal{N}}_{h,3}(\mathcal{K}_h) \equiv -\frac{1}{\varrho_0}\int_{b(y')}^{y_3} \mathcal{P}_{F_h}(\nabla_{F_h}\tilde{p}_h^{(F_h)})(y', z_3, t)\ \mathrm{d}z_3$$
$$+\Big(\sum_{i=1}^{2}\frac{\partial b}{\partial y_i}\Big)^2\mathcal{P}_{F_h}\Big(a^{33}(F_h)\frac{\partial \boldsymbol{u}_h}{\partial y_3}\Big) - \mu_2\mathcal{P}_{F_h}\Big(a^{33}(F_h)\frac{\partial \boldsymbol{u}_h}{\partial y_3}\Big).$$

5.2. Construction of successive approximation

Next, we construct the following successive approximation of the nonlinear problem (5.5)–(5.8) for $(\mathcal{U}'_{h(m)}, u_{3h(m)}, \tilde{\eta}'_{h(m)}, F_{h(m)})$ with $m \geq 0$.

For $m = 0$, we define $(\mathcal{U}'_{h(m)}, \tilde{\eta}'_{h(m)}, u_{3h(m)}, F'_{h(m)}, \tilde{\varrho}_{h(m)}) \equiv (\mathbf{0}, 0,0,0, \varrho(\tilde{p}_0, \bar{\tilde{\theta}}_0, \bar{\tilde{S}}_0))$,

$$\tilde{\eta}_{h(m)} = \int_{b(y')}^{y_3} \mathcal{P}_{F_0}(\bar{\boldsymbol{u}}_0)(y', z_3, t)\ \mathrm{d}z_3.$$

In addition, let $\tilde{p}_{h(m)}^{(F_{h(m)})}$ and

$$\tilde{\varrho}_{h(m)} = \tilde{\varrho}_{h(m)}(y,t) \equiv \varrho(\tilde{p}_{h(m)}^{(F_{h(m)})}(y,t), \tilde{\theta}_{h(m)}(y,t), \tilde{S}_{h(m)}(y,t)) \equiv \varrho(\mathcal{V}_{h(m)}(y,t))$$

satisfy:

$$\begin{cases} a^{33}_{h(m)}\dfrac{\partial \tilde{p}_{h(m)}^{(F_{h(m)})}}{\partial y_3} = -g\tilde{\varrho}_{h(m)} \quad \text{in } \widetilde{\Omega}_T, \\ \tilde{p}_{h(m)}^{(F_{h(m)})}|_{y_3=F_0(y')} = \tilde{p}_0^{(F_{h(m)})}(y', F_0(y'), t) \quad \text{on } \tilde{\Gamma}_{bT}, \end{cases} \tag{5.9}$$

for $m \geq 0$, where $a^{33}_{h(m)} = a^{33}(F_{h(m)})$. Then we consider:

$$\begin{cases} \dfrac{\partial \mathcal{U}'_{h(m+1)}}{\partial t} - \mathcal{L}_{F_{h(m)}}\mathcal{U}'_{h(m+1)} = \mathcal{G}_{1,u_{3h(m)},F_{h(m)}}\mathcal{U}'_{h(m)} + \mathcal{L}_{F_{h(m)}}\overline{\mathcal{U}}_0 - \dfrac{\partial \overline{\mathcal{U}}_0}{\partial t} \\ \equiv \mathcal{E}_{h,1}^{(m)} \quad \text{in } \widetilde{\Omega}_T, \\ \mathcal{B}_{1,F_{h(m)}}\mathcal{U}'_{h(m+1)} = \mathcal{G}_{2,F_{h(m)},\tilde{S}_{h(m)}}\overline{\mathcal{U}}_0 \equiv \mathcal{E}_{h,2}^{(m)} \quad \text{on } \tilde{\Gamma}_T, \\ \mathcal{B}_{2,F_{h(m)}}\mathcal{U}'_{h(m+1)} = \mathcal{G}_{3,F_{h(m)}}\overline{\mathcal{U}}_0 \equiv \mathcal{E}_{h,3}^{(m)} \quad \text{on } \tilde{\Gamma}_{bT}, \\ \mathcal{U}'_{h(m+1)}|_{t=0} = (\mathbf{0}, 0,0) \quad \text{on } \widetilde{\Omega}, \end{cases} \tag{5.10}$$

$$\begin{cases} \dfrac{\partial \tilde{\eta}'_{h(m+1)}}{\partial t} - L_{1,F_{h(m)}} \tilde{\eta}'_{h(m+1)} = \tilde{G}_{4,F_{h(m)}}(\mathcal{K}_{h(m+1)}, \mathcal{K}_{h(m)}, u_{3h(m)}, \tilde{\eta}_{h(m)}) \\ \quad + L_{1,F_{h(m)}} \bar{\tilde{\eta}}_0 - \dfrac{\partial \bar{\tilde{\eta}}_0}{\partial t} \equiv \mathcal{E}_{h,4}^{(m)} \quad \text{in } \tilde{\Omega}_T, \\ a_{h(m)}^{33} \dfrac{\partial \tilde{\eta}'_{h(m+1)}}{\partial y_3} = -a_{h(m)}^{33} \dfrac{\partial \bar{\tilde{\eta}}_0}{\partial y_3} + \displaystyle\int_{b(y')}^{F_0(y')} \mathcal{P}_{F_{h(m)}} \boldsymbol{u}_{h(m+1)}(y', z_3, t)\ \mathrm{d}z_3 \\ \quad \equiv \mathcal{E}_{h,5}^{(m)} \quad \text{on } \tilde{\Gamma}_T, \\ \tilde{\eta}'_{h(m+1)} = 0 \quad \text{on } \tilde{\Gamma}_{bT}, \\ \tilde{\eta}'_{h(m+1)}|_{t=0} = 0 \quad \text{on } \tilde{\Omega}, \end{cases} \tag{5.11}$$

$$u_{3h(m+1)} = -\tilde{\boldsymbol{u}}_{h(m+1)}|_{y_3=b(y')} \cdot \nabla b - \tilde{\eta}_{h(m+1)} \quad \text{in } \tilde{\Omega}_T, \tag{5.12}$$

$$\begin{aligned} F'_{h(m+1)}(y', t) = {} & F_0(X^{-1}_{\boldsymbol{u}_{h(m)}}(y', t)) \\ & + \int_0^t u_{3h(m)}(y', F_0(y'), \tau)\ \mathrm{d}\tau \\ & - \overline{F}_0(y', t) \quad \text{in } \boldsymbol{R}_T^2, \end{aligned} \tag{5.13}$$

where $\mathcal{K}_{h(m)} = (\boldsymbol{u}_{h(m)}, F_{h(m)})$ and

$$\begin{aligned} & \tilde{G}_{4,F_{h(m)}}(\mathcal{K}_{h(m+1)}, \mathcal{K}_{h(m)}, u_{3h(m)}, \tilde{\eta}_{h(m)}) \\ & \equiv -A_1(F_{h(m)}) \frac{\partial \tilde{\eta}_{h(m)}}{\partial y_3} + \tilde{\mathcal{N}}_{h,1}(\mathcal{K}_{h(m)}, u_{3h(m)}) \\ & + \tilde{\mathcal{N}}_{h,2}(\mathcal{K}_{h(m)}) + \tilde{\mathcal{N}}_{h,3}(\mathcal{K}_{h(m+1)}). \end{aligned}$$

Note that the subindices of variables are chosen so that the iteration process converges. In order to prove the well-definedness of (5.9)–(5.13), we first consider the linear problems of them.

5.3. Linear Problems

In this subsection, we consider linear problems for $\mathcal{U}' \equiv (\boldsymbol{u}', \tilde{\theta}', \tilde{S}')^{\mathrm{T}}$ and F'. Each problem will contribute to constructing the successive approximation of the nonlinear problem later.

5.3.1. Linear problem for $\mathcal{U}'$

In this subsection, we consider a linear problem for $\mathcal{U}'$. Let a set of functions $(\boldsymbol{w}, \delta, s, f)$ be provided such that $(\boldsymbol{w}, \delta, s, f) \in \mathcal{W}_{\mathcal{K}}(T)$,

$(\boldsymbol{w}, \delta, s, f)|_{t=0} = (\boldsymbol{v}_0, \theta_0, S_0, F_0)$. Now, consider the following problem:

$$\begin{cases} \dfrac{\partial \mathcal{U}'}{\partial t} - \mathcal{L}_f \mathcal{U}' = \boldsymbol{l}_1 \quad \text{in } \widetilde{\Omega}_T, \\ \mathcal{B}_{1,f}\mathcal{U}' = \boldsymbol{l}_2 \quad \text{on } \widetilde{\Gamma}_T, \\ \mathcal{B}_{2,f}\mathcal{U}' = \boldsymbol{l}_3, \quad \text{on } \widetilde{\Gamma}_{bT}, \\ \mathcal{U}'|_{t=0} = (\mathbf{0}, 0, 0) \quad \text{on } \widetilde{\Omega}, \end{cases} \tag{5.14}$$

where

$$\mathcal{L}_f \mathcal{U}' \equiv (L_{1,f}\boldsymbol{u}',\ L_{2,f}\tilde{\theta}',\ L_{3,f}\tilde{S}')^{\mathrm{T}},$$
$$\mathcal{B}_{1,f}\mathcal{U}' \equiv (\mu_2 D_f \boldsymbol{u}' \cdot \boldsymbol{n}_f,\ \mu_4 D_f \tilde{\theta}' \cdot \boldsymbol{n}_f,\ \mu_6 D_f \tilde{S}' \cdot \boldsymbol{n}_f)^{\mathrm{T}},$$
$$\mathcal{B}_{2,f}\mathcal{U}' \equiv (\mu_2 D_f \boldsymbol{u}' \cdot \boldsymbol{n}_b,\ \mu_4 D_f \tilde{\theta}' \cdot \boldsymbol{n}_b,\ \mu_6 D_f \tilde{S}' \cdot \boldsymbol{n}_b)^{\mathrm{T}},$$

and $\boldsymbol{n}_f$ is provided by (2.4) with F replaced by f.

Theorem 5.1 *Assume the condition* (ix) *in* Theorem 4.1, $F_0 \in W_2^{\frac{5}{2}+l}(\boldsymbol{R}^2)$, $\mathcal{U}_0 \equiv (\boldsymbol{v}_0, \theta_0, S_0) \in \mathcal{W}_0' \equiv W_2^{2+l}(\boldsymbol{R}^3) \times (\overline{W}_2^{2+l}(\boldsymbol{R}^3))^2$. *Let* $T > 0$ *and* f *satisfy the assumption* (A_f), *and let* $\boldsymbol{l}_i \in \mathcal{W}_i(T)$ $(i = 1,2,3)$. *We also assume the compatibility conditions up to the order* 1*:*

$$\begin{cases} [\dfrac{\partial \mathcal{U}'}{\partial t} - \mathcal{L}_f \mathcal{U}']|_{t=0} = \boldsymbol{l}_1|_{t=0} \quad on\ \tilde{\Omega}, \\ \mathcal{B}_{1,f}\mathcal{U}'|_{t=0} = \boldsymbol{l}_2|_{t=0} \quad on\ \tilde{\Gamma}, \\ \mathcal{B}_{2,f}\mathcal{U}'|_{t=0} = \boldsymbol{l}_3|_{t=0} \quad on\ \tilde{\Gamma}_b. \end{cases}$$

Then, there exists $T_{51} \in (0, T]$ *and a unique solution* $\mathcal{U}' \in \mathcal{W}'(T_{51})$ *to* (5.14) *satisfying*

$$\| \mathcal{U}' \|_{\mathcal{W}'(T_{51})} \leq C \sum_{i=1}^{3} \| \boldsymbol{l}_i \|_{\mathcal{W}_i(T_{51})},$$

where C *is a positive constant depending on* $\| \mathcal{U}_0 \|_{\mathcal{W}_0'}$, $\| b \|_{\overline{W}_2^{\frac{5}{2}+l}(\boldsymbol{R}^2)}$ *and* $\| F_0 \|_{W_2^{\frac{5}{2}+l}(\boldsymbol{R}^2)}$. *In addition, if we take a constant* $M > 0$ *such that*

$$\| f \|_{\widetilde{W}_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_T^2)} < M$$

holds, T_{51} *depends only on* M.

Proof. First we show the uniform ellipticity of $\mathcal{L}_f$ over the short time interval. Indeed, the characteristic polynomial of $\mathcal{L}_f$ is provided by

$$\mu_1\{|\xi'|^2 + (\xi' \cdot \boldsymbol{a}^3)\xi_3 + \boldsymbol{a}^3 \cdot \xi'\xi_3 + |\boldsymbol{a}^3|^2\xi_3^2 + \boldsymbol{a}^3 \frac{\partial \boldsymbol{a}^3}{\partial y_3}\xi_3\} + \mu_2(a^{33})^2\xi_3^2$$
$$= \mu_1|\xi' + \boldsymbol{a}^3\xi_3|^2 + \mu_1\boldsymbol{a}^3 \cdot \frac{\partial \boldsymbol{a}^3}{\partial y_3}\xi_3 + \mu_2(a^{33})^2\xi_3^2.$$

Noting that $\lim_{t\to 0}\boldsymbol{a}^3 \cdot \frac{\partial \boldsymbol{a}^3}{\partial y_3}\xi_3 = 0$, $\mathcal{U}_f$ is uniformly elliptic over the short time interval. Then, the regularizer method [16] is applicable if

$$\lim_{t\to 0}\left\|(\nabla \cdot \boldsymbol{a}^3)\frac{\partial \boldsymbol{u}}{\partial y_3}\right\|_{W_2^{1+l,\frac{1+l}{2}}(\widetilde{\Omega}_t)} = 0.$$

This is achieved since

$$\left\|(\nabla \cdot \boldsymbol{a}^3)\frac{\partial \boldsymbol{u}}{\partial y_3}\right\|_{W_2^{1+l,\frac{1+l}{2}}(\widetilde{\Omega}_t)} \le \|\nabla \cdot \boldsymbol{a}^3\|_{W_2^{1+l,\frac{1+l}{2}}(\widetilde{\Omega}_t)}\left\|\frac{\partial \boldsymbol{u}}{\partial y_3}\right\|_{W_2^{1+l,\frac{1+l}{2}}(\widetilde{\Omega}_t)}$$

and

$$\lim_{t\to 0}\|\nabla \cdot \boldsymbol{a}^3\|_{W_2^{1+l,\frac{1+l}{2}}(\widetilde{\Omega}_t)} = 0.$$

This leads to the desired conclusion. □

5.3.2. Linear problem for F

Here, we consider the linear problem for F:

$$\begin{cases}\dfrac{\partial F}{\partial t} - \boldsymbol{w}|_{y_3=F_0(y')} \cdot \nabla F = l_4|_{y_3=F_0(y')} \quad \text{in } \boldsymbol{R}_T^2, \\ F|_{t=0} = F_0 \quad \text{on } \boldsymbol{R}^2. \end{cases} \tag{5.15}$$

Lemma 5.2 *For an arbitrary* $T > 0$, *under the assumptions* $\boldsymbol{w} = (w_1, w_2)^{\mathrm{T}}$, $l_4 \in W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_T)$, $F_0 \in W_2^{\frac{5}{2}+l}(\boldsymbol{R}^2)$ *and* $T\|\boldsymbol{w}\|_{W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_T)} \le \delta_2$ *with a positive constant* δ_2, *we have a unique solution* $F(y',t)$ *to* (5.15) *on a short time interval* $(0, T_{52}]$ *with* $T_{52} \in (0,T]$ *satisfying* $F \in \widetilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_{T_{52}}^2)$ *and*

$$\|F\|_{\widetilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_{T_{52}}^2)} \le C\{1 + (\epsilon + C_\epsilon T_{52})\|\boldsymbol{w}\|_{W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_{T_{52}})} + \|l_4\|_{W_2^{1+l,\frac{1+l}{2}}(\widetilde{\Omega}_{T_{52}})}$$
$$+ T_{52}\|l_4\|_{W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_{T_{52}})}\},$$

where ϵ *is an arbitrary small positive constant, and* C_ϵ, *a positive constant depending on* ϵ.

Proof. We use the characteristic curve to solve (5.15). For any $\xi' \in \boldsymbol{R}^2$, a pair of functions $(\boldsymbol{X_w}(t;\xi'), \overline{F}(t;\xi')) = (X_{\boldsymbol{w}1}(t;\xi'), X_{\boldsymbol{w}2}(t;\xi'), \overline{F}(t;\xi')) \in \boldsymbol{R}^2 \times \boldsymbol{R}$ is provided by

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{X_w}(t;\xi') = \boldsymbol{w}(\boldsymbol{X_w}(t;\xi'), F_0(\boldsymbol{X_w}(t;\xi')), t), \quad \boldsymbol{X_w}(0;\xi') = \xi', \\ \dfrac{d}{dt}\overline{F}(t;\xi') = l_4(\boldsymbol{X_w}(t;\xi'), F_0(\boldsymbol{X_w}(t;\xi')), t), \quad \overline{F}(0;\xi') = F_0(\xi'), \end{cases}$$

which are equivalent to the following forms of the integral equations:

$$\begin{cases} \boldsymbol{X_w}(t;\xi') = \xi' + \displaystyle\int_0^t \boldsymbol{w}\,(\boldsymbol{X_w}(\tau;\xi'), F_0(\boldsymbol{X_w}(\tau;\xi')), \tau)\ \mathrm{d}\tau \\ \qquad\qquad \equiv (X_{\boldsymbol{w}1}(\tau;\xi') \quad X_{\boldsymbol{w}2}(\tau;\xi')))^{\mathrm{T}}, \\ \overline{F}(t;\xi') = F_0(\xi') + \displaystyle\int_0^t l_4\,(\boldsymbol{X_w}(\tau;\xi'), F_0(\boldsymbol{X_w}(\tau;\xi')), \tau)\ \mathrm{d}\tau. \end{cases} \tag{5.16}$$

Now we show that a mapping $\Phi_{\boldsymbol{w}}: (\xi', t) \longmapsto (\boldsymbol{X_w}(t;\xi'), t)$ is a diffeomorphism on $\boldsymbol{R}_T^2$ at each t onto itself on a short time interval $(0, T_{52}]$. Actually, the Jacobi matrix of $\Phi_{\boldsymbol{w}}$ is provided by

$$\boldsymbol{A_w} \equiv \left(\frac{\partial X_{\boldsymbol{w}i}(t;\xi')}{\partial \xi_j}\right)_{i,j=1,2} = \left[\frac{\partial}{\partial \xi_1}\boldsymbol{X_w}(t;\xi')\ \frac{\partial}{\partial \xi_2}\boldsymbol{X_w}(t;\xi')\right],$$

where the ith component of $\frac{\partial}{\partial \xi_j}\boldsymbol{X_w}$, denoted by η_{ij}, is provided by

$$\begin{aligned} \eta_{ij}(t;\xi') &= \delta_{ij} + \int_0^t \frac{\partial}{\partial \xi_j}(w_i(\boldsymbol{X_w}(\tau;\xi'), F_0(\boldsymbol{X_w}(\tau;\xi')), \tau))\ \mathrm{d}\tau \\ &= \delta_{ij} + \int_0^t \left(\nabla w_i + \frac{\partial w_i}{\partial y_3}\nabla F_0\right)\cdot \frac{\partial}{\partial \xi_j}\boldsymbol{X_w}(\tau;\xi')\ \mathrm{d}\tau. \end{aligned} \tag{5.17}$$

From (5.17), we have

$$\frac{d}{dt}\left(\frac{\partial}{\partial \xi_j}\boldsymbol{X_w}(t;\xi')\right) = \boldsymbol{B_w}\frac{\partial}{\partial \xi_j}\boldsymbol{X_w}(t;\xi'), \quad \frac{\partial}{\partial \xi_j}\boldsymbol{X_w}(0;\xi') = (\delta_{1j},\ \delta_{2j})^{\mathrm{T}} \equiv \boldsymbol{d}_{0j}$$

$$(j = 1,2),$$

with a matrix $\boldsymbol{B_w} = \left(\frac{\partial w_k}{\partial y_k} + \frac{\partial w_k}{\partial y_3}\frac{\partial F_0}{\partial y_l}\right)_{k,l=1,2}$. Thus we have

$$\frac{\partial}{\partial \xi_j}\boldsymbol{X_w}(t;\xi') = \boldsymbol{d}_{0j}\exp\left(t\boldsymbol{B_w}\right)\ \ (j = 1,2).$$

Now we show that the determinant of $\boldsymbol{A}_{\boldsymbol{w}}$ is bounded from below on a short time interval. Adding (5.17) with respect to $i = 1,2$ yields

$$\sum_{i=1}^{2} \left|\eta_{ij}(t;\xi')\right| \leq 1 + \int_0^t M(\tau) \sum_{i=1}^{2} \left|\eta_{ij}(t;\xi')\right| \, \mathrm{d}\tau,$$

where

$$M(\tau) \equiv 2 \max_{k,l=1,2} \left\{ \sup_{x\in\tilde{\Omega}} \left| \left(\frac{\partial w_k}{\partial x_l} + \frac{\partial w_k}{\partial x_3}\frac{\partial F_0}{\partial x_l}\right)(x,\tau) \right| \right\}.$$

Applying the Gronwall's inequality yields

$$\sum_{i=1}^{2} \left|\eta_{ij}(t;\xi')\right| \leq \exp\left(\int_0^t M(\tau) \, \mathrm{d}\tau\right) < +\infty.$$

Thus again from (5.17) and the assumption $T \parallel \boldsymbol{w} \parallel_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)} \leq \delta_2$, it is possible to make

$$\left| \int_0^t \frac{\partial}{\partial \xi_j} w_i(X_{\boldsymbol{w}}(\tau;\xi'), F_0(X_{\boldsymbol{w}}(\tau;\xi')), \tau) \, \mathrm{d}\tau \right| \leq \delta_3$$

for an arbitrary small positive constant δ_3 on $t \in (0, T_{52}]$ with T_{52} small enough. This means

$$\det(\boldsymbol{A}_{\boldsymbol{w}}) = \prod_{i=1}^{2} \left\{ 1 + \int_0^t \frac{\partial}{\partial \xi_i} w_i(X_{\boldsymbol{w}}(\tau;\xi'), F_0(X_{\boldsymbol{w}}(\tau;\xi')), \tau) \, \mathrm{d}\tau \right\}$$
$$- \prod_{i,j=1,2,i\neq j} \left(\int_0^t \frac{\partial}{\partial \xi_i} w_j(X_{\boldsymbol{w}}(\tau;\xi'), F_0(X_{\boldsymbol{w}}(\tau;\xi')), \tau) \, \mathrm{d}\tau \right) \geq c_1 > 0,$$

with a positive constant c_1. Thus $X_{\boldsymbol{w}}$, and consequently $\Phi_{\boldsymbol{w}}$ is a diffeomorphism on $(0, T_{52}]$, and for arbitrary (y', t), it is able to take (ξ', t) and $(X_{\boldsymbol{w}}(t;\xi'), \overline{F}(t;\xi'))$ satisfying (5.16). Then, $F(y', t)$ is provided by

$$F(y', t) = F_0(X_{\boldsymbol{w}}^{-1}(y', t)) + \int_0^t l_4(y', F_0(y'), \tau) \, \mathrm{d}\tau.$$

Since $X_{\boldsymbol{w}}^{-1}(y', t) = y' - \int_0^t \boldsymbol{w}(y', F_0(y'), \tau) \, \mathrm{d}\tau$, we have the desired estimate with the aid of the Cauchy–Schwarz inequality, multi-plicative inequality and the trace theorem. □

5.4. Regularity of $\mathcal{P}_f \boldsymbol{u}$

Next, we discuss the regularity of $\mathcal{P}_f \boldsymbol{u}$.

Lemma 5.3 *For arbitrary* $T > 0$, $\boldsymbol{w} \in W_2^{i+l,\frac{i+l}{2}}(\widetilde{\Omega}_T)$ *and* $f \in \widetilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_T^2)$ *satisfying* $f|_{t=0} = F_0$, *the relationships* $\mathcal{P}_f \boldsymbol{w} \in W_2^{i-1+l,\frac{i-1+l}{2}}(\widetilde{\Omega}_T)$ (i=2,3) *hold, satisfying*

$$\| \mathcal{P}_f \boldsymbol{w} \|_{W_2^{i-1+l,\frac{i-1+l}{2}}(\widetilde{\Omega}_T)} \leq C\phi\left(\| f \|_{\widetilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_T^2)} \right) \| \boldsymbol{w} \|_{W_2^{i+l,\frac{i+l}{2}}(\widetilde{\Omega}_T)} \quad (i = 2,3).$$

Proof. A mapping $\Phi: (x,t) \mapsto (y,t)$ is clearly invertible, and hence

$$(x', x_3, t) = (y', X_3^{(f)}(y,t), t)$$

is a one-to-one mapping. Then, $\boldsymbol{v_w} = (v_1, v_2)$ defined by $\boldsymbol{v_w}(x,t) = \boldsymbol{w}(\Phi(x,t))$ is well-defined from $\boldsymbol{w}$, and belongs to $W_2^{i+l,\frac{i+l}{2}}(\Omega_{x,T})$ if $\boldsymbol{w} \in W_2^{i+l,\frac{i+l}{2}}(\widetilde{\Omega}_T)$ $(i = 2,3)$. Now we introdue a notation

$$D_{i,j,k(x)} \equiv \frac{\partial^3}{\partial x_i \partial x_j \partial x_k}$$

for simplicity. Since

$$\begin{aligned}
&\frac{\partial^2}{\partial y_i \partial y_j}\left(\frac{\partial v_1}{\partial x_1}(y' + \sigma h \boldsymbol{e}_1, X_3^{(f)}(y,t))\right) \\
&= D_{1,i,j(x)} v_1(y' + \sigma h \boldsymbol{e}_1, X_3^{(f)}(y,t)) + D_{1,i,3(x)} v_1(y' + \sigma h \boldsymbol{e}_1, X_3^{(f)}(y,t)) \frac{\partial X_3^{(f)}}{\partial y_j} \\
&+\{D_{1,j,3(x)} v_1(y' + \sigma h \boldsymbol{e}_1, X_3^{(f)}(y,t)) \\
&+D_{1,3,3(x)} v_1(y' + \sigma h \boldsymbol{e}_1, X_3^{(f)}(y,t)) \frac{\partial X_3^{(F_h)}}{\partial y_j}\} \frac{\partial X_3^{(F_h)}}{\partial y_i} \\
&+D_{1,3(x)} v_1(y' + \sigma h \boldsymbol{e}_1, X_3^{(f)}(y,t)) \frac{\partial^2 X_3^{(f)}}{\partial y_i \partial y_j},
\end{aligned}$$

we only show the estimate of

$$I_0 \equiv \int_{\widetilde{\Omega}}\int_{\widetilde{\Omega}} | D_{1,i,j(x)} v_1(y^{(1)\prime} + \sigma h \boldsymbol{e}_1, X_3^{(f)}(y^{(1)}, t)) \\ - D_{1,i,j(x)} v_1(y^{(2)\prime} + \sigma h \boldsymbol{e}_1, X_3^{(f)}(y^{(2)}, t))|^2 |y^{(1)} - y^{(2)}|^{-(3+2l)} \, \mathrm{d}y^{(1)} \mathrm{d}y^{(2)},$$

where $|\alpha| = 3$, and $y^{(i)} = (y^{(i)\prime}, y_3^{(i)})$ $(i = 1,2)$. Applying the transform of the variable from $y^{(i)}$ to $q^{(i)} = (q^{(i)\prime}, q_3^{(i)})$ defined by

$$y_1^{(i)\prime} + \sigma h \longmapsto q_1^{(i)\prime},\ y_2^{(i)\prime} \longmapsto q_2^{(i)\prime},\ X_3^{(f)}(y^{(i)}, t) \longmapsto q_3^{(i)}\ (i = 1,2),$$

we have

$$I_0 = \int_{\widetilde{\Omega}}\int_{\widetilde{\Omega}} \frac{|D_{1,i,j(x)} v_1(q^{(1)}, t) - D_{1,i,j(x)} v_1(q^{(2)}, t)|^2}{|q^{(1)} - q^{(2)}|^{3+2l}} \frac{|q^{(1)} - q^{(2)}|^{3+2l}}{|y^{(1)} - y^{(2)}|^{3+2l}} \\ \times \prod_{i=1}^{2} \left(\frac{(b - F_0)(q^{(i)\prime})}{b(q^{(i)\prime}) - f(q^{(i)\prime}, t)} \right) \mathrm{d}q^{(1)} \mathrm{d}q^{(2)}.$$

By virtue of some calculations, we have

$$I_0 \le \phi(\| f \|_{W_2^{\frac{3}{2}+l, \frac{3}{4}+\frac{l}{2}}(\boldsymbol{R}^2)}) \| \boldsymbol{v_w} \|^2_{W_2^{3+l, \frac{3+l}{2}}(\Omega_{x,T})}.$$

Other terms are estimated similarly, and we omit them here. Furthermore, we have the estimate

$$\| \boldsymbol{v_w} \|_{W_2^{3+l, \frac{3+l}{2}}(\Omega_{x,T})} \le \phi(\| f \|_{\widetilde{W}_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_T^2)}) \| \boldsymbol{w} \|_{W_2^{3+l, \frac{3+l}{2}}(\widetilde{\Omega}_T)}.$$

Thus, we have the desired result. □

5.5. Iteration process

For simplicity we hereafter use notations

$$\widetilde{\mathcal{W}}(T) \equiv \mathcal{W}_{\mathcal{U}}(T) \times (W_2^{3+l, \frac{3+l}{2}}(\widetilde{\Omega}_T))^2 \times \widetilde{W}_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_T^2),$$
$$\widetilde{\mathcal{W}}'(T) \equiv \mathcal{W}'_{\mathcal{U}}(T) \times (W_2^{3+l, \frac{3+l}{2}}(\widetilde{\Omega}_T))^2 \times \widetilde{W}_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_T^2),$$
$$\mathcal{K}'_{h(m)} \equiv (\mathcal{U}'_{h(m)}, F'_{h(m)})^{\mathrm{T}}.$$

The iteration process solving (5.9)–(5.13) is executed as follows.

From $(\mathcal{U}_{h(m)}, u_{3h(m)}, \tilde{\eta}_{h(m)}, F_{h(m)}) \in \widetilde{\mathcal{W}}(T)$ provided, we first calculate $\mathcal{U}'_{h(m+1)} \in W_2^{3+l, \frac{3+l}{2}}(\widetilde{\Omega}_T)$ and $F_{h(m+1)} \in \widetilde{W}_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_T^2)$ by virtue of (5.10) and (5.13). Note that $\tilde{p}_{h(m)}^{(F_{h(m)})}$ is calculated in advance. Then, by (5.11) and (5.12), $(u_{3h(m)}, \tilde{\eta}'_{h(m)}) \in$

$(W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_T))^2$ is calculated. For (5.10), the unique existence of $\mathcal{U}'_{h(m+1)}$ on a short time interval $(0,T_{53}]$ is guaranteed by Theorem 5.1. As will be noted later, T_{53} does not depend on m due to the induction. Making use of Lemma 5.4 below, we shall estimate the right-hand sides of (5.10)–(5.13). We introduce new notations:

$$E'_{h(m)}(t) \equiv \| \mathcal{U}'_{h(m)} \|_{\mathcal{W}'_{\mathcal{U}}(t)},$$
$$E_{h(m)}(t) \equiv \| \mathcal{K}'_{h(m)} \|_{\mathcal{W}'_{\mathcal{K}}(t)} + \| \tilde{\eta}'_{h(m)} \|_{W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_T)} + \| u_{3h(m)} \|_{W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_T)}.$$

Lemma 5.4 *Assume* $F_0 \in W_2^{\frac{5}{2}+l}(\boldsymbol{R}^2)$, $\mathcal{U}_0 \equiv (\boldsymbol{v}_0, \theta_0, S_0) \in \mathcal{W}_0$, *and there exists* $T_{54} > 0$ *satisfying* $F_{h(m)}(y',t) - b(y') > c_0 > 0$ *for* $t \in (0, T_{54}]$.

Then, for the right-hand sides of (5.10) *and* (5.11), *following estimates hold for arbitrarily small* $\epsilon > 0$ *and* $t \in (0, T_{54}]$*:*

$$\left\|\mathcal{E}_{h,1}^{(m)}\right\|_{\mathcal{W}_1(t)} \le (\epsilon + C_\epsilon t)\phi(\left\|F_{h(m)}\right\|_{\widetilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)})\left\|\mathcal{U}'_{h(m)}\right\|_{\mathcal{W}'_{\mathcal{U}}(t)} + C,$$

$$\left\|\mathcal{E}_{h,2}^{(m)}\right\|_{\mathcal{W}_2(t)} \le (\epsilon + C_\epsilon t)(1 + \| g_1 \|_{W_2^{2+l,\frac{2+l}{2}}(\boldsymbol{R}_t^3)})\phi(\left\|F_{h(m)}\right\|_{\widetilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)})\left\|\mathcal{U}'_{h(m)}\right\|_{\mathcal{W}'_{\mathcal{U}}(t)}$$
$$+ \sum_{i=1}^{2} \|\tilde{\tau}_i\|_{W_2^{2+l,\frac{2+l}{2}}(\boldsymbol{R}_t^3)} + C,$$

$$\left\|\mathcal{E}_{h,3}^{(m)}\right\|_{\mathcal{W}_3(t)} \le (\epsilon + C_\epsilon t)\left\|F_{h(m)}\right\|_{\widetilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)} + C,$$

$$\left\|\mathcal{E}_{h,4}^{(m)}\right\|_{W_2^{1+l,\frac{1+l}{2}}(\widetilde{\Omega}_t)} \le (\epsilon + C_\epsilon t)\phi_0(E_{h(m)}(t))$$
$$+ \phi(E_{h(m)}(t))(\| F_{h(m+1)} \|_{\widetilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)} + E'_{h(m+1)}(t)),$$

$$\left\|\mathcal{E}_{h,5}^{(m)}\right\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\widetilde{\Gamma}_t)} \le C(1 + (\epsilon + C_\epsilon t) \| F_{h(m)} \|_{\widetilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)})$$
$$\times (1 + \| \boldsymbol{u}'_{h(m+1)} \|_{W_2^{3+l,\frac{3+l}{2}}(\boldsymbol{R}_t^2)}),$$

where ϕ_0 is a homogeneous polynomial of its argument.

Lemma 5.5 *Let* $T > 0$ *be provided. Then, we have the following inequality for any* $t \in (0, T]$ *and* $m \ge 0$*:*

$$|F_{h(m)}(y',t) - F_0(y')| \le CT^{\frac{1}{2}} \| F'_{h(m)} \|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_T^2)},$$
$$|F_{h(m)}(y',t) - b(y')| \ge c_0 - T^{\frac{1}{2}} \| F'_{h(m)} \|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_T^2)}.$$

Proof. This lemma is shown with the aid of the Cauchy–Schwartz inequality and the Sobolev embedding theorem. Actually,

$$|F_{h(m)}(y',t)-F_0(y')| \le \int_0^t \left|\frac{\partial F_{h(m)}}{\partial t}(y',t)\right| \,\mathrm{d}t \le T^{\frac{1}{2}} \parallel F'_{h(m)} \parallel_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_T^2)}$$

holds, which yields the first statement. The proof of the second statement is obvious, and we omit it here. □

By virtue of Lemma 5.5, the assumptions $(A_{F_{h(m)}})$ are satisfied with T_{53} and consequently T_{54} for all $m \in \boldsymbol{N}$.

Now, we show $E_{h(m)}$ $(m = 0,1,2,\dots)$ are bounded from above by a constant independent of h and m. Suppose

$$(\mathcal{U}'_{h(m)}, u_{3h(m)}, \tilde{\eta}'_{h(m)}, F'_{h(m)}) \in \widetilde{\mathcal{W}}'(T)$$

as an assumption of the induction. Note that $\widetilde{\mathcal{W}}'(T)$ is defined at the beginning of the subsection 5.5. We have the following estimates for $t \in (0, T_{54}]$ by using Lemmas 5.2–5.5.

$$\begin{aligned} E'_{h(m+1)}(t) \le C_1(\epsilon + C_\epsilon t)\{\phi_1(E_{h(m)}(t)) \\ + \phi_2(E_{h(m)}(t))E'_{h(m)}(t)\} + C(t), \end{aligned} \tag{5.18}$$

$$\begin{aligned} \parallel u_{3h(m+1)} \parallel_{W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_t)} \le C_2\{\parallel \boldsymbol{u}'_{h(m+1)} \parallel_{W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_t)} + \\ \parallel \tilde{\eta}'_{h(m+1)} \parallel_{W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_t)} + 1\}, \end{aligned} \tag{5.19}$$

$$\begin{aligned} &\parallel \tilde{\eta}'_{h(m+1)} \parallel_{W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_t)} \\ &\le C_3\{(\epsilon + C_\epsilon t)\phi_3(E_{h(m)}(t)) + \phi_4(E_{h(m)}(t)) \parallel F_{h(m+1)} \parallel_{\widetilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)} + 1\} \\ &\quad + \phi_5(E_{h(m)}(t))E'_{h(m+1)}(t), \end{aligned} \tag{5.20}$$

$$\begin{aligned} &\parallel F'_{h(m+1)} \parallel_{\widetilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)} \\ &\le C_4\{(\epsilon + C_\epsilon t)(\parallel \boldsymbol{u}'_{h(m)} \parallel_{W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_t)} + \\ &\quad \parallel \tilde{\eta}'_{h(m)} \parallel_{W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_t)}) + 1\}. \end{aligned} \tag{5.21}$$

Here $\phi_i(\cdot)$ $(i = 1,2,\dots,5)$ are polynomials of their arguments with positive coefficients. Especially ϕ_3 is homogeneous, and $\phi_i(\cdot)$ $(i = 4,5)$ include positive constant terms:

$$\phi_i(\cdot) \geq c_2 > 0 \ (i = 4,5).$$

Adding (5.19) multiplied by $1/2C_2$ and (5.20), we have

$$(\| u'_{3h(m+1)} \|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)} + \| \tilde{\eta}'_{h(m+1)} \|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)})$$
$$\leq C + (\epsilon + C_\epsilon t)\phi_3(E_{h(m)}(t)) + \phi_4(E_{h(m)}(t)) \| F_{h(m+1)} \|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)}$$
$$+\phi_5(E_{h(m)}(t))E'_{h(m+1)}(t). \tag{5.22}$$

After adding (5.21) and (5.22) multiplied by $(2\phi_4(E_{h(m)}(t)))^{-1}$, we have

$$\| u'_{3h(m+1)} \|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)} + \| \tilde{\eta}'_{h(m+1)} \|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)} + \| F'_{h(m+1)} \|_{\widetilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)}$$
$$\leq C + (\epsilon + C_\epsilon t)\phi_3(E_{h(m)}(t)) + \frac{\phi_5(E_{h(m)}(t))}{2c_2}E'_{h(m+1)}(t). \tag{5.23}$$

Finally, adding (5.18) and (5.23) multiplied by $c_2\phi_5(E_{h(m)}(t))^{-1}$ yields

$$E_{h(m+1)}(t) \leq C_3(t) + (\epsilon + C_\epsilon t)\{\phi_6(E_{h(m)}(T_{54})) + \phi_7(E_{h(m)}(T))E'_{h(m)}(T_{54})\}, \tag{5.24}$$

with polynomials ϕ_i $(i = 6,7)$ having same properties as ϕ_i $(i = 1,2,\ldots,5)$. Take $M > 0$ such that $E_{h(m)}(T_{54}) < M$ holds. We first take ϵ so that

$$\epsilon(\phi_6(M) + \phi_7(M)M) < M - C_3(T_{54})$$

holds, and then $T_{55} \in (0, T_{54}]$ so that

$$C_\epsilon T_{55}\{\phi_6(M) + \phi_7(M)M\} < M - C_3(T_{55}) - \epsilon\{\phi_6(M) + \phi_7(M)M\}.$$

Consequently we obtain $E_{h(m+1)}(T_{55}) < M$ from the assumption $E_{h(m)}(T_{55}) < M$. Note that here M is independent of h. Actually, we can take M independent of h such that

$$E_{h(0)} \leq M$$

holds, and due to the induction process above, M is independent of h for all m. In addition, it is to be noted that T_{55} does not depend on m due to Lemma 5.5. By induction, $\{\mathcal{U}'_{h(m)}, u_{3h(m)}, F'_{h(m)}, \tilde{\eta}'_{h(m)}\}_{m=0}^{\infty}$ is well defined in $\widetilde{\mathcal{W}}'(T_{55})$ and $E_{h(m)}(T_{55}) < M$ for $m = 0,1,2,\ldots$.

Next, we prove the convergence of successive sequences. Subtract (5.10)–(5.13) with m replaced by $m-1$ from themselves. Then $\tilde{\mathcal{U}}'_{h(m)} \equiv \mathcal{U}'_{h(m)} - \mathcal{U}'_{h(m-1)}$, $\tilde{\tilde{\eta}}'_{h(m)} \equiv$

$\tilde{\eta}'_{h(m)} - \tilde{\eta}'_{h(m-1)}$, $\tilde{u}_{3h(m)} \equiv u_{3h(m)} - u_{3h(m-1)}$, $\tilde{F}'_{h(m)} \equiv F'_{h(m)} - F'_{h(m-1)}$, satisfies the problem as follows:

$$\begin{cases} \dfrac{\partial \tilde{\mathcal{U}}'_{h(m+1)}}{\partial t} - \mathcal{L}_{F_{h(m)}} \tilde{\mathcal{U}}'_{h(m+1)} \\ = [\mathcal{E}^{(m)}_{h,1} - \mathcal{E}^{(m-1)}_{h,1}] + [\mathcal{L}_{F_{h(m)}} - \mathcal{L}_{F_{h(m-1)}}]\mathcal{U}'_{h(m)} \text{ in } \tilde{\Omega}_{T_{55}}, \\ \mathcal{B}_{1,F_{h(m)}} \tilde{\mathcal{U}}'_{h(m+1)} = \left[\mathcal{E}^{(m)}_{h,2} - \mathcal{E}^{(m-1)}_{h,2}\right] + \left[\mathcal{B}_{1,F_{h(m)}} - \mathcal{B}_{1,F_{h(m-1)}}\right] \mathcal{U}'_{h(m)} \text{ on } \mathrm{I} \\ \mathcal{B}_{2,F_{h(m)}} \tilde{\mathcal{U}}'_{h(m+1)} = \left[\mathcal{E}^{(m)}_{h,3} - \mathcal{E}^{(m-1)}_{h,3}\right] + \left[\mathcal{B}_{2,F_{h(m)}} - \mathcal{B}_{2,F_{h(m-1)}}\right] \mathcal{U}'_{h(m)} \text{ on } \mathrm{I} \\ \tilde{\mathcal{U}}'_{h(m+1)}|_{t=0} = (\mathbf{0}, 0, 0) \quad \text{on } \tilde{\Omega}, \end{cases} \tag{5.25}$$

$$\begin{cases} \dfrac{\partial \tilde{\tilde{\eta}}'_{h(m+1)}}{\partial t} - L_{1,F_{h(m)}} \tilde{\tilde{\eta}}'_{h(m+1)} \\ = [L_{1,F_{h(m)}} - L_{1,F_{h(m-1)}}] \tilde{\tilde{\eta}}'_{h(m+1)} + [\mathcal{E}^{(m)}_{h,4} - \mathcal{E}^{(m-1)}_{h,4}] \quad \text{in } \tilde{\Omega}_{T_{55}}, \\ a^{33}_{h(m)} \dfrac{\partial \tilde{\tilde{\eta}}'_{h(m+1)}}{\partial y_3} = (a^{33}_{h(m)} - a^{33}_{h(m-1)}) \dfrac{\partial \tilde{\tilde{\eta}}'_{h(m)}}{\partial y_3} \\ +[\mathcal{E}^{(m)}_{h,5} - \mathcal{E}^{(m-1)}_{h,5}] \quad \text{on } \tilde{\Gamma}_{T_{55}}, \\ \tilde{\tilde{\eta}}'_{h(m+1)} = 0 \quad \text{on } \tilde{\Gamma}_{bT_{55}}, \\ \tilde{\tilde{\eta}}'_{h(m+1)}|_{t=0} = 0 \quad \text{on } \tilde{\Omega}, \end{cases} \tag{5.26}$$

$$\tilde{u}_{3h(m+1)} = -\tilde{\boldsymbol{u}}_{h(m+1)}|_{y_3=b} \cdot \nabla b - \tilde{\tilde{\eta}}_{h(m+1)} \quad \text{in } \tilde{\Omega}_{T_{55}}, \tag{5.27}$$

$$\tilde{F}'_{h(m+1)} = F_0(X^{-1}_{\boldsymbol{u}_{h(m)}}) - F_0(X^{-1}_{\boldsymbol{u}_{h(m-1)}}) + \int_0^t \tilde{u}_{3h(m)}\,(y', F_0(y'), \tau)\, \mathrm{d}\tau \quad \text{in } \boldsymbol{I} \tag{5.28}$$

Now we introduce following notations for simplicity:

$$\tilde{E}_{h(m)}(t) \equiv \| (\tilde{\mathcal{U}}'_{h(m)}, \tilde{F}'_{h(m)}) \|_{\mathcal{W}'_{\mathcal{K}}(t)} + \| \tilde{\tilde{\eta}}'_{h(m)} \|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)} + \| \tilde{u}_{3h(m)} \|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)},$$

$$\tilde{E}'_{h(m)}(t) \equiv \| \tilde{\mathcal{U}}'_{h(m)} \|_{\mathcal{W}'_{\mathcal{U}}(t)}.$$

Lemma 5.6 *Under the assumptions of* Lemma 5.4, *following estimates hold for any* $t \in (0, T_{55}]$:

$$\left\| \mathcal{E}^{(m)}_{h,1} - \mathcal{E}^{(m-1)}_{h,1} \right\|_{\mathcal{W}_1(t)} \le (\epsilon + C_\epsilon t)\phi(\sum_{i=m-1}^{m} E_{h(i)}\,(t)) \tilde{E}_{h(m)}(t)$$

$$\left\|\mathcal{E}_{h,2}^{(m)}-\mathcal{E}_{h,2}^{(m-1)}\right\|_{\mathcal{W}_2(t)}\le(\epsilon+C_\epsilon t)\phi(\sum_{i=m-1}^{m}E_{h(i)}(t))\parallel\tilde{\mathcal{K}}'_{h(m)}\parallel_{\mathcal{W}'_{\mathcal{K}}(t)}$$

$$\left\|\mathcal{E}_{h,3}^{(m)}-\mathcal{E}_{h,3}^{(m-1)}\right\|_{\mathcal{W}_3(t)}\le(\epsilon+C_\epsilon t)\phi(\sum_{i=m-1}^{m}E_{h(i)}(t))\parallel\tilde{F}_{h(m)}\parallel_{\tilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)}$$

$$\left\|\mathcal{E}_{h,4}^{(m)}-\mathcal{E}_{h,4}^{(m-1)}\right\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}$$

$$\le\phi(E_{h(m)}(t))\tilde{E}_{h(m+1)}(t)+(\epsilon+C_\epsilon t)\phi(\sum_{i=m-1}^{m}E_{h(i)}(t))\tilde{E}_{h(m)}(t),$$

$$\left\|\mathcal{E}_{h,5}^{(m)}-\mathcal{E}_{h,5}^{(m-1)}\right\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_t)}$$

$$\le C\{1+\phi(E_{h(m-1)}(t))\}\parallel\tilde{u}_{h(m+1)}\parallel_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)}$$

$$+(\epsilon+C_\epsilon t)\phi(\sum_{i=m-1}^{m}E_{h(i)}(t))\parallel u_{h(m+1)}\parallel_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)}\parallel\tilde{F}'_{h(m)}\parallel_{\tilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)},$$

Proof. We show the estimate of

$$\left\|\mathcal{G}_{1,u_{3h(m)}F_{h(m)}}\mathcal{U}'_{h(m)}-\mathcal{G}_{1,u_{3h(m-1)}F_{h(m-1)}}\mathcal{U}'_{h(m-1)}\right\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}$$

as an example. We easily have

$$\left\|A_{1(m)}\frac{\partial\boldsymbol{u}_{h(m)}}{\partial y_3}-A_{1(m-1)}\frac{\partial\boldsymbol{u}_{h(m-1)}}{\partial y_3}\right\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}$$

$$\le(\epsilon+C_\epsilon t)\phi(\sum_{i=m-1}^{m}\left\|F_{h(i)}\right\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)})$$

$$\times(\left\|\tilde{\boldsymbol{u}}_{h(m)}\right\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)}+\left\|\tilde{F}_{h(m)}\right\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)}),$$

where $A_{1(m)}=A_1(F_{h(m)})$. Next, we show a part of the estimate of

$$\parallel\tilde{G}_{4,F_{h(m)}}(\mathcal{K}_{h(m+1)},\mathcal{K}_{h(m)},u_{3h(m)},\tilde{\eta}_{h(m)})$$
$$-\tilde{G}_{4,F_{h(m-1)}}(\mathcal{K}_{h(m)},\mathcal{K}_{h(m-1)},u_{3h(m-1)},\tilde{\eta}_{h(m-1)})\parallel_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}.$$

After some calculations, we have

$$\left\|\nabla^2_{F_{h(m+1)}}\tilde{p}_{h(m+1)}^{(F_{h(m+1)})}-\nabla^2_{F_{h(m)}}\tilde{p}_{h(m)}^{(F_{h(m)})}\right\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}$$

$$\leq \phi(\sum_{i=m}^{m+1}(E_{h(i)}(t)+\| F_{h(i)} \|_{\widetilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)}))$$
$$\times (\tilde{E}'_{h(m+1)}+\| \tilde{F}'_{h(m+1)} \|_{\widetilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)}).$$

Other terms can be verified in the similar manner. From these estimates, we have the desired result. □

Lemma 5.7 *For $T > 0$ arbitrarily provided, under the assumption $F_0 \in W_2^{\frac{7}{2}+l}(\boldsymbol{R}^2)$, For $\tilde{F}_{h(m+1)}$, following estimates hold for any $t \in (0,T]$:*

$$\left\|\tilde{F}_{h(m+1)}\right\|_{\widetilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)}$$
$$\leq t \sum_{i=m-1}^{m} \| \tilde{\boldsymbol{u}}_{h(i)} \|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)} + (\epsilon + C_\epsilon t)C(t) \| \tilde{u}_{3h(m)} \|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)},$$

where $C(t)$ goes to 0 as t does.

Proof. Noting that

$$\tilde{F}_{h(m)}(y',t) = F_0(X^{-1}_{\boldsymbol{u}_{h(m)}}(y',t)) - F_0(X^{-1}_{\boldsymbol{u}_{h(m-1)}}(y',t)) + \int_0^t \tilde{u}_{3h(m)}(y',F_0(y'),\tau)\ \mathrm{d}\tau,$$

we first mention that

$$F_0(X^{-1}_{\boldsymbol{u}_{h(m)}}) - F_0(X^{-1}_{\boldsymbol{u}_{h(m-1)}}) = \int_0^1 \nabla F_0(X_\sigma)\cdot(X^{-1}_{\boldsymbol{u}_{h(m)}} - X^{-1}_{\boldsymbol{u}_{h(m-1)}})\ \mathrm{d}\sigma,$$

where $X_\sigma = \sigma X^{-1}_{\boldsymbol{u}_{h(m)}} + (1-\sigma)X^{-1}_{\boldsymbol{u}_{h(m-1)}} \in \boldsymbol{R}^2$. Under the assumption $F_0 \in W_2^{\frac{7}{2}+l}(\boldsymbol{R}^2)$, we have the estimate

$$\| F_0(X^{-1}_{\boldsymbol{u}_{h(m)}}) - F_0(X^{-1}_{\boldsymbol{u}_{h(m-1)}}) \|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)}$$
$$\leq C \| \nabla F_0(X_\sigma) \|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)} \| X^{-1}_{\boldsymbol{u}_{h(m)}} - X^{-1}_{\boldsymbol{u}_{h(m-1)}} \|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)}. \quad (5.29)$$

In order to estimate $\| \nabla F_0(X_\sigma) \|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)}$ in (5.29), we note that

$$|X_\sigma(y_1',t) - X_\sigma(y_2',t)|$$
$$\leq C|y_1'-y_2'|\{1 + t^{\frac{1}{2}}(1+\| F_0 \|_{W_2^{\frac{3}{2}+l}(\boldsymbol{R}^2)}) \sum_{i=m-1}^{m} \| \boldsymbol{u}_{(i)} \|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \}.$$

Therefore, we have

$$\| D^4F_0(X_\sigma)(\cdot,t) \|^2_{\dot{W}_2^{l-\frac{1}{2}}(\boldsymbol{R}^2)} \le \{1+t^{\frac{1}{2}}(1+\| F_0 \|_{W_2^{\frac{3}{2}+l}(\boldsymbol{R}^2)}) \| \boldsymbol{u}_\sigma \|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} \}^{1+2l}$$

$$\times \int_{\boldsymbol{R}^2}\int_{\boldsymbol{R}^2} | D^4F_0(X_\sigma(y^{(1)\prime},t) - D^4F_0(X_\sigma(y^{(2)\prime},t)|^2$$

$$\times |X_\sigma(y^{(1)\prime},t) - X_\sigma(y^{(2)\prime},t)|^{-(1+2l)}\ \mathrm{d}y^{(1)\prime}\mathrm{d}y^{(2)\prime}.$$

In order to calculate the integrand in the right-hand side of the above estimate, let us introduce the transform of the variable $y^{(i)\prime} = X_\sigma(q^{(i)})$ $(i=1,2)$, where $q^{(i)} \in \boldsymbol{R}^2$, whose Jacobian is estimated from above as follows:

$$J(\frac{\partial y^{(i)\prime}}{\partial q^{(i)}}) = \{1 - \int_0^t (\frac{\partial \boldsymbol{u}_\sigma}{\partial x_1} + \frac{\partial \boldsymbol{u}_\sigma}{\partial x_3}\frac{\partial F_0}{\partial x_1})\ \mathrm{d}\tau\}\{1 - \int_0^t (\frac{\partial \boldsymbol{u}_\sigma}{\partial x_2} + \frac{\partial \boldsymbol{u}_\sigma}{\partial x_3}\frac{\partial F_0}{\partial x_2})\ \mathrm{d}\tau\}$$

$$- \int_0^t (\frac{\partial \boldsymbol{u}_\sigma}{\partial x_1} + \frac{\partial \boldsymbol{u}_\sigma}{\partial x_3}\frac{\partial F_0}{\partial x_1})\ \mathrm{d}\tau \int_0^t (\frac{\partial \boldsymbol{u}_\sigma}{\partial x_2} + \frac{\partial \boldsymbol{u}_\sigma}{\partial x_3}\frac{\partial F_0}{\partial x_2})\ \mathrm{d}\tau$$

$$\le Ct^2 \| \boldsymbol{u}_\sigma \|^2_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)} (1+\| F_0 \|_{W_2^{\frac{3}{2}+l}(\boldsymbol{R}^2)})^2,$$

and we have

$$\int_{\boldsymbol{R}^2}\int_{\boldsymbol{R}^2} | D_y^4F_0(X_\sigma(y_1',t)) - D_y^4F_0(X_\sigma(y_2',t))|^2 |X_\sigma(y_1',t) - X_\sigma(y_2',t)|^{-(1+2l)}\ \mathrm{d}y_1'\mathrm{d}y_2'$$

$$\le Ct^2 \| \boldsymbol{u}_\sigma \|^2_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)} (1+\| F_0 \|_{W_2^{\frac{3}{2}+l}(\boldsymbol{R}^2)})^2.$$

In addition, since $\tilde{u}_{3h(m)}|_{t=0} = 0$, it is obvious that

$$\left\| \int_0^t \tilde{u}_{3h(m)}\,(y,\tau)\ \mathrm{d}\tau \right\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)} \le ((\epsilon + C_\epsilon t) + t)\left\|\tilde{u}_{3h(m)}\right\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)}.$$

□

Now, for the solution to (5.25)–(5.28), we have the following inequalities for any $t \in (0,T_{55})$ by virtue of Lemmas 5.6–5.7:

$$\tilde{E}'_{h(m+1)}(t) \le C(\epsilon + C_\epsilon t)\phi(\sum_{i=m-1}^{m} E_{h(i)}\,(T_{55}))\tilde{E}_{h(m)}(t), \tag{5.30}$$

$$\| \tilde{\tilde{\eta}}'_{h(m+1)} \|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)} \le (\epsilon + C_\epsilon t)\phi(\sum_{i=m-1}^{m} E_{h(i)}\,(T_{55}))\phi_0(\tilde{E}_{h(m)}(t))$$

$$+\phi(\sum_{i=m-1}^{m} E_{h(i)}\,(T_{55}))\ \| \tilde{F}_{h(m+1)} \|_{\tilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)}$$

$$+\phi(\sum_{i=m-1}^{m+1} E_{h(i)}(T_{55}))\tilde{E}'_{h(m+1)}(t), \tag{5.31}$$

$$\begin{aligned}\left\|\tilde{u}_{3h(m+1)}\right\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)} \\ \leq C(\left\|\tilde{\boldsymbol{u}}_{h(m+1)}\right\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)} \\ + \left\|\tilde{\tilde{\eta}}_{h(m+1)}\right\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)}),\end{aligned} \tag{5.32}$$

$$\begin{aligned}&\left\|\tilde{F}'_{h(m+1)}\right\|_{\tilde{W}_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\boldsymbol{R}_t^2)} \\ &\leq (\epsilon + C_\epsilon t)\phi(\sum_{i=m-1}^{m} E_{h(i)}(T_{55}))(\| \tilde{\boldsymbol{u}}'_{h(m)} \|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)} + \\ &\| \tilde{u}'_{3h(m)} \|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)}).\end{aligned} \tag{5.33}$$

By the similar process to the derivation of (5.24), we have from (5.30)–(5.33),

$$\tilde{E}_{h(m+1)}(t) \leq (\epsilon + C_\epsilon t)\phi_8(\sum_{i=m-1}^{m+1} E_{h(i)}(T_{55}))\tilde{E}_{h(m)}(t) \tag{5.34}$$

for any $t \in (0, T_{55}]$. Take ϵ small enough so that $\epsilon\phi_8(3M) < 1$ holds, and then $T_{56} \in (0, T_{55}]$ so that $C_\epsilon T_{56}\phi_8(3M) < 1 - \epsilon\phi_8(3M)$ holds. Then we have

$$\tilde{E}_{h(m+1)}(t) \leq r\tilde{E}_{h(m)}(t), \quad r = (\epsilon + C_\epsilon T_{56})\phi_8(2M) \in (0,1), \quad t \in (0, T_{56}].$$

Thus we can verify that

$$\{(\mathcal{U}'_{h(m)}, u_{3h(m)}, \tilde{\eta}_{h(m)}, F'_{h(m)})\}_{m=0}^{\infty}$$

is a Cauchy sequence in $\widetilde{\mathcal{W}}'(T_{56})$. Therefore, the limit

$$(\mathcal{U}'_h, u_{3h}, \tilde{\eta}'_h, F'_h) \equiv \lim_{m\to\infty}(\mathcal{U}'_{h(m)}, u_{3h(m)}, \tilde{\eta}_{h(m)}, F'_{h(m)})$$

exists in $\widetilde{\mathcal{W}}'(T_{56})$. Uniqueness of the solution can be proved by virtue of an analogous inequality to (5.34). Since $(\mathcal{U}'_h, u_{3h}, \tilde{\eta}'_h, F'_h)$ is bounded from above by a constant M independent of h and continuous with respect to h, it is possible to take a limit $\lim_{h\to 0}(\mathcal{U}'_h, u_{3h}, \tilde{\eta}'_h, F'_h) = (\mathcal{U}', u_3, \tilde{\eta}', F')$. Then, noting that

$$\lim_{h\to 0}\mathcal{P}_{F_h}\boldsymbol{u}_h = \nabla_F \cdot \boldsymbol{u},$$

$(\mathcal{U}', u_3, \tilde{\eta}', F')$ obviously satisfies the following problems:

$$\begin{cases} \dfrac{\partial \mathcal{U}'}{\partial t} - \mathcal{L}_F \mathcal{U}' = \mathcal{G}_{1,u_3,F}\mathcal{U}' + \mathcal{L}_F \overline{\mathcal{U}}_0 - \dfrac{\partial \overline{\mathcal{U}}_0}{\partial t} & \text{in } \widetilde{\Omega}_{T_{56}}, \\ \mathcal{B}_{1,F}\mathcal{U}' = \mathcal{G}_{2,F,S}\overline{\mathcal{U}}_0 \quad \text{on } \tilde{\Gamma}_{T_{56}}, \\ \mathcal{B}_{2,F}\mathcal{U}' = \mathcal{G}_{3,F}\overline{\mathcal{U}}_0 \quad \text{on } \tilde{\Gamma}_{bT_{56}}, \\ \mathcal{U}'|_{t=0} = (\mathbf{0}, 0, 0) \quad \text{on } \widetilde{\Omega}, \end{cases}$$

$$\begin{cases} \dfrac{\partial \tilde{\eta}'}{\partial t} - L_{1,F}\tilde{\eta}' = G_{4,F}(\boldsymbol{u}, u_3, \tilde{\eta}, \tilde{\theta}, \tilde{S}) + L_{1,F}\bar{\tilde{\eta}}_0 - \dfrac{\partial \bar{\tilde{\eta}}_0}{\partial t} & \text{in } \widetilde{\Omega}_{T_{56}}, \\ a^{33}(F)\dfrac{\partial \tilde{\eta}'}{\partial y_3} = -a^{33}(F)\dfrac{\partial \bar{\tilde{\eta}}_0}{\partial y_3} + \nabla_F \cdot \boldsymbol{u} \quad \text{on } \tilde{\Gamma}_{T_{56}}, \\ \tilde{\eta}' = 0 \quad \text{on } \tilde{\Gamma}_{bT_{56}}, \\ \tilde{\eta}'|_{t=0} = 0, \end{cases}$$

$$\begin{aligned} &u_3 = -\boldsymbol{u}|_{y_3=b(y')} \cdot \nabla_F b - \tilde{\eta} \quad \text{in } \widetilde{\Omega}_{T_{56}}, \\ &F'(y',t) = F_0(X_{\boldsymbol{u}}^{-1}(y',t)) + \int_0^t u_3\,(y', F_0(y'), \tau)\ \mathrm{d}\tau - \overline{F}_0 \quad \text{in } \boldsymbol{R}^2_{T_{56}}. \end{aligned}$$

Now that $\boldsymbol{u} \in W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_{T_{56}})$, and the unique solutiuon $\tilde{\eta}$ is consistent with

$$\frac{1}{a^{33}(F)} \int_{b(y')}^{y_3} \nabla_F \cdot \boldsymbol{u}(y', z_3, t)\ \mathrm{d}z_3.$$

Thus, $(\mathcal{U}, u_3, F) = (\mathcal{U}' + \overline{\mathcal{U}}_0, u_3, F' + \overline{F}_0)$ is the solution of (4.1)–(4.4). By virtue of the similar arguments in the proof of Lemma 5.3, $(\boldsymbol{v}, w, \theta, S)$ defined by

$$(\boldsymbol{v}, w, \theta, S)(x,t) = (\boldsymbol{u}, u_3, \tilde{\theta}, \tilde{S})(\Phi(x,t))$$

is well-defined from $(\boldsymbol{u}, u_3, \tilde{\theta}, \tilde{S})$, and $(\boldsymbol{v}, w, \theta, S, F)$ belongs to $\mathcal{W}_x(T)$.

Now we shall show $0 < \underline{\theta}_0/2 \le \tilde{\theta}(y,t) < \infty$ and $0 < \underline{S}_0/2 \le \tilde{S}(y,t) < \infty$ by taking the time interval small enough again. Indeed, since $\tilde{\theta}' = \tilde{\theta} - \bar{\tilde{\theta}}_0 \in W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_{T_{66}})$,

$$\begin{aligned} \tilde{\theta}(y,t) &\ge \widetilde{\bar{\theta}_0}|_{t=0}(y) - \left(|\tilde{\theta}'(y,t)| + |\bar{\tilde{\theta}}_0(y,t) - \bar{\tilde{\theta}}_0(y,0)|\right) \\ &\ge \underline{\theta}_0 - t^\gamma (\sup_{y\in\widetilde{\Omega}} |\,\tilde{\theta}'(y,t)|_t^{(\gamma)} + \sup_{y\in\widetilde{\Omega}} |\,\bar{\tilde{\theta}}_0(y,t)|_t^{(\gamma)}), \end{aligned}$$

where $|f|_t^{(\gamma)}$ stands for the Hölder coefficient of f with respect to t with exponent

$0<\gamma<\frac{l}{2}-\frac{1}{4}$. Since Sobolev embedding theorem implies

$$\sup_{y\in\tilde\Omega}|\tilde\theta'(y,t)|_t^{(\gamma)}\le\|\tilde\theta'\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde\Omega_{T_{56}})},\quad \sup_{y\in\tilde\Omega}|\bar{\tilde\theta}_0(y,t)|_t^{(\gamma)}\le\|\bar{\tilde\theta}_0'\|_{\overline{W}_2^{3+l,\frac{3+l}{2}}(\tilde\Omega_{T_{56}})},$$

if we take

$$T_{57}\equiv\underline{\theta}_0^{\frac{1}{\gamma}}2^{-\frac{1}{\gamma}}\left(\|\tilde\theta'\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde\Omega_{T_{56}})}+\|\bar{\tilde\theta}_0'\|_{\overline{W}_2^{3+l,\frac{3+l}{2}}(\tilde\Omega_{T_{56}})}\right)^{-\frac{1}{\gamma}},$$

then we have $\underline{\theta}_0/2\le\tilde\theta(y,t)<\infty$ on $[0,T_{57}]$. A similar argument holds for $\tilde S$. Denote again the time interval by $[0,T_{57}]$ on which both $\underline{\theta}_0/2\le\tilde\theta(y,t)<\infty$ and $\underline{S}_0/2\le\tilde S(y,t)<\infty$ hold. $T^*=\min\{T_{56},T_{57}\}$ provides the desired result, since the boundedness is invariable under the coordinate transform. This completes the proof of the main theorem.

REFERENCES

[1] Beale, J. T. Large-time regularity of viscous surface waves. Arch. Rat. Mech. Anal. **84** (1984), 307–352.

[2] Bleck R. and Boudra D. B. Wind-driven spin-up eddy resolving ocean models formulated in isopycnic and isobaric coordinates. J. Geophys. Res. **91** (1986), 7611–7621.

[3] Bryan K. A numerical method for the study of the circulation of the world ocean. J. Comp. Phys. **4** (1969), 347–376.

[4] Bryan, K. & Cox, M. D. An approximate equation of state for numerical models of ocean circulation. J. Phys. Oceanogr. **2** (1972), 510–514.

[5] Cao, C. and Titi, E. S. Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics. Ann. Math. **166** (2007), 245–267.

[6] Cox, M. D. A primitive equation, three-dimensional model of the ocean. GFDL Ocean Group Technical Rep. (1984).

[7] Crowley, W. P. A numerical model for viscous, free-surface, barotropic wind driven ocean circulations. J. Comp. Phys. **5** (1970), 139–168.

[8] Guillén-González, F. & Rodríguez-Bellido, M. A. A review on the improved regularity for the primitive equations. In: *Regularity and other aspects of the Navier–Stokes equations* (eds. P. Mucha, P. Penel, M. Wiegner and W. Zajaczkowski), Banach Center (2005), 85–103.

[9] Guillén-González F., Masmoudi, M. &Rodríguez-Bellido, M. A. Anisotropic

estimates and strong solutions of the Primitive Equations. Diff. Integ. Equ. (2001), 1381–1408.

[10] Honda, H. and Tani, A. Small-time existence of a strong solution of primitive equations of the coupled atmosphere and the ocean. Sūrikaisekikenkyūsho Kōkyūroku (2009), 12–33.

[11] Honda, H. and Tani, A. Small-time existence of a strong solutionof primitive equations for the atmosphere. Adv. Math. Sci. Appl. (2010), 547–583.

[12] Honda, H. and Tani, A. Small-time existence of a strong solution of primitive equations for the ocean. Tokyo $Math.$ **35**(2012),97– 138.

[13] Honda, H. Solvability of primitive equations for the ocean with vertical mixing. Proc. Japan Acad. Ser. A (2012), 132–137.

[14] Honda, H. Small-time Existence of a Strong Solution of Primitive Equations for the Ocean and the Atmosphere. Ph.D thesis, Keio University in Yokohama, 2011.

[15] Killworth, P. D., Stainforth, D., Webb, D. J. & Peterson, S. M. The Development of a Free-Surface Bryan-Cox-Semtner Ocean Model. J. Phys. Oceanogr. **21** (1991), 1333–1348.

[16] Ladyzenskaja, O. A., Solonnikov, V. A. & Ural'ceva, N. N. Linear and Quasi-linear Equations of Parabolic Type. 1st edition, American Mathematical Society, Rhode Island (1968).

[17] Lions, J. L., Temam, R. & Wang, S. New formulations of the primitive equations of atmosphere and applications. Nonlinearity **5** (1992), 237–288.

[18] Lions, J. L., Temam, R. & Wang, S. On the equations of the large-scale-ocean. Nonlinearity **5** (1992), 1007–1053.

[19] Solonnikov, V. A. On Boundary Value Problems to the Linear Parabolic Systems of Differential Equations of General Form (Russian). Trudy Mat. Inst. Steklov. (1965), 3–163. English Transl. *Proc. Steklov Inst. Math.* (1965), 1–184.

[20] Tanaka, N. & Tani, A. Surface waves for a compressible viscous fluid. J. Math. Fluid Mech. (2003), 303–363.

[21] Temam. R. & Ziane, M. Some mathematical problems in geophysical fluid dynamics. In: *Handbook of Mathematical Fluid Dynamics* (eds. S. J. Friedlander and D. Serre), North-Holland (2004), 535–657.

[22] UNESCO, 1981: Tenth report of the Joint Panel on Oceanographic Tables and Standards, Sidney, B. C. September 1980., Unesco Technical papers in marine science, 36, 25pp.

[23] Washington, W. M. & Parkinson, C. L. An Introduction to Three-Dimensional Climate Modeling. 2 nd edition, Oxford University Press, Berlin (1986).

[24] Wloka, J. Partielle Differentialgleichungen, 2nd edition, B. G. Teubner, Berlin (1982).

[25] Ziane, M. Regularity results for Stokes type systems. Appl. Anal. (1995), 263–292.

[26] Ziane, M. Regularity results for the stationary primitive equations of atmosphere and the ocean. Nonlinear Anal. **28** (1997), 289–313.