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Proposed methods to construct different categories of Radon measure manifolds - An analytical approach

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Abstract

In this paper, S. C. P. Halakatti developed different categories of Radon measure manifolds (M, τ, Σ, μ_R), Quotient Radon measure manifolds $(\mathcal{M}, \tau, \Sigma, \mu_R)$, and Network Radon measure manifolds $((\mathcal{M}, \tau, \Sigma, \mu_R), (G, \circ), (\mathcal{G}, \circ))$ under the measurable homeomorphism and Radon measure structure - invariant map.

AMS subject classification: 28-XX, 54-XX, 57N-XX, 58-XX.
Keywords: Radon measure manifold, Quotient Radon Measure Manifold, Network Radon measure manifold, measurable homeomorphism and Radon measure structure - invariant map, measurable Lie group.

This paper is organized into six sections as follows:

1. Preliminaries

1.1. Introduction

In 2014, a new concept of Measure Manifold (M, τ, Σ, μ) [18] was introduced by S. C. P. Halakatti to study different intrinsic properties of Measure Manifolds. By modeling a Hausdorff second countable topological space (M, τ) onto a measure space (R^n, τ, Σ, μ) [18],[19],[20] with the help of measurable homeomorphism and measure - invariant function ϕ [19], S. C. P. Halakatti has introduced the concept of Measure Manifold (M, τ, Σ, μ) in terms of measure charts and measure atlases along with equivalence relations, viz. $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mathcal{A}_1 \cup \mathcal{A}_2 \in A^k(M)$ induces differentiable structure on (M, τ, Σ, μ) and $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mu(\mathcal{A}_1) = \mu(\mathcal{A}_2)$, induces measure structure on (M, τ, Σ, μ). In paper [23],[24], on the complete measure manifold (M, τ, Σ, μ), three different relations like local path connectedness, internal path connectedness and maximal path connectedness on the three measurable domains (U_i, ϕ_i), $[(U_i, \phi_i) \cup (U_j, \phi_j)]$ and $(\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k)$ respectively were introduced by S. C. P. Halakatti and was proved that they were equivalence relations.

In this paper, based on methodologies IV, V, VI and VII introduced and developed by S. C. P. Halakatti [28] and in the process of development of the theory of measure manifold, continues to analyze these three equivalence relations on metrizable Radon measure manifold (M, τ, Σ, μ_R) and observes that these three equivalence relations induce a partition in the corresponding measurable domains (U_i, ϕ_i), $[(U_i, \phi_i) \cup (U_j, \phi_j)]$ and $(\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k)$. And we have studied this process of development and analyzed these results and contribute some more results and examples towards the development of the theory of measure manifold. Since these three measurable domains are Radon measurable, one can study different patterns of these domains in terms of compact, Lindelof, countably compact, semi-compact, semi-Lindelof and semi-countably compact. Such a study induces different categories of Quotient Radon measure manifolds. S. C. P. Halakatti has proposed a method to generate Quotient Radon measure manifold and construct different categories of Quotient Radon measure manifolds under measurable homeomorphism and Radon measure structure - invariant map by analytical method. Also using two measurable group structures $(G, \circ)$ and $(\mathcal{G}, \circ)$ on Quotient Radon measure manifold, one can introduce the concept of Network Radon measure manifold along with its categories.

1.2. Some basic concepts

The following basic definitions are necessary for the results in this paper.

Definition 1.2.1 Measure chart [18] A measurable chart $((U, \tau_{/U}, \Sigma_{/U}), \phi)$ is called a measure chart, if $\mu_{/U}$ on $((U, \tau_{/U}, \Sigma_{/U}), \phi)$ satisfies the following conditions:

(i) ϕ is homeomorphism,

(ii) ϕ is measurable function
i.e $\phi^{-1}(V) = U = (U, \tau_{/U}, \Sigma_{/U})$, $V \in (R^n, \tau_1, \Sigma_1)$ and $(U, \tau_{/U}, \Sigma_{/U}) \subseteq (M, \tau, \Sigma)$,

(iii) ϕ is measure invariant.
Then, the structure $((U, \tau_{/U}, \Sigma_{/U}, \mu_{/U}), \phi)$ is called as a measure chart.

Definition 1.2.2 Measure atlas [18] By an R^n-measure atlas of class $C^k (k \geq 1)$ on measure manifold (M, τ, Σ, μ), we mean a countable collection $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}}, \mu_{/\mathcal{A}})$ of n-dimensional measure charts
$((U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{/U_i}), \phi_{i_{/U_i}})$ for all $i \in I$ on (M, τ, Σ, μ) satisfying the following conditions:
(a$_1$) $\cup_{i \in I}(U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{/U_i}) = (M, \tau, \Sigma, \mu)$. That is, the countable union of all measure charts in $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}}, \mu_{/\mathcal{A}})$ cover (M, τ, Σ, μ_R) .
(a$_2$) For any pair of measure charts $((U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{/U_i}), \phi_{i_{/U_i}})$ and
$((U_j, \tau_{/U_j}, \Sigma_{/U_j}, \mu_{/U_j}), \phi_{j_{/U_j}})$ in $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}}, \mu_{/\mathcal{A}})$, the transition maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are:

(1) differentiable maps of class C^k $(k \geq 1)$
i.e., $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j) \subseteq (R^n, \tau_1, \Sigma_1, \mu_1)$ and $\phi_j \circ \phi_i^{-1}$:

$\phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j) \subseteq (R^n, \tau_1, \Sigma_1, \mu_1)$ are differentiable maps of class C^k $(k \geq 1)$.

(2) measurable, i.e., the transition maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are measurable functions if,

a) any Borel subset $K \subseteq \phi_i(U_i \cap U_j)$ is measurable in $(R^n, \tau_1, \Sigma_1, \mu_1)$, then $(\phi_i \circ \phi_j^{-1})^{-1}(K) \in \phi_j(U_i \cap U_j)$ is also measurable,

b) $\phi_j \circ \phi_i^{-1}$ is measurable if $S \subseteq \phi_j(U_i \cap U_j)$ is measurable in $(R^n, \tau_1, \Sigma_1, \mu_1)$, then $(\phi_j \circ \phi_i^{-1})^{-1}(S) \in \phi_i(U_i \cap U_j)$ is also measurable.
(a_3) Any two atlases $(\mathcal{A}_1, \tau_{/\mathcal{A}_1}, \Sigma_{/\mathcal{A}_1}, \mu_{/\mathcal{A}_1})$,$(\mathcal{A}_2, \tau_{/\mathcal{A}_2}, \Sigma_{/\mathcal{A}_2}, \mu_{/\mathcal{A}_2})$ are compatible on
(M, τ, Σ, μ) satisfying the two equivalence relations:
i) $\mathcal{A}_1 \sim \mathcal{A}_2$, iff $\mathcal{A}_1 \cup \mathcal{A}_2 \in A^k(M)$,
ii) $\mathcal{A}_1 \sim \mathcal{A}_2$, iff $\mu(\mathcal{A}_1) = \mu(\mathcal{A}_2)$.

Definition 1.2.3 Measure Manifold [18]
A measure space (M, τ, Σ, μ) together with a differentiable structure of class C^k and a measure structure induced by μ is called a Measure Manifold of class C^k.

The concepts of measurable homeomorphism and measure-invariant transformation which were introduced by S. C. P. Halakatti in [7] are necessary to construct different categories of Radon measure Manifolds in this paper.

Definition 1.2.4 Measurable homeomorphism [19]
Let $(M_1, \tau_1, \Sigma_1, \mu_1)$ and $(M_2, \tau_2, \Sigma_2, \mu_2)$ be two measure manifolds. Then the function $F : M_1 \longrightarrow M_2$ is called measurable homeomorphism if,

(i) F is bijective and bi continuous

(ii) F and F^{-1} are measurable.

Definition 1.2.5 Measure invariant function/Pull back Measure [19]
Let $(M_1, \tau_1, \Sigma_1, \mu_1)$ and $(M_2, \tau_2, \Sigma_2, \mu_2)$ be measure manifolds and $F : M_1 \longrightarrow M_2$ be measurable homeomorphism. Then F is said to be **measure-invariant function** if for all measure charts $(U, \phi) \in M_2$ we have, $\mu_1(F^{-1}(U)) = \mu_2(U)$ where $F^{-1}(U) \in M_1$.

Radon measure conditions on Borel subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ [21], [22], [25]
Let $\phi : (M, \tau, \Sigma, \mu_R) \longrightarrow (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ be a measurable homeomorphism and Radon measure-invariant transformation [7], [10]. If $U_i \subset (M, \tau, \Sigma, \mu_R)$ then $\phi(U_i) \subset (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ is a measurable subset of a Radon measure space $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$.

(i) A Radon measure μ_R for a measurable subset $\phi(U) \in \mathcal{B} \in (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ (where $\mathcal{B}$ is a collection of Borel subsets of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$) is a positive Borel measure $\mu :$ $\mathcal{B} \longrightarrow [0, \infty]$ which is finite on Borel compact subsets $\phi(K_i) \in \mathcal{B} \in (R^n, \tau_1, \Sigma_1, \mu_{R_1})$

and is inner regular in the sense that for every measurable compact subset $\phi(K_i) \in \phi(U)$, we have

$$\mu_R(\phi(U)) = \sup\{\mu_R(\phi(K_i)) : \forall i \in I; \phi(K_i) \subseteq \phi(U); \forall \phi(K_i) \in \phi(\mathcal{K})\}, \quad (1.2.1)$$

where $\phi(\mathcal{K}) \subset \phi(U)$ is the collection of Borel compact subsets $\phi(K_i) \in (R^n, \tau_1, \Sigma_1, \mu_{R_1})$.

(ii) Also μ_R is outer regular on a family $\phi(\mathcal{O})$ of measurable/Borel open subsets if, for every measurable compact subset $\phi(U)$ we have,

$$\mu_R(\phi(U)) = inf\{\mu_R(\phi(O_i)) : \forall i \in I; \phi(O_i) \supseteq \phi(U); \forall \phi(O_i) \in \phi(\mathcal{O})\}. \quad (1.2.2)$$

Now the measurable compact subset $\phi(U)$ is Radon measurable in $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$. Therefore, $\phi(U)$ is a **Radon measure subset** of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$.

The concepts of Radon measure chart, Radon measure atlas and Radon measure Manifold were introduced in [10] by S. C. P. Halakatti are used in this paper.

Radon measure conditions on measurable chart [25],[26]

(i) A Radon measure μ_R for a measurable chart $(U, \phi) \in \mathcal{B}$, (where $\mathcal{B}$ is a collection of measure charts of (M, τ, Σ, μ_R) is a positive Borel measure $\mu : \mathcal{B} \longrightarrow [0, \infty]$ which is finite on Borel compact subsets $K_i \in \mathcal{B}\,(M, \tau, \Sigma, \mu_R)$ and is inner regular in the sense that for every measurable chart (U_i, ϕ_i), we have

$$\mu_R(U_i) = \sup\{\mu_R(K_i) : \forall i \in I; K_i \subseteq (U_i, \phi_i); \forall K_i \in \mathcal{K}\},$$

where $\mathcal{K}$ is the collection of Borel compact subsets

$$K_i \subset (U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R), \quad (1.2.3)$$

(ii) Also μ_R is outer regular on a family $A^k(M)$ of measurable atlases $\mathcal{A}_i$ if, for every measurable chart $(U_i, \phi_i) \in A^k(M)$ we have,

$$\mu_R(U_i) = inf\{\mu_R(\mathcal{A}_i) : \forall i \in I; \mathcal{A}_i \supseteq (U_i, \phi_i); \mathcal{A}_i \in A^k(M)\}. \quad (1.2.4)$$

Definition 1.2.6 Radon measure chart [22],[26]
A measurable chart $((U_i, \tau_{/U_i}, \Sigma_{/U_i}), \phi_i)$ of (M, τ, Σ) equipped with a Radon measure $\mu_{R_{/U_i}}$ satisfying the Radon measure conditions 1.2.3 and 1.2.4 is called a Radon measure chart denoted by $((U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{R_{/U_i}}), \phi)$.
Since $\cup_{i=1}^{\infty}(U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{/U_i}) = (M, \tau, \Sigma, \mu)$, we can measure measurable manifold (M, τ, Σ) by Radon measure.

Radon measure conditions on measurable atlas [26]
A measurable atlas $\mathcal{A}_i$ is Radon measurable if it satisfies the following Radon measure conditions:

(i) Let $\mathcal{F} = \{\cup_{i=I}(U_i, \phi_i)\}$ be a family of all Radon measure charts of (M, τ, Σ, μ_R). A Radon measure μ_R of a measurable atlas $\mathcal{A}_i$ is a positive Borel measure $\mu : \mathcal{B} \longrightarrow [0, \infty]$ which is finite on measurable compact charts (U_i, ϕ_i) and is inner regular in the sense that for every measurable atlas $\mathcal{A}_i$, we have

$$\mu_R(\mathcal{A}_i) = \sup\{\mu_R(U_i) : \forall i \in I; U_i \subseteq \mathcal{A}_i; \forall U_i \in \mathcal{F}\}, \qquad (1.2.5)$$

(ii) Also μ_R is outer regular on a family $\mathcal{O}$ of measurable charts if, for every measurable atlas $\mathcal{A}_i \in A^k(\mathcal{M})$ we have,

$$\mu_R(\mathcal{A}_i) = inf\{\mu_R(\mathcal{O}) : \mathcal{O} \supseteq \mathcal{A}_i, \mathcal{O} \in \mathcal{M}\}. \qquad (1.2.6)$$

Definition 1.2.7 Radon measure atlas [26]

By an R^n-Radon measure atlas of class $C^k(k \geq 1)$ on a measurable manifold (M, τ, Σ), we mean a countable collection $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}}, \mu_{/\mathcal{A}})$ of n-dimensional Radon measure charts

$((U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{R_{/U_i}}), \phi_{i_{/U_i}})$ for all $i \in I$ on (M, τ, Σ, μ_R) satisfying the following conditions:

$$(\mathbf{a_1})\ \cup_{i\in I}\ (U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{R_{/U_i}}) = (M, \tau, \Sigma, \mu_R).$$

That is, the countable union of all Radon measure charts in

$$(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}}, \mu_{/\mathcal{A}}) \text{ cover } (M, \tau, \Sigma, \mu_R).$$

($\mathbf{a_2}$) For any pair of Radon measure charts $((U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{R_{/U_i}}), \phi_{i_{/U_i}})$ and $((U_j, \tau_{/U_j}, \Sigma_{/U_j}, \mu_{R_{1_{/U_j}}}), \phi_{j_{/U_j}})$ in $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}}, \mu_{/\mathcal{A}})$, the transition maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are:

(1) differentiable maps of class C^k $(k \geq 1)$
i.e., $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j) \subseteq (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j) \subseteq (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ are differentiable maps of class C^k $(k \geq 1)$.

(2) Radon measurable:
Transition maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are Radon measurable functions if,

a) any Borel subset $K \subseteq \phi_i(U_i \cap U_j)$ is Radon measurable in $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$, then $(\phi_i \circ \phi_j^{-1})^{-1}(K) \in \phi_j(U_i \cap U_j)$ is also Radon measurable.

b) $\phi_j \circ \phi_i^{-1}$ is Radon measurable if $S \subseteq \phi_j(U_i \cap U_j)$ is Radon measurable in $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$, then $(\phi_j \circ \phi_i^{-1})^{-1}(S) \in \phi_i(U_i \cap U_j)$ is also Radon measurable.
($\mathbf{a_3}$) Any two measure atlases $(\mathcal{A}_1, \tau_{/\mathcal{A}_1}, \Sigma_{/\mathcal{A}_1}, \mu_{/\mathcal{A}_1})$,$(\mathcal{A}_2, \tau_{/\mathcal{A}_2}, \Sigma_{/\mathcal{A}_2}, \mu_{/\mathcal{A}_2})$ are compatible on (M, τ, Σ, μ_R) satisfying the two equivalence relations:

i) $\mathcal{A}_1 \sim \mathcal{A}_2$, iff $\mathcal{A}_1 \cup \mathcal{A}_2 \in A^k(M)$,
ii) $\mathcal{A}_1 \sim \mathcal{A}_2$, iff $\mu_R(\mathcal{A}_1) = \mu_R(\mathcal{A}_2)$.

Radon measure structure on measure manifold [15]
For any two Radon measure atlases $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{A}^k(M) \subset (M, \tau, \Sigma, \mu_R)$, we say that $\mathcal{A}_1 \sim \mathcal{A}_2$ is an equivalence relation on (M, τ, Σ, μ_R) if and only if $\mu_R(\mathcal{A}_1) = \mu_R(\mathcal{A}_2)$. This equivalence relation induces a Radon measure-structure on (M, τ, Σ, μ_R).

Definition 1.2.8 Radon Measure Manifold [22]
A Radon measure space (M, τ, Σ, μ_R) with the above differentiable structure of class C^k and a Radon measure structure induced by μ_R forms a Radon measure manifold of class C^k.

In the following the path connectedness property is studied on Radon measure manifold.

Definition 1.2.9 [4]
A **path** in a topological space (X, τ) is a continuous function $f : I \longrightarrow X$. The path f is also called a path from $f(0)$ to $f(1)$. The points $f(0)$ and $f(1)$ are called the endpoints of f, and f is said to join the initial point $f(0)$ and the terminal point $f(1)$. Then path $\bar{f}(x) = f(1 - x)$ for each $x \in I$ is called the **reverse path of** f.

To develop the analysis on Radon measure manifold (M, τ, Σ, μ_R), S. C. P. Halakatti has used local, internal and maximal path connectedness relations on (M, τ, Σ, μ_R) which was introduced first in [11], [12].

Definition 1.2.10 Local path connectedness on Radon Measure Manifold [24]
A Radon measure manifold (M, τ, Σ, μ_R) is locally path connected if $\exists$ a measurable C^∞ path $\gamma_i : [0, 1] \longrightarrow (U_i, \phi_i)$ such that
$\gamma_i(0) = p_i \in (U_i, \phi_i)$ and
$\gamma_i(1) = p_j \in (U_i, \phi_i)$, for which $\mu_R(U_i) > 0$.
In other words, every p_i is locally path connected to every p_j in $(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R)$ such that local path connected measurable C^∞ paths γ_i form a non-empty set, say, $G = \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \in (U_i, \phi_i) \subset (M, \tau, \Sigma, \mu_R)$.

Note 1.2.11 Let $A_1 = \{p_i, q_i \in [p] \in (U_i, \phi_i)$: $\exists\, \gamma_i \in [\gamma]$: p_i is locally path connected to q_i by $\gamma_i \in G\}$ where $\mu_R(A_1) > 0$. If $\mu_R(A_1) = 0$ then A_1 represents the **dark region** of $(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R)$.

Definition 1.2.12 Internal path connectedness on Radon Measure Manifold [24]
A Radon measure manifold (M, τ, Σ, μ_R) is internally path connected if $\exists$ a measurable C^∞ path $\gamma_i : [0, 1] \longrightarrow [(U_i, \phi_i) \cup (U_j, \phi_j)] \in \mathcal{A}_i \in A^k(M)$ such that
$\gamma_i(0) = p_i \in (U_i, \phi_i)$, for which $\mu_R(U_i) > 0$ and
$\gamma_i(1) = p_j \in (U_j, \phi_j)$, for which $\mu_R(U_j) > 0$.
In other words, every $p_i \in (U_i, \phi_i)$ is internally path connected to every $p_j \in (U_j, \phi_j)$

of (M, τ, Σ, μ_R) such that internal path connected measurable C^∞ paths γ_i form a non-empty set, say, $G = \{\gamma_1, \gamma_2, \dots, \gamma_n\} \in [(U_i, \phi_i) \cup (U_j, \phi_j)] \subset (M, \tau, \Sigma, \mu_R)$.

Note 1.2.13 Let $A_2 = \{p_i, q_i \in [p] \in [(U_i, \phi_i) \cup (U_j, \phi_j)]$: $\exists\, \gamma_i \in [\gamma]$: p_i is internally path connected to q_i by $\gamma_i \in G\}$ where $\mu_R(A_2) > 0$. If $\mu_R(A_2) = 0$ then A_2 represents the **dark region** of $[(U_i, \phi_i) \cup (U_j, \phi_j)] \in (M, \tau, \Sigma, \mu_R)$.

Definition 1.2.14 Maximal path connectedness on Radon Measure Manifold [24]
A Radon measure manifold (M, τ, Σ, μ_R) is maximally path connected if $\exists$ a measurable C^∞ path $\gamma_i : [0, 1] \longrightarrow (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k) \in A^k(M)$ such that
$\gamma_i(0) = p_i \in (U_i, \phi_i) \in \mathcal{A}_i$, for which $\mu_R(U_i) > 0$, $\mu_R(\mathcal{A}_i) > 0$ and
$\gamma_i\left(\frac{1}{2}\right) = p_j \in (U_j, \phi_j) \in \mathcal{A}_j$, for which $\mu_R(U_j) > 0$, $\mu_R(\mathcal{A}_j) > 0$ and
$\gamma_i(1) = p_k \in (U_k, \phi_k) \in \mathcal{A}_k$, for which $\mu_R(U_k) > 0$, $\mu_R(\mathcal{A}_k) > 0$.

In other words, every $p_i \in (U_i, \phi_i) \in \mathcal{A}_i$ is maximally path connected to every $p_j \in (U_j, \phi_j) \in \mathcal{A}_j$ for $(\mathcal{A}_i \cup \mathcal{A}_j) \in A^k(M)$, $\mu_R(\mathcal{A}_i \cup \mathcal{A}_j) > 0$ and every $p_j \in (U_j, \phi_j) \in \mathcal{A}_j$ is maximally path connected to every $p_k \in (U_k, \phi_k) \in \mathcal{A}_k$ for $(\mathcal{A}_j \cup \mathcal{A}_k) \in A^k(M)$, $\mu_R(\mathcal{A}_j \cup \mathcal{A}_k) > 0$ of (M, τ, Σ, μ_R) such that maximal path connected measurable C^∞ paths γ_i form a non-empty set, say, $G = \{\gamma_1, \gamma_2, ..., \gamma_n\} \in (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k) \subset (M, \tau, \Sigma, \mu_R)$.

Then (M, τ, Σ, μ_R) is maximally path connected for which $\mu_R(\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k) > 0$.

Note 1.2.15 Let $A_3 = \{p_i, q_i \in [p] \in (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k)$: $\exists\, \gamma_i \in [\gamma]$: p_i is maximally path connected to q_i by $\gamma_i \in G\}$ where $\mu_R(A_3) > 0$. If $\mu_R(A_3) = 0$ then A_3 represents the **dark region** of $(\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k) \in (M, \tau, \Sigma, \mu_R)$.

Let us introduce different measurable subsets of measure space $(R^n, \tau, \Sigma, \mu_R)$ [6], [19]:

Definition 1.2.16 Compact Radon measure subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ **[7], [20], [25]**
A measurable subset $\phi(U) \in (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ is said to be measurable compact if every Borel open cover, say, $\{\cup_{i \in I} \phi(U_i)\}$ for $\phi(U) \in (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ such that $\phi(U) \subseteq \{\cup_{i \in I} \phi(U_i)\}$, has a finite Borel sub cover, say, $\{\cup_{j=1}^n \phi(U_{i_j})\}$ for $\phi(U)$ i.e., $\phi(U) \subseteq \{\cup_{j=1}^n \phi(U_{i_j})\}$ for $j \in J$, $J \subset I$, then $\phi(U)$ is a measurable compact subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ and is Radon measurable if it satisfies the **Radon measure conditions (1.2.1) and (1.2.2)**. Then $\phi(U)$ is called a Compact Radon measure subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$.

Definition 1.2.17 Lindelof Radon measure subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ **[7], [20], [25]**
A measurable subset $\phi(U) \in (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ is said to be measurable Lindelof if every Borel open cover, say, $\{\cup_{i \in I} \phi(U_i)\}$ for $\phi(U) \subset (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ such that $\phi(U) \subseteq \{\cup_{i \in I} \phi(U_i)\}$, has a countable Borel sub cover, say, $\{\cup_{j=1}^\infty \phi(U_{i_j})\}$ for $\phi(U)$ i.e., $\phi(U) \subseteq \{\cup_{j=1}^\infty \phi(U_{i_j})\}$ for $j \in J$, $J \subset I$, then $\phi(U)$ is measurable Lindelof subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ and is Radon measurable if it satisfies the **Radon measure con-**

ditions (1.2.1) and (1.2.2). Then $\phi(U)$ is called a Lindelof Radon measure subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$.

Definition 1.2.18 Countably compact Radon measure subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ **[7], [20], [25], [28]**
A measurable subset $\phi(U) \in (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ is said to be measurable countably compact if every countable Borel open cover, say, $\{\cup_{i\in I}\phi(U_i)\}$ for $\phi(U) \subset (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ such that $\phi(U) \subseteq \{\cup_{i\in I}\phi(U_i)\}$, has a finite Borel sub cover, say, $\{\cup_{j=1}^{n}\phi(U_{i_j})\}$ for $\phi(U)$ i.e., $\phi(U)) \subseteq \{\cup_{j=1}^{n}\phi(U_{i_j})\}$ for $j \in J$, $J \subset I$, then $\phi(U)$ is a measurable countably compact subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ and is Radon measurable if it satisfies the **Radon measure conditions (1.2.1) and (1.2.2)**. Then $\phi(U)$ is called a countably compact Radon measure subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$.

Similarly, different patterns of the measurable subsets like measurable semi-compact subsets, measurable semi-Lindelof subsets and measurable semi-countably compact subsets are defined as follows:

Definition 1.2.19 Semi-open measurable subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ **[13], [30]**
A measurable subset $\phi(U) \in (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ is semi-open if $\phi(U) \subseteq cl(int(\phi(U)))$ such that $\mu(\phi(U)) \leq \mu(cl(int(\phi(U)))$.

Definition 1.2.20 Semi-compact Radon measure subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ **[12], [14], [30]**
A measurable subset $\phi(U) \in (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ is said to be measurable semi-compact if every Borel semi-open cover, say, $\{\cup_{i\in I}\phi(U_i)\}$ for $\phi(U)$ of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ such that $\phi(U) \subseteq \{\cup_{i\in I}\phi(U_i)\}$, has a finite Borel semi-open sub cover, say, $\{\cup_{j=1}^{n}\phi(U_{i_j})\}$ for $\phi(U)$ i.e., $\phi(U)) \subseteq \{\cup_{j=1}^{n}\phi(U_{i_j})\}$ for $j \in J$, $J \subset I$, then $\phi(U)$ is a measurable semi-compact subset of $(R^n, \tau_1, \Sigma_1,$
$\mu_{R_1})$ and is Radon measurable if it satisfies the **Radon measure conditions (1.2.1) and (1.2.2)**. Then $\phi(U)$ is called a semi-compact Radon measure subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$.

Definition 1.2.21 Semi-Lindelof Radon measure subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ **[24]**
A measurable subset $\phi(U) \in (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ is said to be measurable semi-Lindelof if every Borel semi-open cover, say, $\{\cup_{i\in I}\phi(U_i)\}$ for $\phi(U)$ of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ such that $\phi(U) \subseteq \{\cup_{i\in I}\phi(U_i)\}$, has a countable Borel semi-open sub cover, say, $\{\cup_{j=1}^{\infty}\phi(U_{i_j})\}$ for $\phi(U)$ i.e., $\phi(U)) \subseteq \{\cup_{j=1}^{\infty}\phi(U_{i_j})\}$ for $j \in J$, $J \subset I$, then $\phi(U)$ is a measurable semi-Lindelof subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ and is Radon measurable if it satisfies the **Radon measure conditions (1.2.1) and (1.2.2)**. Then $\phi(U)$ is called a Semi-Lindelof Radon measure subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$.

Definition 1.2.22 Semi-countably compact Radon measure subset of $(R^n, \tau_1, \Sigma_1,$
$\mu_{R_1})$ **[7]**
A measurable subset $\phi(U) \in (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ is said to be measurable semi-compact if every countable Borel semi-open cover, say, $\{\cup_{i\in I}\phi(U_i)\}$ for $\phi(U)$ of $(R^n, \tau, \Sigma, \mu_R)$ such that $\phi(U) \subseteq \{\cup_{i\in I}\phi(U_i)\}$, has a finite Borel semi-open sub cover, say, $\{\cup_{j=1}^{n}\phi(U_{i_j})\}$

for $\phi(U)$ i.e., $\phi(U) \subseteq \{\cup_{j=1}^{n} \phi(U_{i_j})\}$ for $j \in J$, $J \subset I$, then $\phi(U)$ is a measurable semi-countably compact subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ and is Radon measurable if it satisfies the **Radon measure conditions (1.2.1) and (1.2.2)**. Then $\phi(U)$ is called a semi-countably compact Radon measure subset of $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$.

The advantage of the study of Radon measure manifolds is that, for every different patterns of measurable subsets $\phi(U) \subset (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ there exist corresponding different patterns of measurable charts $(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R)$ under measurable homeomorphism and Radon measure structure-invariant map $\phi : (M, \tau, \Sigma, \mu_R) \longrightarrow (R^n, \tau_1, \Sigma_1, \mu_{R_1})$. The purpose of the discussion is to construct different categories of Radon measure manifolds, Quotient Radon measure manifolds and Network Radon measure manifolds.

2. Different categories of Radon measure manifolds

In the previous paper, S. C. P. Halakatti has used the measurable homeomorphism and measure-invariant function to introduce three different patterns of measurable charts [8] on (M, τ, Σ, μ_R) to generate corresponding compact Radon measure manifold, Lindelof Radon measure manifolds and countably compact Radon measure manifolds [26]. On the similar line the other patterns of measurable charts on (M, τ, Σ, μ_R) are introduced in this paper to generate few more categories of Radon measure manifolds namely semi-compact Radon measure manifolds, semi-Lindelof Radon measure manifolds and semi-countably compact Radon measure manifolds which remain invariant under measurable homeomorphism and Radon measure structure - invariant map.

In the previous paper [26], it has been shown that extended Heine Borel property, extended countably compact property and extended Lindelof property, say, P holds $\mu_R - a.e.$, on Radon measure manifold (M, τ, Σ, μ_R).

In this paper the first author has shown that the above three properties remain invariant under measurable homeomorphism and Radon measure-structure-invariant maps as follows:

2.1. Compact Radon measure Manifold

Here the Radon measure - structure is induced on measure manifold and Radon measure structure - invariance has been developed to generate different categories of Radon measure manifolds as follows:

Definition 2.1.1 Radon measure structure on measure manifold
For any two Radon measure atlases $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{A}^k(M) \subset (M, \tau, \Sigma, \mu_R)$, we say that $\mathcal{A}_1 \sim \mathcal{A}_2$ is an equivalence relation on (M, τ, Σ, μ_R) if and only if $\mu_R(\mathcal{A}_1) = \mu_R(\mathcal{A}_2)$. This equivalence relation induces a Radon measure-structure on (M, τ, Σ, μ_R).

Definition 2.1.2 Radon measure structure - invariant map
Suppose $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Radon measure manifolds. Let $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be a measurable homeomorphism then,

F is Radon measure structure-invariant if and only if $\mathcal{A}_1 \sim \mathcal{A}_2$ with the condition $\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2) \Leftrightarrow F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ with the condition $\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2))$.

Based on the above concepts, the following different categories of Radon measure manifolds are introduced and developed by S. C. P. Halakatti.

Definition 2.1.3 Compact Radon measure chart
We say (U_i, ϕ_i) is measurable compact, if for every Borel open cover, say, $\{\cup_{i\in I} S_i\}$ of (U_i, ϕ_i) if $\exists$ a finite Borel sub cover, say, $\{\cup_{j=1}^{n} S_{i_j}\}$ for $j \in J$, $J \subset I$, such that $(U_i, \phi_i) \subseteq \{\cup_{j=1}^{n} S_{i_j}\}$ where S_i are Borel subsets of (U_i, ϕ_i) and (U_i, ϕ_i) is Radon measurable if it satisfies the **Radon measure conditions (1.2.3) and (1.2.4)**. Then (U_i, ϕ_i) is a compact Radon measure chart of (M, τ, Σ, μ_R).

Definition 2.1.4 Compact Radon measure atlas
By a compact Radon measure atlas $\mathcal{A} \in A^k(M)$ of class C^k $(k \geq 1)$ on a Radon measure manifold, we mean a collection of n-dimensional compact Radon measure charts (U_i, ϕ_i) satisfying the following conditions:
(a₁) $\cup_{i\in I}(U_i, \phi_i) \subset A^k(M) \subset (M, \tau, \Sigma, \mu_R)$,
(a₂) For any pair of Quotient Radon measure charts (U_i, ϕ_i) and (U_j, ϕ_j) there exists transition maps $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$ and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ are differentiable and Radon measurable.
(a₃) Any two measure atlases are compatible on (M, τ, Σ, μ_R) satisfying two equivalence relations:

(i) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $(\mathcal{A}_1 \cup \mathcal{A}_2) \in A^k(M)$.

(ii) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mu_R(\mathcal{A}_1) = \mu_R(\mathcal{A}_2)$.

(a₄) For every Borel cover $\{\cup_{i\in I}(U_i, \phi_i)\}$ for $\mathcal{A}$ i.e., $\mathcal{A} \subseteq \{\cup_{i\in I}(U_i, \phi_i)\}$ if there exists a Borel sub cover, say, $\{\cup_{j=1}^{n}(U_{i_j}, \phi_i)\}$, then $\mathcal{A}$ is a compact measure atlas of (M, τ, Σ, μ_R). If $\mathcal{A}$ is Radon measurable then $\mathcal{A}$ is a compact Radon measure atlas of (M, τ, Σ, μ_R).

Definition 2.1.5 Compact Radon measure Manifold
If every Borel cover $\{\cup_{i\in I}(\mathcal{A}_i)\}$ of $A^k(M)$ has a finite Borel sub cover $\{\cup_{j=1}^{n}\mathcal{A}_{i_j}\}$ of $A^k(M)$ then $A^k(M)$ is a compact measure manifold. If $A^k(M)$ is covered by compact Radon measure atlases $\{\cup_{i\in I}\mathcal{A}_i\}$ such that $A^k(M) \subseteq \{\cup_{i\in I}\mathcal{A}_i\}$ and if there exists finite Borel sub cover $\{\cup_{j=1}^{n}\mathcal{A}_{i_j}\}$ for $A^k(M)$ i.e. $A^k(M) \subseteq \{\cup_{j=1}^{n}\mathcal{A}_{i_j}\}$, then $A^k(M) \subseteq (M, \tau, \Sigma, \mu_R)$ is a compact Radon measure manifold.

Note 2.1.6 Let $A = \{(U_i, \phi_i)$: $\forall\, i \in I : (U_i, \phi_i) \subset \mathcal{A}_i \subset (M, \tau, \Sigma, \mu_R)$ are measurable compact chart$\}$ where $\mu_R(A) > 0$. If $\mu_R(A) = 0$ then A represents the **dark region** of $A^k(M) \subseteq (M, \tau, \Sigma, \mu_R)$.

Theorem 2.1.7 Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and Radon measure structure-invariant map. If extended Heine Borel property holds $\mu_{R_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then the property also holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Proof. Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and Radon measure structure-invariant map. We show that if extended Heine Borel property, say, P_1 holds $\mu_{R_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_1 also holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$. Suppose $\mathcal{A}_1, \mathcal{A}_2 \in A^k(M) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ be two Radon measure atlases of $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$.

Let P_1 holds $\mu_{R_1} - a.e.$, on $\mathcal{A}_1, \mathcal{A}_2$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1}) : \mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ according to definition 2.1.1.
Now since $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is measurable homeomorphism and Radon measure structure-invariant map, according to definition 2.1.2, for every $\mathcal{A}_1, \mathcal{A}_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $\mu_{R_1}(\mathcal{A}_1) > 0$, $\mu_{R_1}(\mathcal{A}_2) > 0 \; \exists \; F(\mathcal{A}_1), F(\mathcal{A}_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$, $\mu_{R_2}(F(\mathcal{A}_1)) > 0$, $\mu_{R_2}(F(\mathcal{A}_2)) > 0 : \mathcal{A}_1 \sim \mathcal{A}_2$ if and only if $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ then $\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ if and only if $\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2))$.
Therefore, extended Heine Borel property holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$. ■

Corollary 2.1.8 Let $M_1, \dots, M_n$ be Radon measure manifolds and $F_1 : M_1 \longrightarrow M_2, \dots, F_n : M_{n-1} \longrightarrow M_n$ be measurable homeomorphisms and Radon measure structure-invariant maps. Then, if extended Heine Borel property holds $\mu_{R_1} - a.e.$, on M_1 then it also holds $\mu_{R_n} - a.e.$, on M_n under the composition of $F_1, \dots, F_n$.

Remark 2.1.9 Using the above results, one can generate a new **category of compact Radon measure manifold** $(\Pi, (G, \circ))$ where $\Pi = \{M_1 \times M_2 \times \dots \times M_n\}$ which is closed under the group action $(G, \circ) = \{F_1, F_2, \dots, F_n\}$ of C^∞ measurable homeomorphisms and Radon measure structure-invariant maps.

2.2. Lindelof Radon measure Manifold

Definition 2.2.1 Lindelof Radon measure chart
We say (U_i, ϕ_i) is measurable Lindelof, if for every Borel open cover, say, $\{\cup_{i \in I} S_i\}$ of (U_i, ϕ_i) if $\exists$ a countable Borel sub cover, say, $\{\cup_{j=1}^{\infty} S_{i_j}\}$ for $j \in J$, $J \subset I$, such that $(U_i, \phi_i) \subseteq \{\cup_{j=1}^{\infty} S_{i_j}\}$ where S_i are Borel subsets of (U_i, ϕ_i) and is Radon measurable if it satisfies the **Radon measure conditions (1.2.3) and (1.2.4)**. Then (U_i, ϕ_i) is a Lindelof Radon measure chart of (M, τ, Σ, μ_R).

Definition 2.2.2 Lindelof Radon measure atlas
By a Lindelof Radon measure atlas of class C^k, $(k \geq 1)$ on a Radon measure manifold, we mean a collection of n-dimensional Lindelof Radon measure charts (U_i, ϕ_i) satisfying the following conditions:
($\mathbf{a_1}$) $\cup_{i \in I}(U_i, \phi_i) \subset A^k(M) \subset (M, \tau, \Sigma, \mu_R)$,
($\mathbf{a_2}$) For any pair of Lindelof Radon measure charts (U_i, ϕ_i) and (U_j, ϕ_j) there exists transition maps $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$ and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow$

$\phi_j(U_i \cap U_j)$ are differentiable and Radon measurable.
(**a$_3$**) Any two measure atlases are compatible on (M, τ, Σ, μ_R) satisfying two equivalence relations:

(i) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $(\mathcal{A}_1 \cup \mathcal{A}_2) \in A^k(M)$.

(ii) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mu_R(\mathcal{A}_1) = \mu_R(\mathcal{A}_2)$.

(**a$_4$**) For every cover $\{\cup_{i\in I}(U_i, \phi_i)\}$ for $\mathcal{A}$ i.e., $\mathcal{A} \subseteq \{\cup_{i\in I}(U_i, \phi_i)\}$ if there exists a countable Borel sub cover, say, $\{\cup_{j=1}^{\infty}(U_{i_j}, \phi_i)\}$, then $\mathcal{A}$ is a Lindelof measure atlas of (M, τ, Σ, μ_R). If $\mathcal{A}$ is Radon measurable then $\mathcal{A}$ is a Lindelof Radon measure atlas of (M, τ, Σ, μ_R).

Definition 2.2.3 Lindelof Radon measure Manifold
If every Borel cover $\{\cup_{i\in I}(\mathcal{A}_i)\}$ of $A^k(M)$ has a countable Borel sub cover $\{\cup_{j=1}^{\infty}\mathcal{A}_{i_j}\}$ of $A^k(M)$ then $A^k(M)$ is a Lindelof measure manifold. If $A^k(M)$ is covered by Lindelof Radon measure atlases $\{\cup_{i\in I}\mathcal{A}_i\}$ such that $A^k(M) \subseteq \{\cup_{j=1}\mathcal{A}_{i_j}\}$ and if there exists countable Borel sub cover $\{\cup_{j=1}^{\infty}\mathcal{A}_{i_j}\}$ for $A^k(M)$ i.e. $A^k(M) \subseteq M \subseteq \{\cup_{j=1}^{\infty}\mathcal{A}_{i_j}\}$, then $A^k(M)$ is a Lindelof Radon measure manifold.

Note 2.2.4 Let $A = \{(U_i, \phi_i) : \forall i \in I (U_i, \phi_i) \subset \mathcal{A}_i \subset (M, \tau, \Sigma, \mu_R)$ are measurable Lindelof charts$\}$ where $\mu_R(A) > 0$. If $\mu_R(A) = 0$ then A represents the **dark region** of $A^k(M) \subseteq (M, \tau, \Sigma, \mu_R)$.

Theorem 2.2.5 Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and Radon measure structure-invariant map. If extended Lindelof property holds $\mu_{R_1} - a.e.$, on $(M_1, \tau_1,$
$\Sigma_1, \mu_{R_1})$ then the property also holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Proof. Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and Radon measure structure-invariant map. We show that, if extended Lindelof property, say, P_2 holds $\mu_{R_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_2 also holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$. Suppose $\mathcal{A}_1, \mathcal{A}_2 \in A^k(M) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ be two Radon measure atlases of $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$.

Let P_2 holds $\mu_{R_1} - a.e.$, on $\mathcal{A}_1, \mathcal{A}_2$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1}) : \mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ according to definition 2.1.1.

Now since $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is measurable homeomorphism and Radon measure structure-invariant map, for every $\mathcal{A}_1, \mathcal{A}_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $\mu_{R_1}(\mathcal{A}_1) > 0$, $\mu_{R_1}(\mathcal{A}_2) > 0 \;\exists\; F(\mathcal{A}_1), F(\mathcal{A}_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$, $\mu_{R_2}(F(\mathcal{A}_1)) > 0$, $\mu_{R_2}(F(\mathcal{A}_2)) > 0 : \mathcal{A}_1 \sim \mathcal{A}_2$ if and only if $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ then $\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ if and only if $\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2)$. Therefore, extended Lindelof property holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$. ■

Corollary 2.2.6 Let $M_1, \ldots, M_n$ be Radon measure manifolds and $F_1 : M_1 \longrightarrow M_2, \ldots, F_n : M_{n-1} \longrightarrow M_n$ be measurable homeomorphisms and Radon measure structure-invariant functions. Then, if extended Lindelof property holds $\mu_{R_1} - a.e.$, on M_1 then it also holds $\mu_{R_n} - a.e.$, on M_n under the composition of $F_1, \ldots, F_n$.

Remark 2.2.7 Using the above results, one can generate a new **category of Lindelof Radon measure manifold** $(\Pi, (G, \circ))$ where $\Pi = \{M_1 \times M_2 \times ... \times M_n\}$ which is closed under the group action $(G, \circ) = \{F_1, F_2, ..., F_n\}$ of C^∞ measurable homeomorphisms and Radon measure structure-invariant maps.

2.3. Countably compact Radon measure manifold

Definition 2.3.1 Countably compact Radon measure chart
We say (U_i, ϕ_i) measurable is countably compact, if for every countable Borel open cover, say, $\{\cup_{i\in I} S_i\}$ of (U_i, ϕ_i) if $\exists$ a finite Borel sub cover, say, $\{\cup_{j=1}^{n} S_{i_j}\}$ for $j \in J$, $J \subset I$, such that $(U_i, \phi_i) \subseteq \{\cup_{j=1}^{n} S_{i_j}\}$ where S_i are Borel subsets of (U_i, ϕ_i) and is Radon measurable if it satisfies the **Radon measure conditions (1.2.3) and (1.2.4)**. Then (U_i, ϕ_i) is a countably compact Radon measure chart of (M, τ, Σ, μ_R).

Definition 2.3.2 Countably compact Radon measure atlas
By a countably compact Radon measure atlas of class C^k, $(k \geq 1)$ on a Radon measure manifold, we mean a collection of n-dimensional countably compact Radon measure charts (U_i, ϕ_i) satisfying the following conditions:
($\mathbf{a_1}$) $\cup_{i\in I}(U_i, \phi_i) \subset A^k(M) \subset (M, \tau, \Sigma, \mu_R)$,
($\mathbf{a_2}$) For any pair of (U_i, ϕ_i) and (U_j, ϕ_j) there exists transition maps $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$ and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ are differentiable and Radon measurable.
($\mathbf{a_3}$) Any two measure atlases are compatible on (M, τ, Σ, μ_R) satisfying two equivalence relations:

(i) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $(\mathcal{A}_1 \cup \mathcal{A}_2) \in A^k(M)$.

(ii) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mu_R(\mathcal{A}_1) = \mu_R(\mathcal{A}_2)$.

($\mathbf{a_4}$) For every countable cover, say, $\{\cup_{i\in I}(U_i, \phi_i)\}$ for $\mathcal{A}$ i.e., $\mathcal{A} \subseteq \{\cup_{i\in I}(U_i, \phi_i)\}$ if there exists a finite Borel sub cover, say, $\{\cup_{j=1}^{n}(U_{i_j}, \phi_i)\}$, then $\mathcal{A}$ is a countably compact measure atlas of (M, τ, Σ, μ_R). If $\mathcal{A}$ is Radon measurable then $\mathcal{A}$ is a countably compact Radon measure atlas of (M, τ, Σ, μ_R).

Definition 2.3.3 Countably compact Radon measure Manifold
If every countable measurable cover $\{\cup_{i\in I}(\mathcal{A}_i)\}$ of $A^k(M)$ has a finite Borel sub cover, say, $\{\cup_{j=1}^{n}\mathcal{A}_{i_j}\}$ of $A^k(M)$ for $j \in J$, $J \subset I$, then $A^k(M)$ is a countably compact measure manifold. If $A^k(M)$ is covered by countably compact Radon measure atlases $\{\cup_{i\in I}\mathcal{A}_i\}$ such that $A^k(M) \subseteq \{\cup_{j=1}\mathcal{A}_{i_j}\}$ and if there exists finite Borel sub cover say $\{\cup_{j=1}^{n}\mathcal{A}_{i_j}\}$ for $A^k(M)$ i.e. $A^k(M) \subseteq M \subseteq \{\cup_{j=1}^{n}\mathcal{A}_{i_j}\}$ for $j \in J$, $J \subset I$, then (M, τ, Σ, μ_R) is a countably compact Radon measure manifold.

Note 2.3.4 Let $A = \{(U_i, \phi_i){:}\forall\, i \in I, (U_i, \phi_i) \subset \mathcal{A}_i \subset (M, \tau, \Sigma, \mu_R)$ are measurable countably compact charts$\}$ where $\mu_R(A) > 0$. If $\mu_R(A) = 0$ then A represents the **dark region** of $A^k(M) \subseteq (M, \tau, \Sigma, \mu_R)$.

Theorem 2.3.5 Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and Radon measure structure-invariant function. If extended countably compact property holds $\mu_{R_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then the property also holds $\mu_{R_2} - a.e$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Proof. Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and Radon measure structure-invariant map. We show that if extended countably compact property, say, P_3 holds $\mu_{R_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_3 also holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$. Suppose $\mathcal{A}_1, \mathcal{A}_2 \in A^k(M) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ be two Radon measure atlases of $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$. Let P_3 holds $\mu_{R_1} - a.e.$, on $\mathcal{A}_1, \mathcal{A}_2$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$: $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ according to definition 2.1.1. Now since $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is measurable homeomorphism and Radon measure structure-invariant map, for every $\mathcal{A}_1, \mathcal{A}_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $\mu_{R_1}(\mathcal{A}_1) > 0$, $\mu_{R_1}(\mathcal{A}_2) > 0 \;\exists\; F(\mathcal{A}_1), F(\mathcal{A}_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$, $\mu_{R_2}(F(\mathcal{A}_1)) > 0$, $\mu_{R_2}(F(\mathcal{A}_2)) > 0$: $\mathcal{A}_1 \sim \mathcal{A}_2$ if and only if $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ then $\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ if and only if $\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2))$. Therefore, extended countably compact property holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$. ■

Corollary 2.3.6 Let $M_1, \ldots, M_n$ be Radon measure manifolds and $F_1 : M_1 \longrightarrow M_2, \ldots, F_n : M_{n-1} \longrightarrow M_n$ be measurable homeomorphisms and Radon measure structure-invariant functions. Then, if extended countably compact property holds $\mu_{R_1} - a.e.$, on M_1 then it also holds $\mu_{R_n} - a.e.$, on M_n under the composition of $F_1, \ldots, F_n$.

Remark 2.3.7 Using the above results, one can generate a new **category of countably compact Radon measure manifold** $(\Pi, (G, \circ))$ where $\Pi = \{M_1 \times M_2 \times \ldots \times M_n\}$ which is closed under the group action $(G, \circ) = \{F_1, F_2, \ldots, F_n\}$ of C^∞ measurable homeomorphisms and Radon measure-invariant maps.

Definition 2.3.8 Measurable semi-open chart: Let (M, τ, Σ, μ) be a measure manifold. A measurable chart $(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R)$ is said to be semi-open if

$$(U_i, \phi_i) \subseteq cl(int((U_i, \phi_i))$$

$$\Rightarrow \mu(U_i, \phi_i) \leq \mu(cl(int((U_i, \phi_i)))$$

2.4. Semi-compact Quotient Radon measure Manifold

Definition 2.4.1 Semi-compact Radon measure chart

We say (U_i, ϕ_i) is measurable semi-compact, if for every Borel semi-open cover, say,

$\{\cup_{i\in I} S_i\}$ of (U_i, ϕ_i) if $\exists$ a finite Borel semi-open sub cover, say, $\{\cup_{j=1}^{n} S_{i_j}\}$ for $j \in J$, $J \subset I$, such that $(U_i, \phi_i) \subseteq \{\cup_{j=1}^{n} S_{i_j}\}$ where S_i are Borel semi-open subsets of (U_i, ϕ_i) and is Radon measurable if it satisfies the **Radon measure conditions (1.2.3) and (1.2.4)**. Then (U_i, ϕ_i) is a semi-compact Radon measure chart of (M, τ, Σ, μ_R).

Definition 2.4.2 Semi-compact Radon measure atlas
By a semi-compact Radon measure atlas of class C^k, $(k \geq 1)$ on a Radon measure manifold, we mean a collection of n-dimensional semi-compact Radon measure charts (U_i, ϕ_i) satisfying the following conditions:
(**a$_1$**) $\cup_{i\in I}(U_i, \phi_i) \subset A^k(M) \subset (M, \tau, \Sigma, \mu_R)$,
(**a$_2$**) For any pair of (U_i, ϕ_i) and (U_j, ϕ_j) there exists transition maps $\phi_i \circ \phi_j^{-1}$: $\phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$ and $\phi_j \circ \phi_i^{-1}$: $\phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ are differentiable and Radon measurable.
(**a$_3$**) Any two measure atlases are compatible on (M, τ, Σ, μ_R) satisfying two equivalence relations:

(i) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $(\mathcal{A}_1 \cup \mathcal{A}_2) \in A^k(M)$.

(ii) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mu_R(\mathcal{A}_1) = \mu_R(\mathcal{A}_2)$.

(**a$_4$**) For every Borel semi-open cover, say, $\{\cup_{i\in I}(U_i, \phi_i)\}$ for $\mathcal{A}$ i.e., $\mathcal{A} \subseteq \{\cup_{i\in I}(U_i, \phi_i)\}$ if there exists a finite Borel semi-open sub cover, say, $\{\cup_{j=1}^{n}(U_{i_j}, \phi_i)\}$, then $\mathcal{A}$ is a semi-compact measure atlas of (M, τ, Σ, μ_R). If $\mathcal{A}$ is Radon measurable then $\mathcal{A}$ is a semi-compact Radon measure atlas of (M, τ, Σ, μ_R).

Definition 2.4.3 Semi-compact Radon measure Manifold
If every Borel semi-open cover $\{\cup_{i\in I}(\mathcal{A}_i)\}$ of $A^k(M)$ has a finite Borel semi-open sub cover $\{\cup_{j=1}^{n}\mathcal{A}_{i_j}\}$ of $A^k(M)$ then $A^k(M)$ is a semi-compact measure manifold. If $A^k(M)$ is covered by semi-compact Radon measure atlases $\{\cup_{i\in I}\mathcal{A}_i\}$ such that $A^k(M) \subseteq \{\cup_{j=1}\mathcal{A}_{i_j}\}$ and if there exists finite Borel semi-open sub cover $\{\cup_{j=1}^{n}\mathcal{A}_{i_j}\}$ for $A^k(M)$ i.e. $A^k(M) \subseteq M \subseteq \{\cup_{j=1}^{n}\mathcal{A}_{i_j}\}$, then (M, τ, Σ, μ_R) is a semi-compact Radon measure manifold.

Note 2.4.4 Let $A = \{(U_i, \phi_i)$: $\forall\ i \in I\ (U_i, \phi_i) \subset \mathcal{A}_i \subset (M, \tau, \Sigma, \mu_R)$ are measurable semi-compact charts$\}$ where $\mu_R(A) > 0$. If $\mu_R(A) = 0$ then A represents the **dark region** of $A^k(M) \subseteq (M, \tau, \Sigma, \mu_R)$.

Theorem 2.4.5 Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and Radon measure structure-invariant function. If extended semi-compactness property holds $\mu_{R_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then the property also holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Proof. Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds and F : $(M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and

Radon measure structure-invariant map. We show that if extended semi-compactness property, say, P_4 holds $\mu_{R_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_4 also holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$. Suppose $\mathcal{A}_1, \mathcal{A}_2 \in A^k(M) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ be two Radon measure atlases of $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$. Let P_4 holds $\mu_{R_1} - a.e.$, on $\mathcal{A}_1, \mathcal{A}_2$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$: $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ according to definition 2.1.1. Now since $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is measurable homeomorphism and Radon measure structure-invariant map, for every $\mathcal{A}_1, \mathcal{A}_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $\mu_{R_1}(\mathcal{A}_1) > 0$, $\mu_{R_1}(\mathcal{A}_2) > 0 \; \exists \; F(\mathcal{A}_1), F(\mathcal{A}_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$, $\mu_{R_2}(F(\mathcal{A}_1)) > 0$, $\mu_{R_2}(F(\mathcal{A}_2)) > 0$: $\mathcal{A}_1 \sim \mathcal{A}_2$ if and only if $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ then $\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ if and only if $\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2))$. Therefore, extended semi-compactness property holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$. ■

Corollary 2.4.6 Let $M_1, \ldots, M_n$ be Radon measure manifolds and $F_1 : M_1 \longrightarrow M_2, ..., F_n : M_{n-1} \longrightarrow M_n$ be measurable homeomorphisms and Radon measure structure-invariant functions. Then, if extended semi-compactness property holds $\mu_{R_1} - a.e.$, on M_1 then it also holds $\mu_{R_n} - a.e.$, on M_n under the composition of $F_1, \ldots, F_n$.

Remark 2.4.7 Using the above results, one can generate a new **category of semi-compact Radon measure manifold** $(\Pi, (G, \circ))$ where $\Pi = \{M_1 \times M_2 \times ... \times M_n\}$ which is closed under the group action $(G, \circ) = \{F_1, F_2, ..., F_n\}$ of C^∞ measurable homeomorphisms and Radon measure structure-invariant maps.

2.5. Semi-Lindelof Radon measure Manifold

Definition 2.5.1 Semi-Lindelof Radon measure chart
We say (U_i, ϕ_i) is measurable Semi-Lindelof, if for every Borel semi-open cover, say, $\{\cup_{i \in I} S_i\}$ of (U_i, ϕ_i) if $\exists$ a countable Borel semi-open sub cover, say, $\{\cup_{j=1}^{\infty} S_{i_j}\}$ for $j \in J$, $J \subset I$, such that $(U_i, \phi_i) \subseteq \{\cup_{j=1}^{\infty} S_{i_j}\}$ where S_i are semi-open Borel subsets of (U_i, ϕ_i) and is Radon measurable if it satisfies the **Radon measure conditions (1.2.3) and (1.2.4)**. Then (U_i, ϕ_i) is a semi-Lindelof Radon measure chart of (M, τ, Σ, μ_R).

Definition 2.5.2 Semi-Lindelof Radon measure atlas
By a semi-Lindelof Radon measure atlas of class C^k, $(k \geq 1)$ on a Radon measure manifold, we mean a collection of n-dimensional semi-Lindelof Radon measure charts (U_i, ϕ_i) satisfying the following conditions:
(a_1) $\cup_{i \in I}(U_i, \phi_i) \subset A^k(M) \subset (M, \tau, \Sigma, \mu_R)$,
(a_2) For any pair of semi-Lindelof Radon measure charts (U_i, ϕ_i) and (U_j, ϕ_j) there exists transition maps $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$ and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ are differentiable and Radon measurable.
(a_3) Any two measure atlases are compatible on (M, τ, Σ, μ_R) satisfying two equivalence relations:

(i) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $(\mathcal{A}_1 \cup \mathcal{A}_2) \in A^k(M)$.

(ii) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mu_R(\mathcal{A}_1) = \mu_R(\mathcal{A}_2)$.

(**a₄**) For every Borel semi-open cover $\{\cup_{i\in I}(U_i, \phi_i)\}$ for $\mathcal{A}$ i.e., $\mathcal{A} \subseteq \{\cup_{i\in I}(U_i, \phi_i)\}$ if there exists a countable Borel semi-open sub cover, say, $\{\cup_{j=1}^{\infty}(U_{i_j}, \phi_i)\}$, then $\mathcal{A}$ is a Lindelof measure atlas of (M, τ, Σ, μ_R). If $\mathcal{A}$ is Radon measurable then $\mathcal{A}$ is a semi-Lindelof Radon measure atlas of (M, τ, Σ, μ_R).

Definition 2.5.3 Semi-Lindelof Radon measure Manifold
If every Borel semi-open cover, say, $\{\cup_{i\in I}(\mathcal{A}_i)\}$ of (M, τ, Σ, μ_R) has a countable Borel semi-open sub cover $\{\cup_{j=1}^{\infty}\mathcal{A}_{i_j}\}$ of (M, τ, Σ, μ_R) then (M, τ, Σ, μ_R) is a semi-Lindelof measure manifold. If (M, τ, Σ, μ_R) is covered by semi-Lindelof Radon measure atlases $\{\cup_{i\in I}\mathcal{A}_i\}$ such that $(M, \tau, \Sigma, \mu_R) \subseteq \{\cup_{j=1}\mathcal{A}_{i_j}\}$ and if there exists countable Borel semi-open sub cover, say, $\{\cup_{j=1}^{\infty}\mathcal{A}_{i_j}\}$ for (M, τ, Σ, μ_R) i.e. $M \subseteq \{\cup_{j=1}^{\infty}\mathcal{A}_{i_j}\}$, then (M, τ, Σ, μ_R) is a semi-Lindelof Radon measure manifold.

Note 2.5.4 Let $A = \{(U_i, \phi_i) : (U_i, \phi_i) \subset \mathcal{A}_i \subset (M, \tau, \Sigma, \mu_R)$ are measurable semi-Lindelof charts$\}$ where $\mu_R(A) > 0$. If $\mu_R(A) = 0$ then A represents the **dark region** of $A^k(M) \subseteq (M, \tau, \Sigma, \mu_R)$.

Theorem 2.5.5 Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and Radon measure structure-invariant function. If extended semi-Lindelof property holds $\mu_{R_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then the property also holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Proof. Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and Radon measure structure-invariant map. We show that if extended semi-Lindelof property, say, P_5 holds $\mu_{R_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_5 also holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$. Suppose $\mathcal{A}_1, \mathcal{A}_2 \in A^k(M) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ be two Radon measure atlases of $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$. Let P_5 holds $\mu_{R_1} - a.e.$, on $\mathcal{A}_1, \mathcal{A}_2$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$: $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ according to definition 2.1.1. Now since $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is measurable homeomorphism and Radon measure structure-invariant map, for every $\mathcal{A}_1, \mathcal{A}_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $\mu_{R_1}(\mathcal{A}_1) > 0$, $\mu_{R_1}(\mathcal{A}_2) > 0 \; \exists \; F(\mathcal{A}_1), F(\mathcal{A}_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$, $\mu_{R_2}(F(\mathcal{A}_1)) > 0$, $\mu_{R_2}(F(\mathcal{A}_2)) > 0$: $\mathcal{A}_1 \sim \mathcal{A}_2$ if and only if $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ then $\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ if and only if $\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2))$. Therefore, extended semi-Lindelof property holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$. ■

Corollary 2.5.6 Let $M_1, \ldots, M_n$ be Radon measure manifolds and $F_1 : M_1 \longrightarrow M_2, ..., F_n : M_{n-1} \longrightarrow M_n$ be measurable homeomorphisms and Radon measure structure-invariant functions. Then, if extended semi-Lindelof property holds $\mu_{R_1} - a.e.$, on M_1 then it also holds $\mu_{R_n} - a.e.$, on M_n under the composition of $F_1, \ldots, F_n$.

Remark 2.5.7 Using the above results, one can generate a new **category of semi-Lindelof Radon measure manifold** $(\Pi, (G, \circ))$ where $\Pi = \{M_1 \times M_2 \times ... \times M_n\}$ which is closed under the group action $(G, \circ) = \{F_1, F_2, ..., F_n\}$ of C^{∞} measurable homeomorphisms and Radon measure structure-invariant maps.

2.6. Semi-countably compact Radon measure manifold

Definition 2.6.1 Semi-countably compact Radon measure chart
We say (U_i, ϕ_i) is measurable semi-countably compact, if for every countable Borel semi-open cover, say, $\{\cup_{i \in I} S_i\}$ of (U_i, ϕ_i) if $\exists$ a finite Borel semi-open sub cover, say, $\{\cup_{j=1}^{n} S_{i_j}\}$ for $j \in J$, $J \subset I$, such that $(U_i, \phi_i) \subseteq \{\cup_{j=1}^{n} S_{i_j}\}$ where S_i are Borel subsets of (U_i, ϕ_i) and is Radon measurable if it satisfies the **Radon measure conditions (1.2.3) and (1.2.4)**. Then (U_i, ϕ_i) is a semi-countably compact Radon measure chart of (M, τ, Σ, μ_R).

Definition 2.6.2 Semi-countably compact Radon measure atlas
By a semi-countably compact Radon measure atlas of class C^k, $(k \geq 1)$ on a Radon measure manifold, we mean a collection of n-dimensional semi-countably compact Radon measure charts (U_i, ϕ_i) satisfying the following conditions:
($\mathbf{a_1}$) $\cup_{i \in I}(U_i, \phi_i) \subset A^k(M) \subset (M, \tau, \Sigma, \mu_R)$,
($\mathbf{a_2}$) For any pair of semi-countably compact Radon measure charts (U_i, ϕ_i) and (U_j, ϕ_j) there exists transition maps $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$ and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ are differentiable and Radon measurable.
($\mathbf{a_3}$) Any two measure atlases are compatible on (M, τ, Σ, μ_R) satisfying two equivalence relations:

(i) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $(\mathcal{A}_1 \cup \mathcal{A}_2) \in A^k(M)$.

(ii) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mu_R(\mathcal{A}_1) = \mu_R(\mathcal{A}_2)$.

($\mathbf{a_4}$) For every countable Borel semi-open cover $\{\cup_{i \in I}(U_i, \phi_i)\}$ for $\mathcal{A}$ i.e., $\mathcal{A} \subseteq \{\cup_{i \in I}(U_i, \phi_i)\}$ if there exists a finite Borel semi-open sub cover, say, $\{\cup_{j=1}^{n}(U_{i_j}, \phi_i)\}$, then $\mathcal{A}$ is a semi-countably compact measure atlas of (M, τ, Σ, μ_R). If $\mathcal{A}$ is Radon measurable then $\mathcal{A}$ is a semi-countably compact Radon measure atlas of (M, τ, Σ, μ_R).

Definition 2.6.3 Semi-countably compact Radon measure Manifold
If every countable Borel semi-open cover $\{\cup_{i \in I}(\mathcal{A}_i)\}$ of (M, τ, Σ, μ_R) has a finite Borel semi-open sub cover $\{\cup_{j=1}^{n}\mathcal{A}_{i_j}\}$ of (M, τ, Σ, μ_R) then (M, τ, Σ, μ_R) is a semi-countably compact measure manifold. If (M, τ, Σ, μ_R) is covered by semi-countably compact Radon measure atlases $\{\cup_{i \in I}\mathcal{A}_i\}$ such that $(M, \tau, \Sigma, \mu_R) \subseteq \{\cup_{j=1}\mathcal{A}_{i_j}\}$ and if there exists finite Borel semi-open sub cover $\{\cup_{j=1}^{n}\mathcal{A}_{i_j}\}$ for (M, τ, Σ, μ_R) i.e. $M \subseteq \{\cup_{j=1}^{n}\mathcal{A}_{i_j}\}$, then (M, τ, Σ, μ_R) is a semi-countably compact Radon measure manifold.

Note 2.6.4 Let $A = \{(U_i, \phi_i) : (U_i, \phi_i) \subset \mathcal{A}_i \subset (M, \tau, \Sigma, \mu_R)$ are measurable semi-countably compact chart$\}$ where $\mu_R(A) > 0$. If $\mu_R(A) = 0$ then A represents the **dark region** of (M, τ, Σ, μ_R).

Theorem 2.6.5 Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and Radon measure structure-invariant function. If extended semi-countable compactness property holds $\mu_{R_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then the property also holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Proof. Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and Radon measure structure-invariant map. We show that if extended semi-countable compactness property, say, P_6 holds $\mu_{R_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_6 also holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$. Suppose $\mathcal{A}_1, \mathcal{A}_2 \in A^k(M) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ be two Radon measure atlases of $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$.

Let P_6 holds $\mu_{R_1} - a.e.$, on $\mathcal{A}_1, \mathcal{A}_2$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1}) : \mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ according to definition 2.1.1.

Now since $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is measurable homeomorphism and Radon measure structure-invariant map, for every $\mathcal{A}_1, \mathcal{A}_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $\mu_{R_1}(\mathcal{A}_1) > 0$, $\mu_{R_1}(\mathcal{A}_2) > 0 \; \exists \; F(\mathcal{A}_1), F(\mathcal{A}_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$, $\mu_{R_2}(F(\mathcal{A}_1)) > 0$, $\mu_{R_2}(F(\mathcal{A}_2) > 0$: $\mathcal{A}_1 \sim \mathcal{A}_2$ if and only if $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ then $\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ if and only if $\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2))$. Therefore, extended semi-countable compact property holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$. ■

Corollary 2.6.6 Let $M_1, \ldots, M_n$ be Radon measure manifolds and $F_1 : M_1 \longrightarrow M_2, ..., F_n : M_{n-1} \longrightarrow M_n$ be measurable homeomorphisms and Radon measure structure-invariant functions. Then, if extended semi-countably compact property holds $\mu_{R_1} - a.e.$, on M_1 then it also holds $\mu_{R_n} - a.e.$, on M_n under the composition of $F_1, \ldots, F_n$.

Remark 2.6.7 Using the above results, one can generate a new **category of semi-countably compact Radon measure manifold** $(\Pi, (G, \circ))$ where $\Pi = \{M_1 \times M_2 \times ... \times M_n\}$ which is closed under the group action $(G, \circ) = \{F_1, F_2, ..., F_n\}$ of C^∞ measurable homeomorphisms and Radon measure structure-invariant maps.

Thus, in this section, the first author has generated different categories of Radon measure manifolds namely - compact Radon measure manifold, Lindelof Radon measure manifold, countably compact Radon measure manifold, semi-compact Radon measure manifold, semi-Lindelof Radon measure manifold and semi-countably compact Radon measure manifolds - under the group action $G = \{F_1, F_2, \ldots, F_n\}$ of C^∞ measurable homeomorphism and Radon measure structure - invariant maps.

3. Proposed method to generate a new class of Quotient Radon measure Manifolds - An Analytical Approach

In this section, the first author has proposed an analytical method to generate class of Quotient Radon measure manifolds by inducing three different equivalence relations to define Quotient Radon measure chart, Quotient Radon measure atlas and Quotient Radon measure manifold.

In this section, we use the concept of metrizable measure manifold (M, τ, Σ, μ) on which the analysis is carried on as follows:

Define the non-empty sets $E_1 = \{p_i, p_j \in (U_i, \phi_i) \in (M, \tau, \Sigma, \mu)$: $d(p_i, p_j) = 0$ if and only if $p_i = p_j\}$, where the reflexive property, say, P_1 holds $\mu - a.e.$, on $E_1 \in (M, \tau, \Sigma, \mu)$ such that $\mu(E_1) > 0$. If $\mu(E_1) = 0$ then E_1 is a **dark region** on (M, τ, Σ, μ);

$E_2 = \{p_i, p_j \in (U_i, \phi_i) \in (M, \tau, \Sigma, \mu)$: $d(p_i, p_j) = d(p_j, p_i)\}$ where the symmetric property, say, P_2 holds $\mu - a.e.$, on $E_2 \in (M, \tau, \Sigma, \mu)$ such that $\mu(E_2) > 0$. If $\mu(E_2) = 0$ then E_2 is a **dark region** on (M, τ, Σ, μ), and

$E_3 = \{p_i, p_j \in (U_i, \phi_i) \in (M, \tau, \Sigma, \mu)$: $d(p_i, p_k) \leq d(p_i, p_j) + d(p_j, p_k)\}$ where the triangular inequality property, say, P_3 holds $\mu - a.e.$, on $E_3 \in (M, \tau, \Sigma, \mu)$ such that $\mu(E_3) > 0$. If $\mu(E_3) = 0$ then E_3 is a **dark region** on (M, τ, Σ, μ).

3.1. Locally path connected Radon measure manifold

Definition 3.1.1 Metrizable measure manifold
The measure manifold (M, τ, Σ, μ) is said to be metrizable if it admits a continuous measurable function $d : MXM \longrightarrow [0, \infty]$ satisfying reflexive property P_1, symmetric property P_2 and triangular inequality property P_3 on (M, τ, Σ, μ):

(i) the reflexive property P_1 holds $\mu - a.e.$, on $E_1 = \{p_i, p_j \in (U_i, \phi_i) \in (M, \tau, \Sigma, \mu)$: $d(p_i, p_j) = 0$ if and only if $p_i = p_j\}$, $\mu(E_1) > 0$,

(ii) the symmetric property P_2 holds $\mu - a.e.$, on $E_2 = \{p_i, p_j \in (U_i, \phi_i) \in (M, \tau, \Sigma, \mu)$: $d(p_i, p_j) = d(p_j, p_i)\}$, $\mu(E_2) > 0$ and

(iii) the triangular inequality property P_3 holds $\mu - a.e.$, on $E_3 = \{p_i, p_j \in (U_i, \phi_i) \in (M, \tau, \Sigma, \mu)$: $d(p_i, p_k) \leq d(p_i, p_j) + d(p_j, p_k)\}$, $\mu(E_3) > 0$.

In this paper, we study the Radon measure structure on metrizable measure manifold (M, τ, Σ, μ) viz. metrizable Radon measure manifold (M, τ, Σ, μ_R) and develops a method to study the intrinsic properties of (M, τ, Σ, μ_R) in terms of three different partitions induced by three different equivalence relations namely locally, internally and maximally path connectedness on (M, τ, Σ, μ_R) which generates a **new class of Quotient Radon measure manifolds**. To continue the analysis, it is necessary to show that all the three types of relations define correspondingly three different equivalence relations on metrizable Radon measure manifold (M, τ, Σ, μ_R).

Theorem 3.1.2 If (M, τ, Σ, μ_R) is a metrizable Radon measure manifold then local path connectedness is an equivalence relation on (M, τ, Σ, μ_R).

Proof. Let $(M, \tau_1, \Sigma_1, \mu_R)$ be a metrizable Radon measure manifold such that for every point $p \in (M, \tau, \Sigma, \mu_R)$ there exists a measurable chart $(U_i, \phi_i) \in (M, \tau_1, \Sigma_1, \mu_{R_1})$. Let $A_1 = \{p_i, p_j \in (U_i, \phi_i)$; $\exists\, \gamma_i \in G$ / $p_i \sim p_j$ by γ_i; $\forall\, i, j \in I\}$ be the non-empty

set belonging to $(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R)$ and $G = \{\gamma_1, \gamma_2, ..., \gamma_n\}$ be the non-empty collection of measurable C^∞ paths $\gamma_i \in (U_i, \phi_i)$ such that $p_i \sim p_j$ by γ_i. Let us prove that local path connected relation is an equivalence relation:
We know that a metrizable Radon measure manifold is local path connected if $\exists$ a measurable C^∞ path $\gamma_i \in G : \gamma_i : [0, 1] \longrightarrow (U_i, \phi_i) \in A^k(M)$ such that
$\gamma_i(0) = p_i \in (U_i, \phi_i)$ for which $\mu_R(U_i, \phi_i) > 0$ and
$\gamma_i(1) = p_j \in (U_i, \phi_i)$ for which $\mu_R(U_i, \phi_i) > 0$ where, $p_i, p_j \in P \in (U_i, \phi_i)$.
(i) '$\sim$' is reflexive :
Since (M, τ, Σ, μ_R) is a metrizable Radon measure manifold satisfying reflexive property of metrizable Radon measure manifold, that is, if $d(p_i, p_j) = 0$ if and only if $p_i = p_j$ then $\exists$ a measurable C^∞ path $\gamma_i : [0, 1] \longrightarrow (U_i, \phi_i)$ such that
$\gamma_i(0) = p_i \in (U_i, \phi_i)$ for which $\mu_R(U_i, \phi_i) > 0$ and
$\gamma_i(1) = p_j \in (U_i, \phi_i)$ for which $\mu_R(U_i, \phi_i) > 0$.
Since by reflexive property of metrizable Radon measure manifold, $p_i = p_j$. Then $\gamma : [0, 1] \longrightarrow (U_i, \phi_i)$ is a measurable C^∞ path from p_i to p_i itself which is a constant path. That is, $p_i \sim p_i$ by γ_i
Therefore, '$\sim$' is reflexive.
(ii)'$\sim$' is symmetric :
By symmetric property of metrizable Radon measure manifold, that is, if $d(p_i, p_j) = d(p_j, p_i)$, then $\exists$ a measurable C^∞ path $\gamma_i : [0, 1] \longrightarrow (U_i, \phi_i)$ such that
$\gamma_i(0) = p_i \in (U_i, \phi_i)$ for which $\mu_R(U_i, \phi_i) > 0$ and
$\gamma_i(1) = p_j \in (U_i, \phi_i)$ for which $\mu_R(U_i, \phi_i) > 0$.
Since by symmetric property of metrizable Radon measure manifold and definition 1.2.14 [4], for every $\gamma_i : [0, 1] \longrightarrow (U_i, \phi_i)$ there exists inverse of γ_i i.e. $\gamma_i^{-1} : [0, 1] \longrightarrow (U_i, \phi_i)$ defined by $\gamma_i^{-1}(p) = \gamma_i(1-p)$ and $\gamma_i^{-1}(q) = \gamma_i(1-q)$ for each $p, q \in [0, 1]$.
Therefore, '$\sim$' is symmetric.
(iii) '$\sim$' is transitive :
From the triangular inequality property of metrizable Radon measure manifold, if $d(p_i, p_k) \leq d(p_i, p_j) + d(p_j, p_k)$ then there exists measurable C^∞ paths $\gamma_i, \gamma_j \in G$
$\gamma_i : [0, 1] \longrightarrow (U_i, \phi_i)$ from p_i to p_j such that
$\gamma_i(0) = p_i \in (U_i, \phi_i)$
$\gamma_i(\frac{1}{2}) = p_j \in (U_i, \phi_i)$ and
$\gamma_j : [0, 1] \longrightarrow (U_i, \phi_i)$ from p_j to p_k such that $\gamma_j(\frac{1}{2}) = p_j \in (U_i, \phi_i)$ for which $\mu_R(U_i, \phi_i) > 0$
$\gamma_j(1) = p_k \in (U_i, \phi_i)$ for which $\mu_R(U_i, \phi_i) > 0$ and a composition of measurable C^∞ paths $\gamma_i, \gamma_j \in G$ such that
$\gamma_j \circ \gamma_i : [0, 1] \longrightarrow (U_i, \phi_i)$ from p_i to p_k such that
$\gamma_j \circ \gamma_i(0) = p_i \in (U_i, \phi_i)$ and
$\gamma_j \circ \gamma_i(1) = p_k \in (U_i, \phi_i)$.
Hence, if $p_i \sim p_j$ by γ_i and $p_j \sim p_k$ by γ_j then $p_i \sim p_k$ by $\gamma_j \circ \gamma_i$
Therefore, '$\sim$' is transitive.
Hence, local path connectedness relation '$\sim$' is an equivalence relation on $(U_i, \phi_i) \subset$

(M, τ, Σ, μ_R). ■

This equivalence relation further partitions the measurable chart $(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R)$ into disjoint equivalence classes. Then $(U_i, \phi_i)/_\sim$ is a **Quotient measure chart** on (M, τ, Σ, μ_R) admitting Radon measure. Similarly, we develop the concept of Quotient Radon measure atlas and then Quotient Radon measure manifold by adopting local path connectedness property, internal path connectedness property and maximal path connectedness property as invariant properties under measurable homeomorphism and Radon measure structure-invariant map as follows:

Theorem 3.1.3 Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be metrizable Radon measure manifolds. Then local path connectedness property is invariant under a C^∞ measurable homeomorphism and Radon measure structure - invariant map $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Proof. Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be metrizable Radon measure manifolds.
Let the local path connectedness property, say, P_7 holds $\mu_{R_1} - a.e.$, on the non-empty sets $A_1 \subset (U_i, \phi_i) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $A_2 \subset (U_j, \phi_j) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ where $A_1 = \{p_i, p_j \in [p] \in (U_i, \phi_i)$: $\exists\, \gamma_i \in [\gamma]$: p_i is locally path connected to p_j by $\gamma_i \in G\}$, $\mu_{R_1}(A_1) > 0$ and $A_2 = \{q_i, q_j \in [p] \in (U_i, \phi_i)$: $\exists\, \gamma_i \in [\gamma]$: q_i is locally path connected to q_j by $\gamma_i \in G\}$, $\mu_{R_1}(A_2) > 0$.
According to Radon measure structure-invariant map [definition 2.2], we say that $A_1 \sim A_2$ if and only if $\mu_{R_1}(A_1) = \mu_{R_1}(A_2)$.
We show that if P_7 holds $\mu_{R_1} - a.e.$, on $A_1, A_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_7 also holds $\mu_{R_2} - a.e.$, on $F(A_1), F(A_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$.
Suppose P_7 holds $\mu_{R_1} - a.e.$, on $A_1, A_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then $A_1 \sim A_2$ if and only if $\mu_{R_1}(A_1) = \mu_{R_1}(A_2)$.
Now if $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is C^∞ measurable homeomorphism and Radon measure structure - invariant map, then for every $A_1, A_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$,
$\mu_{R_1}(A_1) > 0, \mu_{R_1}(A_2) > 0$, there exists $F(A_1), F(A_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2}), \mu_{R_2}(F(A_1)) > 0, \mu_{R_2}(F(A_2)) > 0$:
$A_1 \sim A_2 \Rightarrow F(A_1) \sim F(A_2)$ with measure structure condition
$\mu_{R_1}(A_1) = \mu_{R_1}(A_2) \Rightarrow \mu_{R_2}(F(A_1)) = \mu_{R_2}(F(A_2))$ according to definition 2.1.1.
This implies local path connectedness property holds $\mu_{R_2} - a.e.$, on $F(A_1)$,$F(A_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$. ■

Theorem 3.1.4 Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ be metrizable Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $G : (M_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ be C^∞ measurable homeomorphisms and Radon measure structure-invariant maps. Then local path connectedness property is invariant under the composition map $G \circ F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$.

Proof. Let $(M_1, \tau_1, \Sigma_1, \mu_{R1})$, $(M_2, \tau_2, \Sigma_2, \mu_{R2})$ and $(M_3, \tau_3, \Sigma_3, \mu_{R3})$ be metrizable Radon measure manifolds.
Let $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R2})$ and $G : (M_2, \tau_2, \Sigma_2, \mu_{R2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R3})$ be C^∞ measurable homeomorphisms and Radon measure structure-invariant maps. We show that if local path connectedness property, say, P_7 holds $\mu_{R_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_7 also holds $\mu_{R_3} - a.e.$, on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ under measurable homeomorphism and Radon measure structure - invariant map $G \circ F$.
Let the local path connectedness property P_7 holds $\mu_{R_1} - a.e.$, on the non empty Borel sets $A_1 \subset (U_i, \phi_i) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $A_2 \subset (U_j, \phi_j) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ where (U_i, ϕ_i) and (U_j, ϕ_j) are Quotient Radon measure charts:
$A_1 \sim A_2$ if and only if $\mu_{R_1}(A_1) = \mu_{R_1}(A_2)$ by definition 2.1.1. (1)
By above theorem 3.1.3, if $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is C^∞ measurable homeomorphism and Radon measure structure - invariant map, then for every $A_1, A_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $\mu_{R_1}(A_1) > 0$, $\mu_{R_1}(A_2) > 0$, $\exists\ F(A_1), F(A_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$, $\mu_{R_2}(F(A_1)) > 0$, $\mu_{R_2}(F(A_2)) > 0$:
$A_1 \sim A_2 \Rightarrow F(A_1) \sim F(A_2)$ with measure structure condition:
$\mu_{R_1}(A_1) = \mu_{R_1}(A_2) \Rightarrow \mu_{R_2}(F(A_1)) = \mu_{R_2}(F(A_2))$ according to definition 2.1.2. (2)
Similarly, we show that under $G : (M_2, \tau_2, \Sigma_2, \mu_{R2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R3})$, if P_7 holds $\mu_{R_2} - a.e.$, on the non-empty Borel sets $F(A_1) \subset F(U_i, \phi_i)$ and $F(A_2) \subset F(Uj, \phi_j)$ on $(M_2, \tau_2, \Sigma_2, \mu_{R2})$ then P_7 also holds $\mu_{R_3} - a.e.$, on the non-empty Borel sets $(G \circ F)(A_1) \subset (G \circ F)(U_i, \phi_i)$ and $(G \circ F)(A_2) \subset (G \circ F)(U_j, \phi_j)$ on $(M_3, \tau_3, \Sigma_3, \mu_{R3})$ where, $(G \circ F(A_1)) = \{G \circ F(p_i), G \circ F(p_j)) \in [p] \in G \circ F(U_i, \phi_i)) :\exists\ G \circ F(\gamma_i)) \in G \circ F([\gamma]) : G \circ F(p_i)$ is locally path connected to $G \circ F(p_j)$ by $G \circ F(\gamma_i) \in G \circ F(G)\}$ where $\mu_{R_3}(G \circ F(A_1)) > 0$ and $(G \circ F(A_2)) = \{G \circ F(q_i), G \circ F(q_j) \in [p] \in G \circ F(U_j, \phi_j) :\exists\ G \circ F(\gamma_i) \in G \circ F([\gamma]) : G \circ F(q_i)$ is locally path connected to $G \circ F(q_j)$ by $G \circ F(\gamma_i)) \in G \circ F(G)\}$ where $\mu_{R_3}(G \circ F(A_2)) > 0$ on $(M_3, \tau_3, \Sigma_3, \mu_{R3})$:
$F(A_1) \sim F(A_2) \Rightarrow (G \circ F)(A_1) \sim (G \circ F)(A_2)$ with the Radon measure condition:
$\mu_{R_2}(F(A_1)) = \mu_{R_2}(F(A_2)) \Rightarrow \mu_{R_3}(G \circ F(A_1)) = \mu_{R_3}(G \circ F(A_2))$
according to definition 2.1.2. (3)
This implies under $G \circ F : (M_1, \tau_1, \Sigma_1, \mu_{R1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R3})$, if P_1 holds $\mu_{R_1} - a.e.$, on $A_1, A_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_7 also holds $\mu_{R_3} - a.e.$, on $(G \circ F(A_1)), (G \circ F(A_2)) \subset (M_3, \tau_3, \Sigma_3, \mu_{R3})$:
$A_1 \sim A_2 \Rightarrow (G \circ F(A_1)) \sim (G \circ F(A_2))$ with the Radon measure condition:
$\mu_{R_1}(A_1) = \mu_{R_1}(A_2)$ implies $\mu_{R_3}(G \circ F(A_1)) = \mu_{R_3}(G \circ F(A_2))$.
Therefore, from (1), (2) and (3), if local path connectedness property holds $\mu_{R_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{R1})$ then it also holds $\mu_{R_3} - a.e.$, on $(M_3, \tau_3, \Sigma_3, \mu_{R3})$. ■

Corollary 3.1.5 Let $M_1, \ldots, M_n$ be metrizable Radon measure manifolds. If $F_1 : M_1 \longrightarrow M_2, ..., F_n : M_{n-1} \longrightarrow M_n$ are measurable homeomorphism and Radon measure structure-invariant maps, then, if local path connectedness property holds $\mu_{R_1} - a.e.$, on M_1 then it also holds $\mu_{R_n} - a.e.$, on M_n under composition maps.

Remark 3.1.6 Using the above results, one can generate a new **class of locally path connected Quotient Radon measure manifold** $(\Pi, (G, \circ))$ where $\Pi = \{M_1 \times M_2 \times ... \times M_n\}$ which is closed under the group action $(G, \circ) = \{F_1, F_2, ..., F_n\}$ of C^∞ measurable homeomorphisms and Radon measure-invariant maps and S. C. P. Halakatti constructs different categories of Quotient Radon measure manifolds on different patterns of charts in next section 4.

Similar to theorem 3.1.2, one can show that the internal path connectedness relation '$\sim$' defined on the measurable domain $[(U_i, \phi_i) \cup (U_j, \phi_j)] \in \mathcal{A}_i \in A^k(M)$ is an equivalence relation.

3.2. Internally path connected Radon measure manifold

Theorem 3.2.1 If (M, τ, Σ, μ_R) is a metrizable Radon measure manifold then internal path connectedness relation is an equivalence relation on (M, τ, Σ, μ_R).

Proof. Let $(M, \tau_1, \Sigma_1, \mu_R)$ be a metrizable Radon measure manifold such that for every point $p \in (M, \tau, \Sigma, \mu_R)$ there exists a measurable domain $[(U_i, \phi_i) \cup (U_j, \phi_j)] \in (M, \tau_1, \Sigma_1, \mu_{R_1})$.
Let $A_2 = \{p_i, p_j \in [(U_i, \phi_i) \cup (U_j, \phi_j)];\ \exists\ \gamma_i \in G\ /\ p_i \sim p_j$ by $\gamma_i;\ \forall\ i, j \in I\}$ be the non-empty set belonging to $[(U_i, \phi_i) \cup (U_j, \phi_j)] \in (M, \tau, \Sigma, \mu_R)$ and $G = \{\gamma_1, \gamma_2, ..., \gamma_n\}$ be the non-empty collection of measurable C^∞ paths $\gamma_i \in [(U_i, \phi_i) \cup (U_j, \phi_j)]$ such that $p_i \sim p_j$ by γ_i.
Let us prove that internally path connected relation is an equivalence relation:
We know that a metrizable Radon measure manifold is internally path connected if $\exists$ a measurable C^∞ path $\gamma_i \in G : \gamma_i : [0, 1] \longrightarrow [(U_i, \phi_i) \cup (U_j, \phi_j)] \in A^k(M)$ such that
$\gamma_i(0) = p_i \in (U_i, \phi_i)$ for which $\mu_R(U_i) > 0$ and
$\gamma_i(1) = p_j \in (U_j, \phi_j)$ for which $\mu_R(U_j) > 0$ where, $p_i, p_j \in [(U_i, \phi_i) \cup (U_i, \phi_i)]$.
Now we show that the relation '$\sim$' is an equivalence relation on the measurable domain $[(U_i, \phi_i) \cup (U_j, \phi_j)]$ of (M, τ, Σ, μ_R).
(i) '$\sim$' is reflexive :
Since (M, τ, Σ, μ_R) is a metrizable Radon measure manifold satisfying reflexive property of metrizable Radon measure manifold, that is, if $d(p_i, p_j) = 0$ if and only if $p_i = p_j$ then $\exists$ a measurable C^∞ path $\gamma_i : [0, 1] \longrightarrow [(U_i, \phi_i) \cup (U_j, \phi_j)]$ such that
$\gamma_i(0) = p_i \in (U_i, \phi_i)$ for which $\mu_R(U_i) > 0$ and
$\gamma_i(1) = p_j \in (U_j, \phi_j)$ for which $\mu_R(U_j) > 0$.
Since by reflexive property of metrizable Radon measure manifold, $p_i = p_j$. Then $\gamma : [0, 1] \longrightarrow [(U_i, \phi_i) \cup (U_j, \phi_j)]$ is a measurable C^∞ path from p_i to p_i itself which is a constant path. That is, $p_i \sim p_i$ by γ_i
Therefore, '$\sim$' is reflexive.
(ii)'$\sim$' is symmetric:
By symmetric property of metrizable Radon measure manifold, that is, if $d(p_i, p_j) = d(p_j, p_i)$, then $\exists$ a measurable C^∞ path $\gamma_i : [0, 1] \longrightarrow [(U_i, \phi_i) \cup (U_j, \phi_j)]$ such that
$\gamma_i(0) = p_i \in (U_i, \phi_i)$ for which $\mu_R(U_i) > 0$ and
$\gamma_i(1) = p_j \in (U_j, \phi_j)$ for which $\mu_R(U_j) > 0$.

Since by symmetric property of metrizable Radon measure manifold, for every γ_i : $p_i \longrightarrow p_j$ there exists $\gamma_j : p_j \longrightarrow p_i$ such that there exists a measurable C^∞ path
$\gamma_j : [0, 1] \longrightarrow [(U_i, \phi_i) \cup (U_j, \phi_j)]$ such that
$\gamma_j(0) = p_j \in (U_j, \phi_j)$ and
$\gamma_j(1) = p_i \in (U_i, \phi_i)$ for which $\mu_R(U_i) > 0$.
Therefore, '$\sim$' is symmetric.
(iii) '$\sim$' is transitive :
From the triangular inequality property of metrizable Radon measure manifold, if $d(p_i, p_k) \leq d(p_i, p_j)+d(p_j, p_k)$ then there exists measurable C^∞ paths $\gamma_i, \gamma_j \in G$ $\gamma_i : [0, 1] \longrightarrow [(U_i, \phi_i) \cup (U_j, \phi_j)]$ from p_i to p_j such that
$\gamma_i(0) = p_i \in [(U_i, \phi_i) \cup (U_j, \phi_j)]$
$\gamma_i(\frac{1}{2}) = p_j \in [(U_i, \phi_i) \cup (U_j, \phi_j)]$ and
$\gamma_j : [0, 1] \longrightarrow [(U_i, \phi_i) \cup (U_j, \phi_j)]$ from p_j to p_k such that
$\gamma_j(\frac{1}{2}) = p_j \in [(U_i, \phi_i) \cup (U_j, \phi_j)]$ for which $\mu_R(U_i \cup U_j) > 0$
$\gamma_j(1) = p_k \in [(U_i, \phi_i) \cup (U_j, \phi_j)]$ for which $\mu_R(U_i \cup U_j) > 0$ and a composition of measurable C^∞ paths $\gamma_i, \gamma_j \in G$ such that
$\gamma_j \circ \gamma_i : [0, 1] \longrightarrow [(U_i, \phi_i) \cup (U_j, \phi_j)]$ from p_i to p_k such that
$\gamma_j \circ \gamma_i(0) = p_i \in (U_i, \phi_i)$ and
$\gamma_j \circ \gamma_i(1) = p_k \in (U_j, \phi_j)$.
Hence, if $p_i \sim p_j$ by γ_i and $p_j \sim p_k$ by γ_j then $p_i \sim p_k$ by $\gamma_j \circ \gamma_i$. Therefore, '$\sim$' is transitive. Hence, internal path connectedness relation '$\sim$' is an equivalence relation on $[(U_i, \phi_i) \cup (U_j, \phi_j)] \subset (M, \tau, \Sigma, \mu_R)$. ■

Further, this equivalence relation partitions the measurable domain $[(U_i, \phi_i) \cup (U_j, \phi_j)] \in \mathcal{A}_i \in A^k(M)$ into disjoint equivalence classes. such that $(U_i \cup U_j)/\sim$ is Quotient Radon measure domain in $\mathcal{A}_i \subset (M, \tau, \Sigma, \mu_R)$.

Theorem 3.2.2 Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be metrizable Radon measure manifolds. Then internal path connectedness property is invariant under a C^∞ measurable homeomorphism and Radon measure structure - invariant map $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow$

$(M_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Proof. Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be metrizable Radon measure manifolds.
Let the internal path connectedness property, say, P_8 holds $\mu_{R_1} - a.e.$, on the non-empty sets $A_1 \subset [(U_i, \phi_i) \cup (U_j, \phi_j)] \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $A_2 \subset [(U_k, \phi_k) \cup (U_l, \phi_l)] \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ where $A_1 = \{p_i, p_j \in [p] \in [(U_i, \phi_i) \cup (U_j, \phi_j)]$: $\exists\ \gamma_i \in [\gamma]$: p_i is internally path connected to p_j by $\gamma_i \in G\}$, $\mu_{R_1}(A_1) > 0$ and $A_2 = \{q_i, q_j \in [p] \in [(U_k, \phi_k) \cup (U_l, \phi_l)]$: $\exists\ \gamma_i \in [\gamma]$: q_i is internally path connected to q_j by $\gamma_i \in G\}$, $\mu_{R_1}(A_2) > 0$.
According to Radon measure structure condition [definition 2.1.1], we say that $\mathcal{A}_1 \sim \mathcal{A}_2$

if and only if $\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$.
We show that if P_8 holds $\mu_{R_1} - a.e.$, on $A_1, A_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_8 also holds $\mu_{R_2} - a.e.$, on $F(A_1), F(A_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$.
Now if $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is C^∞ measurable homeomorphism and Radon measure structure - invariant map, then for every $A_1, A_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$,
$\mu_{R_1}(A_1) > 0, \mu_{R_1}(A_2) > 0$, there exists $F(A_1), F(A_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2}), \mu_{R_2}(F(A_1)) > 0, \mu_{R_2}(F(A_2)) > 0$:
$A_1 \sim A_2$ implies $F(A_1) \sim F(A_2)$ with Radon measure structure condition
$\mu_{R_1}(A_1) = \mu_{R_1}(A_2)$ implies $\mu_{R_2}(F(A_1)) = \mu_{R_2}(F(A_2))$ according to definition 2.1.2.
This implies internal path connectedness property holds $\mu_{R_2} - a.e.$, on $F(A_1)$,$F(A_2)$ $\subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$. ■

Theorem 3.2.3 Let $(M_1, \tau_1, \Sigma_1, \mu_{R1})$, $(M_2, \tau_2, \Sigma_2, \mu_{R2})$ and $(M_3, \tau_3, \Sigma_3, \mu_{R3})$ be metrizable Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R2})$ and $G : (M_2, \tau_2, \Sigma_2, \mu_{R2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R3})$ be C^∞ measurable homeomorphisms and Radon measure structure-invariant maps. Then internal path connectedness property is invariant under the composition map $G \circ F : (M_1, \tau_1, \Sigma_1, \mu_{R1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R3})$.

Proof. Let $(M_1, \tau_1, \Sigma_1, \mu_{R1})$, $(M_2, \tau_2, \Sigma_2, \mu_{R2})$ and $(M_3, \tau_3, \Sigma_3, \mu_{R3})$ be metrizable Radon measure manifolds.
Let $F : (M_1, \tau_1, \Sigma_1, \mu_{R1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R2})$ and $G : (M_2, \tau_2, \Sigma_2, \mu_{R2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R3})$ be C^∞ measurable homeomorphisms and Radon measure structure-invariant maps.
We show that if internal path connectedness property, say, P_8 holds $\mu_{R_1} - a.e.$, on non-empty sets $A_1 \subset [(U_i, \phi_i) \cup (U_j, \phi_j)]$, $A_2 \subset [(U_k, \phi_k) \cup (U_l, \phi_l)] \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_8 also holds $\mu_{R_3} - a.e.$, on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ under measurable homeomorphism and Radon measure structure-invariant map $G \circ F$.
Let the internal path connectedness property P_8 holds $\mu_{R_1} - a.e.$, on the non empty sets $A_1 \subset [(U_i, \phi_i) \cup (U_j, \phi_j)] \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $A_2 \subset [(U_k, \phi_k) \cup (U_l, \phi_l)] \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ where $[(U_i, \phi_i) \cup (U_j, \phi_j)]$ and $[(U_k, \phi_k) \cup (U_l, \phi_l)]$ are Quotient Radon measure domains:
$A_1 \sim A_2$ if and only if $\mu_{R_1}(A_1) = \mu_{R_1}(A_2)$ by definition 2.1.1.
By above theorem 3.2.2, if $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is C^∞ measurable homeomorphism and Radon measure structure - invariant map, then for every $A_1, A_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $\mu_{R_1}(A_1) > 0$, $\mu_{R_1}(A_2) > 0$, $\exists\ F(A_1), F(A_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2}), \mu_{R_2}(F(A_1)) > 0, \mu_{R_2}(F(A_2)) > 0$:
$A_1 \sim A_2$ implies $F(A_1) \sim F(A_2)$ with Radon measure structure condition
$\mu_{R_1}(A_1) = \mu_{R_1}(A_2)$ implies $\mu_{R_2}(F(A_1)) = \mu_{R_2}(F(A_2))$ according to definition 2.1.2. (1)
Similarly, we show that under $G : (M_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$, if P_8 holds $\mu_{R_2} - a.e.$, on the non-empty Borel sets $F(A_1) \subset F[(U_i, \phi_i) \cup (U_j, \phi_j)]$ and

$F(A_2) \subset F[(U_k, \phi_k) \cup (U_l, \phi_l)]$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ then P_8 also holds $\mu_{R_3} - a.e.$, on the non-empty Borel sets $(G \circ F)(A_1) \subset (G \circ F)[(U_i, \phi_i) \cup (U_j, \phi_j)]$ and $(G \circ F)(A_2) \subset (G \circ F)[(U_k, \phi_k) \cup (U_l, \phi_l)]$ on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ where,
$(G \circ F(A_1)) = \{G \circ F(p_i)), G \circ F(p_j)) \in [p] \in G \circ F[(U_i, \phi_i) \cup (U_j, \phi_j)] : \exists$ $G \circ F(\gamma_i)) \in G \circ F([\gamma]) : G \circ F(p_i)$ is internally path connected to $G \circ F(p_j)$ by $G \circ F(\gamma_i) \in G \circ F(G)\}$ where $\mu_{R_3}(G \circ F(A_1)) > 0$ and
$(G \circ F(A_2)) = \{G \circ F(q_i), G \circ F(q_j) \in [p] \in G \circ F[(U_k, \phi_k) \cup (U_l, \phi_l)] : \exists$ $G \circ F(\gamma_i) \in G \circ F([\gamma]) : G \circ F(q_i)$ is internally path connected to $G \circ F(q_j)$ by $G \circ F(\gamma_i)) \in G \circ F(G)\}$ where $\mu_{R_3}(G \circ F(A_2)) > 0$ on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$:
$F(A_1) \sim F(A_2)$ implies $(G \circ F)(A_1) \sim (G \circ F)(A_2)$ with the Radon measure condition:
$\mu_{R_2}(F(A_1)) = \mu_{R_2}(F(A_2)$ implies $\mu_{R_3}(G \circ F(A_1)) = \mu_{R_3}(G \circ F(A_2))$
according to definition 2.1.2. (2)
This implies under $G \circ F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$, if P_8 holds $\mu_{R_1} - a.e.$, on A_1, $A_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_8 also holds $\mu_{R_3} - a.e.$, on $(G \circ F(A_1)), (G \circ F(A_2)) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3})$:
$A_1 \sim A_2$ implies $(G \circ F(A_1)) \sim (G \circ F(A_2))$ with the Radon measure condition:
$\mu_{R_1}(A_1) = \mu_{R_1}(A_2)$ implies

$$\mu_{R_3}(G \circ F(A_1)) = \mu_{R_3}(G \circ F(A_2)). \quad (3)$$

Therefore, from (1), (2) and (3), if internal path connectedness property holds $\mu_{R_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then it also holds $\mu_{R_3} - a.e.$, on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$. ■

Corollary 3.2.4 Let $M_1, \ldots, M_n$ be metrizable Radon measure manifolds. If $F_1 : M_1 \longrightarrow M_2$,...,$F_n : M_{n-1} \longrightarrow M_n$ are measurable homeomorphism and Radon measure structure-invariant maps, then, if internally path connectedness property holds $\mu_{R_1} - a.e.$, on M_1 then it also holds $\mu_{R_n} - a.e.$, on M_n under composition maps.

Remark 3.2.5 Using the above results, one can generate a new **class of internally path connected Quotient Radon measure manifold** $(\Pi, (G, \circ))$ where $\Pi = \{M_1 \times M_2 \times \ldots \times M_n\}$ which is closed under the group action $(G, \circ) = \{F_1, F_2, \ldots, F_n\}$ of C^∞ measurable homeomorphisms and Radon measure-invariant maps and S. C. P. Halakatti constructs different categories of Quotient Radon measure manifolds on different patterns of charts in next section 4.

3.3. Maximally path connected Radon measure manifold

Now, the first author has introduced, developed and defined the concept of Quotient Radon measure atlas and proved the following results:

Definition 3.3.1 Quotient Radon measure Atlas
By an R^n-Quotient Radon measure atlas of class $C^k (k \geq 1)$ on a measure manifold (M, τ, Σ, μ_R), we mean a countable collection $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}}, \mu_{/\mathcal{A}})$ of n-dimensional Quotient Radon measure charts $((U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{R_{/U_i}}), \phi_{i_{/U_i}})$ for all $i \in I$ on (M, τ, Σ, μ_R) satisfying the following conditions:

(**a₁**) $\cup_{i \in I}(U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{R_{/U_i}}) = (M, \tau, \Sigma, \mu_R)$. That is, the countable union of all Quotient Radon measure charts in $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}}, \mu_{/\mathcal{A}})$ cover (M, τ, Σ, μ_R) .
(**a₂**) For any pair of Quotient Radon measure charts (U_i, ϕ_i) and (U_j, ϕ_j) there exists transition maps $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$ and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ are differentiable and Radon measurable.
(**a₃**) Any two Quotient Radon measure atlases are compatible on (M, τ, Σ, μ_R) satisfying two equivalence relations:

(i) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $(\mathcal{A}_1 \cup \mathcal{A}_2) \in A^k(M)$.

(ii) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mu_R(\mathcal{A}_1) = \mu_R(\mathcal{A}_2)$.

Since (M, τ, Σ, μ_R) is maximally path connected if and only if it is locally and internally path connected [2], maximal path connectedness relation '$\sim$' defined on the measurable domain $(\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k) \in A^k(M)$ is an equivalence relation.

Theorem 3.3.2 If (M, τ, Σ, μ_R) is a metrizable Radon measure manifold then maximal path connectedness relation is an equivalence relation on (M, τ, Σ, μ_R).

Proof. Let $(M, \tau_1, \Sigma_1, \mu_R)$ be a metrizable Radon measure manifold such that for every point $p \in (M, \tau, \Sigma, \mu_R)$ there exists a measurable domain $(\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k) \in (M, \tau_1, \Sigma_1, \mu_{R_1})$.
Let $A_3 = \{p_i, p_j \in (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k) \ ; \exists \ \gamma_i \in G \ / \ p_i \sim p_j$ by $\gamma_i ; \forall \ i, j, k \in I\}$ be the non-empty set belonging to $(\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k) \in (M, \tau, \Sigma, \mu_R)$ and $G = \{\gamma_1, \gamma_2, ..., \gamma_n\}$ be the non-empty collection of measurable C^∞ paths $\gamma_i \in G$ such that $p_i \sim p_j$ by γ_i.
Let us prove that maximal path connected relation is an equivalence relation:
We know that a metrizable Radon measure manifold is maximally path connected if $\exists$ a measurable C^∞ path $\gamma_i \in G : \gamma_i : [0, 1] \longrightarrow (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k) \in A^k(M)$ such that
$\gamma_i(0) = p_i \in (U_i, \phi_i) \in \mathcal{A}_i$ for which $\mu_R(U_i) > 0, \mu_R(\mathcal{A}_i) > 0$ and
$\gamma_i(1) = p_j \in (U_j, \phi_j) \in \mathcal{A}_j$ for which $\mu_R(U_j) > 0, \mu_R(\mathcal{A}_j) > 0$ where, $p_i, p_j \in (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k)$.
Now we show that the relation '$\sim$' is an equivalence relation on the measurable domain $(\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_j)$ of (M, τ, Σ, μ_R).
(i) '$\sim$' is reflexive :
Since (M, τ, Σ, μ_R) is a metrizable Radon measure manifold satisfying reflexive property of metrizable Radon measure manifold, that is, if $d(p_i, p_j) = 0$ if and only if $p_i = p_j$ then $\exists$ a measurable C^∞ path $\gamma_i : [0, 1] \longrightarrow (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k)$ such that
$\gamma_i(0) = p_i \in (U_i, \phi_i) \in \mathcal{A}_i$ for which $\mu_R(U_i) > 0, \mu_R(\mathcal{A}_i) > 0$ and
$\gamma_i(1) = p_j \in (U_j, \phi_j) \in \mathcal{A}_j$ for which $\mu_R(U_j) > 0, \mu_R(\mathcal{A}_j) > 0$.
Since by reflexive property of metrizable Radon measure manifold, $p_i = p_j$. Then $\gamma : [0, 1] \longrightarrow (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k)$ is a measurable C^∞ path from p_i to p_i itself which is a constant path. That is, $p_i \sim p_i$ by γ_i
Therefore, '$\sim$' is reflexive.
(ii)'$\sim$' is symmetric :
By symmetric property of metrizable Radon measure manifold, that is, if $d(p_i, p_j) =$

$d(p_j, p_i)$, then $\exists$ a measurable C^∞ path $\gamma_i : [0, 1] \longrightarrow (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k)$ such that
$\gamma_i(0) = p_i \in (U_i, \phi_i) \in \mathcal{A}_i$ for which $\mu_R(U_i) > 0$ and
$\gamma_i(1) = p_j \in (U_j, \phi_j) \in \mathcal{A}_j$ for which $\mu_R(U_j) > 0$.
Since by symmetric property of metrizable Radon measure manifold, for every $\gamma_i : p_i \longrightarrow p_j$ there exists $\gamma_j : p_j \longrightarrow p_i$ such that there exists a measurable C^∞ path $\gamma_j : [0, 1] \longrightarrow (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k)$ such that
$\gamma_j(0) = p_j \in (U_i, \phi_i) \in \mathcal{A}_i$ and
$\gamma_j(1) = p_i \in (U_j, \phi_j) \in \mathcal{A}_j$.
Therefore, '$\sim$' is symmetric.
(iii) '$\sim$' is transitive:
From the triangular inequality property of metrizable Radon measure manifold, if $d(p_i, p_k) \leq d(p_i, p_j) + d(p_j, p_k)$ then there exists measurable C^∞ paths $\gamma_i, \gamma_j \in G$
$\gamma_i : [0, 1] \longrightarrow (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k)$ from p_i to p_j such that
$\gamma_i(0) = p_i \in (U_i, \phi_i) \in \mathcal{A}_i$
$\gamma_i(\frac{1}{2}) = p_j \in (U_j, \phi_j) \in \mathcal{A}_j$ and
$\gamma_j : [0, 1] \longrightarrow (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k)$ from p_j to p_k such that
$\gamma_j(\frac{1}{2}) = p_j \in (U_j, \phi_j) \in \mathcal{A}_j$ for which $\mu_R(U_j) > 0$, $\mu_R(\mathcal{A}_j) > 0$
$\gamma_j(1) = p_k \in (U_k, \phi_k) \in \mathcal{A}_k$ for which $\mu_R(U_k) > 0$, $\mu_R(\mathcal{A}_k) > 0$ and a composition of measurable C^∞ paths $\gamma_i, \gamma_j \in G$ such that
$\gamma_j \circ \gamma_i : [0, 1] \longrightarrow (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k)$ from p_i to p_k such that
$\gamma_j \circ \gamma_i(0) = p_i \in (U_i, \phi_i) \in \mathcal{A}_i$ and $\gamma_j \circ \gamma_i(1) = p_k \in (U_k, \phi_k) \in \mathcal{A}_k$.
Hence, if $p_i \sim p_j$ by γ_i and $p_j \sim p_k$ by γ_j then $p_i \sim p_k$ by $\gamma_j \circ \gamma_i$.
Therefore, '$\sim$' is transitive. Hence, maximally path connectedness relation '$\sim$' is an equivalence relation on $[(U_i, \phi_i) \cup (U_i, \phi_i)] \subset (M, \tau, \Sigma, \mu_R)$. ∎

Hence, this equivalence relation partitions the measurable domain $(A_i \cup A_j \cup A_k) \in A^k(M)$ into disjoint equivalence classes. Then $(M, \tau, \Sigma, \mu_R)/_\sim$ is a **Quotient Radon measure Manifold** denoted by $(\mathcal{M}, \tau, \Sigma, \mu_R)$.

Theorem 3.3.3 Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be metrizable Radon measure manifolds. Then maximal path connectedness property is invariant under a C^∞ measurable homeomorphism and Radon measure structure-invariant map $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Proof. Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be metrizable Radon measure manifolds.
Let the maximal path connectedness property, say, P_9 holds $\mu_{R_3} - a.e.$, on Radon measure atlases $\mathcal{A}_1 \subset (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $\mathcal{A}_2 \subset (\mathcal{A}_l \cup \mathcal{A}_m \cup \mathcal{A}_n) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ where $\mathcal{A}_1 = \{p_i, p_j \in [p] \in (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k)$: $\exists\, \gamma_i \in [\gamma]$: p_i is maximally path connected to p_j by $\gamma_i \in G\}$, $\mu_{R_1}(\mathcal{A}_1) > 0$ and $\mathcal{A}_2 = \{q_i, q_j \in [p] \in (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k)$: $\exists\, \gamma_i \in [\gamma]$: q_i is maximally path connected to q_j by $\gamma_i \in G\}$, $\mu_{R_1}(\mathcal{A}_2) > 0$.

According to Radon measure structure condition [definition 2.1.1], we say that $\mathcal{A}_1 \sim \mathcal{A}_2$ if and only if $\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$.
We show that if P_9 holds $\mu_{R_1} - a.e.$, on $\mathcal{A}_1, \mathcal{A}_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_9 also holds $\mu_{R_2} - a.e.$, on $F(\mathcal{A}_1), F(\mathcal{A}_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$.
Now if $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is C^∞ measurable homeomorphism and Radon measure structure - invariant map, then for every $\mathcal{A}_1, \mathcal{A}_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$,
$\mu_{R_1}(\mathcal{A}_1) > 0, \mu_{R_1}(\mathcal{A}_2) > 0, \exists\ F(\mathcal{A}_1), F(\mathcal{A}_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2}), \mu_{R_2}(F(\mathcal{A}_1)) > 0, \mu_{R_2}(F(\mathcal{A}_2)) > 0$: $\mathcal{A}_1 \sim \mathcal{A}_2$ implies $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ with Radon measure structure condition
$\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ implies $\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2))$ according to definition 2.1.1.
Therefore, from (1), (2) and (3), if maximal path connectedness property holds $\mu_{R_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then it also holds $\mu_{R_3} - a.e.$, on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$. ■

Theorem 3.3.4 Let $(M_1, \tau_1, \Sigma_1, \mu_{R1})$, $(M_2, \tau_2, \Sigma_2, \mu_{R2})$ and $(M_3, \tau_3, \Sigma_3, \mu_{R3})$ be metrizable Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R2})$ and $G : (M_2, \tau_2, \Sigma_2, \mu_{R2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R3})$ be C^∞ measurable homeomorphism and Radon
measure structure-invariant maps. Then maximal path connectedness property is invariant under the composition map $G \circ F : (M_1, \tau_1, \Sigma_1, \mu_{R1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R3})$.

Proof. Let $(M_1, \tau_1, \Sigma_1, \mu_{R1})$, $(M_2, \tau_2, \Sigma_2, \mu_{R2})$ and $(M_3, \tau_3, \Sigma_3, \mu_{R3})$ be metrizable Radon measure manifolds.
Let $F : (M_1, \tau_1, \Sigma_1, \mu_{R1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R2})$ and $G : (M_2, \tau_2, \Sigma_2, \mu_{R2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R3})$ be C^∞ measurable homeomorphisms and Radon measure structure-invariant maps.
We show that if maximal path connectedness property, say, P_9 holds $\mu_{R_1} - a.e.$, on Radon measure atlases $\mathcal{A}_1 \subset (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k)$, $\mathcal{A}_2 \subset (\mathcal{A}_l \cup \mathcal{A}_m \cup \mathcal{A}_n) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_9 also holds $\mu_{R_3} - a.e.$, on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ under measurable homeomorphism and Radon measure structure-invariant map $G \circ F$.
Let the maximal path connectedness property P_9 holds $\mu_{R_1} - a.e.$, on Radon measure atlases $\mathcal{A}_1 \subset (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $\mathcal{A}_2 \subset (\mathcal{A}_l \cup \mathcal{A}_m \cup \mathcal{A}_n) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ where $(\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k)$ and $(\mathcal{A}_l \cup \mathcal{A}_m \cup \mathcal{A}_n)$ are Quotient Radon measure atlases:
$\mathcal{A}_1 \sim \mathcal{A}_2$ if and only if $\mu_{R_1}\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ by definition 2.1.1.
By above theorem 3.3.3, if $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is C^∞ measurable homeomorphism and Radon measure structure - invariant map, then for every $\mathcal{A}_1, \mathcal{A}_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1}), \mu_{R_1}(\mathcal{A}_1) > 0, \mu_{R_1}(\mathcal{A}_2) > 0$, there exists $F(\mathcal{A}_1), F(\mathcal{A}_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2}), \mu_{R_2}(F(\mathcal{A}_1)) > 0, \mu_{R_2}(F(\mathcal{A}_2)) > 0$:
$\mathcal{A}_1 \sim \mathcal{A}_2$ implies $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ with Radon measure structure condition:
$\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ implies $\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2))$ according to definition 2.1.2.
(1)
Similarly, we show that under $G : (M_2, \tau_2, \Sigma_2, \mu_{R2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R3})$, if P_9

holds $\mu_{R_2} - a.e.$, on the non-empty Borel sets $F(\mathcal{A}_1) \subset F(\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k)$ and $F(\mathcal{A}_2) \subset F(\mathcal{A}_l \cup \mathcal{A}_m \cup \mathcal{A}_n)$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ then P_9 also holds $\mu_{R_3} - a.e.$, on the non-empty Borel sets $(G \circ F)(\mathcal{A}_1) \subset (G \circ F)(\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k)$ and $(G \circ F)(\mathcal{A}_2) \subset (G \circ F)(\mathcal{A}_l \cup \mathcal{A}_m \cup \mathcal{A}_n)$ on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ where,
$(G \circ F(\mathcal{A}_1)) = \{G \circ F(p_i)), G \circ F(p_j)) \in [p] \in G \circ F(\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k) : \exists\, G \circ F(\gamma_i)) \in G \circ F([\gamma]) : G \circ F(p_i)$ is maximally path connected to $G \circ F(p_j)$ by $G \circ F(\gamma_i) \in G \circ F(G)\}$ where $\mu_{R_3}(G \circ F(\mathcal{A}_1)) > 0$ and $(G \circ F(\mathcal{A}_2)) = \{G \circ F(q_i), G \circ F(q_j) \in [p] \in G \circ F(\mathcal{A}_l \cup \mathcal{A}_m \cup \mathcal{A}_n) : \exists\, G \circ F(\gamma_i) \in G \circ F([\gamma]) : G \circ F(q_i)$ is maximally path connected to $G \circ F(q_j)$ by $G \circ F(\gamma_i)) \in G \circ F(G)\}$ where $\mu_{R_3}(G \circ F(\mathcal{A}_2)) > 0$ on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$:
$F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ implies $(G \circ F)(\mathcal{A}_1) \sim (G \circ F)(\mathcal{A}_2)$ with the Radon measure condition:
$\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2)$ implies $\mu_{R_3}(G \circ F(\mathcal{A}_1)) = \mu_{R_3}(G \circ F(\mathcal{A}_2))$
according to definition 2.1.2. (2)
This implies under $G \circ F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$, if P_9 holds $\mu_{R_3} - a.e.$, on $\mathcal{A}_1, \mathcal{A}_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_9 also holds $\mu_{R_3} - a.e.$, on $(G \circ F(\mathcal{A}_1)), (G \circ F(\mathcal{A}_2)) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3})$:
$\mathcal{A}_1 \sim \mathcal{A}_2$ implies $(G \circ F(\mathcal{A}_1)) \sim (G \circ F(\mathcal{A}_2))$ with the Radon measure condition:
$\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ implies $\mu_{R_3}(G \circ F(\mathcal{A}_1)) = \mu_{R_3}(G \circ F(\mathcal{A}_2))$. (3)
This implies maximal path connectedness property holds $\mu_{R_3} - a.e.$, on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$.

■

Corollary 3.3.5 Let $M_1, \ldots, M_n$ be metrizable Radon measure manifolds. If $F_1 : M_1 \longrightarrow M_2$,...,$F_n : M_{n-1} \longrightarrow M_n$ are measurable homeomorphism and Radon measure structure-invariant maps, then, if maximal path connectedness property holds $\mu_{R_3} - a.e.$, on M_1 then it also holds $\mu_{R_n} - a.e.$, on M_n under composition maps.

Remark 3.3.6 Using the above results, one can generate a new **class of maximally path connected Quotient Radon measure manifold** $(\Pi, (G, \circ))$ where $\Pi = \{M_1 \times M_2 \times \ldots \times M_n\}$ which is closed under the group action $(G, \circ) = \{F_1, F_2, ..., F_n\}$ of C^∞ measurable homeomorphisms and Radon measure-invariant maps and S. C. P. Halakatti constructs different categories of Quotient Radon measure manifolds on different patterns of charts in the next section.

Definition 3.3.7 Quotient Radon measure Manifold
Let $(M, \tau, \Sigma, \mu_R) \subseteq \{\cup_{i \in I} \mathcal{A}_i\}$ be a metrizable Radon measure manifold and (U_i, ϕ_i) and $\mathcal{A}_i$ (where $i \in I$) are Quotient Radon measure charts and Quotient Radon measure atlas respectively. If (M, τ, Σ, μ_R) admits the local, internal and maximal path connectedness relations under the measurable C^∞ canonical surjective projection map $q : (M, \tau, \Sigma, \mu_R) \longrightarrow (M, \tau, \Sigma, \mu_R)/_\sim$ such that for all $(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R)/_\sim$, $q^{-1}(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R)$ where the quotient topology is defined as :
$\tau_{\mathcal{M}} = \{(U_i, \phi_i) \in (\mathcal{M}, \tau, \Sigma, \mu_R) : q^{-1}(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R); \forall\, i \in I\}$. Then $(\mathcal{M}, \tau, \Sigma, \mu_R)$ is said to be a Quotient Radon measure Manifold.

4. Proposed method to construct different categories of Quotient Radon measure Manifolds

In the following analysis, the first author develops different categories of Quotient Radon measure Manifolds $(\mathcal{M}, \tau, \Sigma, \mu_R)$ based on six different patterns of measurable charts and measurable atlases.

Now using different patterns of locally path connected Quotient Radon measure charts and Quotient Radon measure atlases, different categories of Quotient Radon measure manifolds are generated, where all the six properties like extended compactness, extended Lindelof, extended countably compactness, extended semi-compactness, extended semi-Lindelof and extended semi-countably compactness properties remain invariant under measurable homeomorphism and Radon measure structure - invariant map $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Quotient Radon measure manifolds.

4.1. Compact Quotient Radon measure Manifold

Let $(\mathcal{M}, \tau, \Sigma, \mu_R)$ be a Quotient Radon measure manifold. Let $A^k(\mathcal{M}) \subseteq (\mathcal{M}, \tau, \Sigma, \mu_R)$. For every Borel cover, say, $\{\cup_{i\in I}\mathcal{A}_i\}$ i.e., $A^k(\mathcal{M}) \subseteq \{\cup_{i\in I}\mathcal{A}_i\}$ $A^k(\mathcal{M})$ i.e., $A^k(\mathcal{M}) \subseteq \{\cup_{i\in I}\mathcal{A}_i\}$, if $\exists$ a finite Borel sub cover, say, $\{\cup_{j=1}^{n}(U_{i_j}, \mathcal{A}_{i_j})\}$ for $j \in J$, $J \subset I$, such that $A^k(\mathcal{M}) \subseteq \{\cup_{j=1}^{n}(U_{i_j}, \mathcal{A}_{i_j})\}$, then $(\mathcal{M}, \tau, \Sigma, \mu_R)$ such that $A^k(\mathcal{M}) \subseteq (\mathcal{M}, \tau, \Sigma, \mu_R)$ satisfying the **Radon measure conditions (1.2.5) and (1.2.6)** is called a **Compact Quotient Radon measure manifold**.

Note 4.1.1 Let $A = \{(U_i, \phi_i) : \forall\, i \in I$: $(U_i, \phi_i) \subset \mathcal{A}_i \subset (M, \tau, \Sigma, \mu_R)$ are measurable compact Quotient charts$\}$ where $\mu_R(A) > 0$. If $\mu_R(A) = 0$ then A represents the **dark region** of $A^k(M) \subseteq (\mathcal{M}, \tau, \Sigma, \mu_R)$.

According to theorem 2.1.7, it has been shown that extended Heine Borel property i.e., compactness property is invariant under measurable homeomorphism and Radon measure structure - invariant map $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Radon measure manifolds.

Also, according to theorems 3.1.3, 3.2.2 and 3.3.3, it has been shown that local, internal and maximal path connectedness properties remain invariant under measurable homeomorphism and Radon measure structure - invariant map $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Quotient Radon measure manifolds. Then the Radon measure charts and Radon measure atlases becomes Quotient Radon measure charts and Quotient Radon measure atlases and since (M, τ, Σ, μ_R) is covered by Quotient Radon measure atlases $\{\cup_{i\in I}\mathcal{A}_i\}$ i.e., $(M, \tau, \Sigma, \mu_R) \subseteq \{\cup_{i\in I}\mathcal{A}_i\}$, then (M, τ, Σ, μ_R) under the measurable C^∞ canonical surjective projection map
$q : (M, \tau, \Sigma, \mu_R) \longrightarrow (M, \tau, \Sigma, \mu_R)/_\sim$ such that for all $(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R)/_\sim$, $q^{-1}(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R)$ where the quotient topology is defined as:
$\tau_{\mathcal{M}} = \{(U_i, \phi_i) \in (\mathcal{M}, \tau, \Sigma, \mu_R) : q^{-1}(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R); \forall\, i \in I\}$. Hence

they generate Quotient Radon measure manifold $(\mathcal{M}, \tau, \Sigma, \mu_R)$.

Now, we prove that extended compactness property remains invariant under measurable homeomorphism and Radon measure structure - invariant map $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Quotient Radon measure manifolds.

Theorem 4.1.2 Let $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Quotient Radon measure manifolds. Then if extended compactness property holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then the property also holds $\mu_{R_2} - a.e.$, on $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ under measurable homeomorphism and Radon measure structure - invariant map $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2,$
$\mu_{R_2})$.

Proof. Let $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Quotient Radon measure manifolds and $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and Radon measure structure - invariant map.
We show that if extended compactness property say P_1 holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_1 also holds $\mu_{R_2} - a.e.$, on $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$.
Since S. C. P. Halakatti has proved that the extended Heine Borel property holds $\mu_{R_3} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then it also holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ under measurable homeomorphism and Radon measure structure - invariant map $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Radon measure manifolds.
That is, if for any two Radon measure atlases $\mathcal{A}_1, \mathcal{A}_2 \in A^k(\mathcal{M}_1) \subset (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$
$\exists\, F(\mathcal{A}_1), F(\mathcal{A}_2) \in \mathcal{A}^k(\mathcal{M}_2) \subset (\mathcal{M}_2, \tau_1, \Sigma_1, \mu_{R_2})$:
$\mathcal{A}_1 \sim \mathcal{A}_2$ implies $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ with Radon measure structure condition:
$\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ implies $\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2))$ according to definition 2.1.1. (1)
According to theorems 3.1.3, 3.2.2 and 3.3.3, local, internal and maximal path connectedness properties remain invariant under measurable homeomorphism and Radon measure structure - invariant map $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Radon measure manifolds. Then the Radon measure charts and Radon measure atlases becomes Quotient Radon measure charts and Quotient Radon measure atlases and since (M, τ, Σ, μ_R) is covered by Quotient Radon measure atlases $\{\cup_{i \in I} \mathcal{A}_i\}$ i.e., $(M, \tau, \Sigma, \mu_R) \subseteq \{\cup_{i \in I} \mathcal{A}_i\}$, then (M, τ, Σ, μ_R) under the measurable C^∞ canonical surjective projection map $q : (M, \tau, \Sigma, \mu_R) \longrightarrow (M, \tau, \Sigma, \mu_R)/\sim$ such that for all $(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R)/\sim, q^{-1}(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R)$ where the quotient topology is defined as :
$\tau_{\mathcal{M}} = \{(U_i, \phi_i) \in (\mathcal{M}, \tau, \Sigma, \mu_R) : q^{-1}(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R); \forall\, i \in I\}$. Hence they generate Quotient Radon measure manifold $(\mathcal{M}, \tau, \Sigma, \mu_R)$.
Therefore, if for any two Quotient Radon measure atlases $\mathcal{A}_1, \mathcal{A}_2 \in A^k(\mathcal{M}_1) \subset (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ $\exists\, F(\mathcal{A}_1), F(\mathcal{A}_2) \in \mathcal{A}^k(\mathcal{M}_2) \subset (\mathcal{M}_2, \tau_1, \Sigma_1, \mu_{R_2})$:
$\mathcal{A}_1 \sim \mathcal{A}_2$ implies $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ with Radon measure structure condition:
$\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ implies $\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2))$ according to definition 2.1.1.

(2).
This implies, since local, internal and maximal path connectedness properties holds $\mu_{R_3} - a.e.$, on $\mathcal{A}_1, \mathcal{A}_2 \subset (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then this property also holds $\mu_{R_2} - a.e.$, on $F(\mathcal{A}_1)$, $F(\mathcal{A}_2) \subset (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Quotient Radon measure manifolds.
Therefore, from (1) and (2), if extended compactness property holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then it also holds $\mu_{R_2} - a.e.$, on $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ under measurable homeomorphism and Radon measure structure - invariant map where $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Quotient Radon measure manifolds. ■

Theorem 4.1.3 Let $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$, $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ be Quotient Radon measure manifolds. Let $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $G : (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ be C^∞ measurable homeomorphisms and Radon measure structure - invariant maps. Then if extended compactness property holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}$ then the property also holds $\mu_{R_3} - a.e.$, on $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ under the composition mapping $G \circ F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$.

Proof. Let $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R1})$, $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R2})$ and $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R3})$ be Quotient Radon measure manifolds.
We show that if extended compactness property, say, P_1 holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_1 also holds $\mu_{R_3} - a.e.$, on $(\mathcal{M}_3, \tau_2, \Sigma_2, \mu_{R_2})$.
From theorem 4.1.2, if extended compactness property say P_{10} holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1,$
$\Sigma_1, \mu_{R_1})$ then P_1 also holds $\mu_{R_2} - a.e.$, on $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R2})$:
That is, if for any two Quotient Radon measure atlases $\mathcal{A}_1, \mathcal{A}_2 \in A^k(\mathcal{M}_1) \subset (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ $\exists\ F(\mathcal{A}_1), F(\mathcal{A}_2) \in \mathcal{A}^k(\mathcal{M}_2) \subset (\mathcal{M}_2, \tau_1, \Sigma_1, \mu_{R_2})$:
$\mathcal{A}_1 \sim \mathcal{A}_2$ implies $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ with Radon measure structure condition:
$\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ implies $\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2))$ according to definition 2.1.1. (1)
Similarly, since $G : (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ is C^∞ measurable homeomorphisms and Radon measure structure - invariant map, if for any two Quotient Radon measure atlases $F(\mathcal{A}_2) \subset (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ $\exists\ (G \circ F)(\mathcal{A}_1)$, $(G \circ F)(\mathcal{A}_2) \subset (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ with Radon measure structure condition
$\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2))$ implies $\mu_{R_3}((G \circ F)(\mathcal{A}_1)) = \mu_{R_3}((G \circ F)(\mathcal{A}_2))$ (2)
This implies under $G \circ F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R3})$, if P_1 holds $\mu_{R_1} - a.e.$, on $\mathcal{A}_1, \mathcal{A}_2 \subset (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R1})$ then P_1 also holds $\mu_{R_3} - a.e.$, on $(G \circ F)(\mathcal{A}_1)$, $(G \circ F)(\mathcal{A}_2) \subset (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$:
$\mathcal{A}_1 \sim \mathcal{A}_2$ implies $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ with Radon measure structure condition
$\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ implies $\mu_{R_3}(G \circ F(\mathcal{A}_1)) = \mu_{R_3}(G \circ F(\mathcal{A}_2))$ (3)
according to definition 2.1.1. Therefore, from (1), (2) and (3), if extended compactness property holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then it also holds $\mu_{R_3} - a.e.$, on $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R3})$ under measurable homeomorphism and Radon measure structure - invariant map $G \circ F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ where $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$,

$(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ are Quotient Radon measure manifolds. ■

Remark 4.1.4 By using above method, one can show that the extended compactness property remains invariant $\mu_R - a.e.$, on the non-empty set of Quotient Radon measure manifolds $\mathcal{M}_{R_1}, \mathcal{M}_{R_2}, ..., \mathcal{M}_{R_n}$ under the composition of functions $F_1 : \mathcal{M}_{R_1} \longrightarrow \mathcal{M}_{R_2}, F_2 : \mathcal{M}_{R_2} \longrightarrow \mathcal{M}_{R_3}, ..., F_n : \mathcal{M}_{n-1} \longrightarrow \mathcal{M}_{R_n}$ and by using this, one can generate a new **category of compact Quotient Radon measure manifold** $(\Pi, (G, \circ))$ where $\Pi = \{M_1 \times M_2 \times ... \times M_n\}$ which is closed under the group action $(G, \circ) = \{F_1, F_2, ..., F_n\}$ of C^∞ measurable homeomorphisms and Radon measure structure-invariant maps.

4.2. Lindelof Quotient Radon measure Manifold

Let $(\mathcal{M}, \tau, \Sigma, \mu_R)$ be a Quotient Radon measure manifold. Let $A^k(\mathcal{M}) \subseteq (\mathcal{M}, \tau, \Sigma, \mu_R)$. For every Borel cover, say, $\{\cup_{i\in I}\mathcal{A}_i\}$ i.e., $A^k(\mathcal{M}) \subseteq \{\cup_{i\in I}\mathcal{A}_i\}$, if $\exists$ a countable Borel sub cover, say, $\{\cup_{j=1}^{\infty}(\mathcal{A}_{i_j}\}$ such that $A^k(\mathcal{M}) \subseteq \{\cup_{j=1}^{\infty}\mathcal{A}_{i_j}\}$ for $j \in J$, $J \subset I$, then $(\mathcal{M}, \tau, \Sigma, \mu_R)$ such that $A^k(\mathcal{M}) \subseteq (\mathcal{M}, \tau, \Sigma, \mu_R)$ satisfying the **Radon measure conditions (1.2.5) and (1.2.6)** is called a **Lindelof Quotient Radon measure manifold**.

Note 4.2.1 Let $A = \{(U_i, \phi_i) : \forall\, i \in I{:}(U_i, \phi_i) \subset A_i \subset (M, \tau, \Sigma, \mu_R)$ are measurable Lindelof Quotient charts$\}$ where $\mu_R(A) > 0$. If $\mu_R(A) = 0$ then A represents the **dark region** of $A^k(M) \subseteq (\mathcal{M}, \tau, \Sigma, \mu_R)$.

According to theorem 2.2.5, it has been shown that extended Lindelof property is invariant under measurable homeomorphism and Radon measure structure - invariant map $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Radon measure manifolds.

Now, we prove that extended Lindelof property remains invariant measurable homeomorphism and Radon measure structure - invariant map $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Quotient Radon measure manifolds.

Theorem 4.2.2 Let $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Quotient Radon measure manifolds. Then if extended Lindelof property holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then the property also holds $\mu_{R_2} - a.e.$, on $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ under measurable homeomorphism and Radon measure structure - invariant map $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Proof. Let $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Quotient Radon measure manifolds and $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and Radon measure structure - invariant map.
We show that if extended Lindelof property say P_2 holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_2 also holds $\mu_{R_2} - a.e.$, on $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$.
Since S. C. P. Halakatti has proved that the extended Lindelof property holds $\mu_{R_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then it also holds $\mu_{R_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ un-

der measurable homeomorphism and Radon measure structure - invariant map F : $(M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$.
That is, if for any two Lindelof Radon measure atlases $\mathcal{A}_1, \mathcal{A}_2 \in A^k(\mathcal{M}_1) \subset (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ $\exists\ F(\mathcal{A}_1), F(\mathcal{A}_2) \in \mathcal{A}^k(\mathcal{M}_2) \subset (\mathcal{M}_2, \tau_1, \Sigma_1, \mu_{R_2})$:
$\mathcal{A}_1 \sim \mathcal{A}_2$ implies $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ with Radon measure structure condition:
$\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ implies $\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2))$ according to definition 2.1.1. (1)
According to theorems 3.1.3, 3.2.2 and 3.3.3, local, internal and maximal path connectedness properties remain invariant under measurable homeomorphism and Radon measure structure - invariant map $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Radon measure manifolds. Then the Radon measure charts and Radon measure atlases becomes Quotient Radon measure charts and Quotient Radon measure atlases and since (M, τ, Σ, μ_R) is covered by Quotient Radon measure atlases $\{\cup_{i\in I}\mathcal{A}_i\}$ i.e., $(M, \tau, \Sigma, \mu_R) \subseteq \{\cup_{i\in I}\mathcal{A}_i\}$, then (M, τ, Σ, μ_R) under the measurable C^∞ canonical surjective projection map $q : (M, \tau, \Sigma, \mu_R) \longrightarrow (M, \tau, \Sigma, \mu_R)/\sim$ such that for all $(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R)/\sim, q^{-1}(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R)$ where the quotient topology is defined as :
$\tau_{\mathcal{M}} = \{(U_i, \phi_i) \in (\mathcal{M}, \tau, \Sigma, \mu_R) : q^{-1}(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R);\ \forall\ i \in I\}$. Hence they generate Quotient Radon measure manifold $(\mathcal{M}, \tau, \Sigma, \mu_R)$.
Therefore, if for any two Quotient Radon measure atlases $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{A}^k(\mathcal{M}_1) \subset (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})\ \exists\ F(\mathcal{A}_1), F(\mathcal{A}_2) \in \mathcal{A}^k(\mathcal{M}_2) \subset (\mathcal{M}_2, \tau_1, \Sigma_1, \mu_{R_2})$:
$\mathcal{A}_1 \sim \mathcal{A}_2$ implies $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ with Radon measure structure condition:
$\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ implies $\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2))$ according to definition 2.1.1. (2)
This implies, since local, internal and maximal path connectedness properties holds μ_{R_1} on $\mathcal{A}_1, \mathcal{A}_2$ then these properties also holds μ_{R_2} on Quotient Radon measure manifolds. Therefore, from (1) and (2), if extended Lindelof property holds $\mu_{R_1}-a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then it also holds $\mu_{R_2} - a.e.$, on $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ under measurable homeomorphism and Radon measure structure - invariant map where $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Quotient Radon measure manifolds. ■

Theorem 4.2.3 Let $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$, $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ be Quotient Radon measure manifolds. Let $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $G : (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ be C^∞ measurable homeomorphisms and Radon measure structure - invariant maps. Then if extended Lindelof property holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then the property also holds $\mu_{R_3} - a.e.$, on $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ under the composition mapping $G \circ F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$.

Proof. Let $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R1})$, $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R2})$ and $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R3})$ be Quotient Radon measure manifolds.
We show that if extended Lindelof property say P_2 holds $\mu_{R_1}-a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_2 also holds $\mu_{R_3} - a.e.$, on $(\mathcal{M}_3, \tau_2, \Sigma_2, \mu_{R_2})$.
From theorem 4.1.2, if extended compactness property say P_2 holds $\mu_{R_1} - a.e.$, on

$(\mathcal{M}_1, \tau_1, \Sigma_1,$
$\mu_{R1})$ then P_2 also holds $\mu_{R_2} - a.e.$, on $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R2})$:
That is, if for any two Quotient Radon measure atlases $\mathcal{A}_1, \mathcal{A}_2 \in A^k(\mathcal{M}_1) \subset (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$
$\exists\ F(\mathcal{A}_1), F(\mathcal{A}_2) \in A^k(\mathcal{M}_2) \subset (\mathcal{M}_2, \tau_1, \Sigma_1, \mu_{R_2})$:
$\mathcal{A}_1 \sim \mathcal{A}_2$ implies $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ with Radon measure structure condition:
$\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ implies $\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2))$ according to definition 2.1.1. (1)
Similarly, since $G : (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ is C^∞ measurable homeomorphisms and Radon measure structure - invariant map, if for ant two Quotient Radon measure atlases $F(\mathcal{A}_2) \subset (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})\ \exists\ (G \circ F)(\mathcal{A}_1), (G \circ F)(\mathcal{A}_2) \subset (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ with Radon measure structure condition:
$\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2))$ implies $\mu_{R_3}((G \circ F)(\mathcal{A}_1)) = \mu_{R_3}((G \circ F)(\mathcal{A}_2))$ (2)
This implies under $G \circ F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$, if P_2 holds $\mu_{R_1} - a.e.$, on $\mathcal{A}_1, \mathcal{A}_2 \subset (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_2 also holds $\mu_{R_3} - a.e.$, on $(G \circ F)(\mathcal{A}_1)$, $(G \circ F)(\mathcal{A}_2) \subset (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$:
$\mathcal{A}_1 \sim \mathcal{A}_2$ implies $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ with Radon measure structure condition
$\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ implies $\mu_{R_3}(G \circ F(\mathcal{A}_1)) = \mu_{R_3}(G \circ F(\mathcal{A}_2))$ (3)
according to definition 2.1.1. Therefore, from (1), (2) and (3), if extended Lindelof property holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then it also holds $\mu_{R_3} - a.e.$, on $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ under measurable homeomorphism and Radon measure structure - invariant map $G \circ F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ where $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$, $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R2})$ and $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ are Quotient Radon measure manifolds. ■

Remark 4.2.4 By using above method, one can show that the extended Lindelof property remains invariant $\mu_R - a.e.$, on the non-empty set of metrizable Quotient Radon measure manifolds $\mathcal{M}_{R_1}, \mathcal{M}_{R_2}, ..., \mathcal{M}_{R_n}$ under the composition of functions $F_1 : \mathcal{M}_{R_1} \longrightarrow \mathcal{M}_{R_2}$, $F_2 : \mathcal{M}_{R_2} \longrightarrow \mathcal{M}_{R_3}, ..., F_n : \mathcal{M}_{R_{n-1}} \longrightarrow \mathcal{M}_{R_n}$ and by using this, one can generate a new **category of Lindelof Quotient Radon measure manifold** $(\Pi, (G, \circ))$ where $\Pi = \{M_1 \times M_2 \times ... \times M_n\}$ which is closed under the group action $(G, \circ) = \{F_1, F_2, ..., F_n\}$ of C^∞ measurable homeomorphisms and Radon measure structure-invariant maps.

4.3. Countably compact Quotient Radon measure Manifold

Let $(\mathcal{M}, \tau, \Sigma, \mu_R)$ be a Quotient Radon measure manifold. Let $\mathcal{A}^k(\mathcal{M}) \subseteq (\mathcal{M}, \tau, \Sigma, \mu_R)$. For every countable Borel cover, say, $\{\cup_{i \in I} \mathcal{A}_i\}$ i.e., $A^k(\mathcal{M}) \subseteq \{\cup_{i \in I} \mathcal{A}_i\}\ A^k(\mathcal{M})$ i.e., if $\exists$ a finite Borel sub cover, say, $\{\cup_{j=1}^{n} \mathcal{A}_{i_j}\}$ such that $A^k(\mathcal{M}) \subseteq \{\cup_{j=1}^{n} \mathcal{A}_{i_j}\}$ for $j \in J$, $J \subset I$, then $(\mathcal{M}, \tau, \Sigma, \mu_R)$ such that $A^k(\mathcal{M}) \subseteq (\mathcal{M}, \tau, \Sigma, \mu_R)$ satisfying the textbfRadon measure conditions (1.2.5) and (1.2.6) is called a **Countably compact Quotient Radon measure manifold**.

Note 4.3.1 Let $A = \{(U_i, \phi_i) : \forall\ i \in I$: $(U_i, \phi_i) \subset \mathcal{A}_i \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ are measurable countably compact Quotient charts$\}$ where $\mu_R(A) > 0$. If $\mu_R(A) = 0$ then A represents the **dark region** of $A^k(M) \subseteq (\mathcal{M}, \tau, \Sigma, \mu_R)$.

According to theorem 2.3.5, it has been shown that extended countable compactness property is invariant under measurable homeomorphism and Radon measure structure - invariant map $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Radon measure manifolds.

Now, we prove that extended countable compactness property remains invariant under measurable homeomorphism and Radon measure structure - invariant map $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Quotient Radon measure manifolds.

Theorem 4.3.2 Let $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Quotient Radon measure manifolds. Then if extended countably compactness property holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1,$
$\mu_{R_1})$ then the property also holds $\mu_{R_2} - a.e.$, on $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ under measurable homeomorphism and Radon measure structure - invariant map $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow$
$(\mathcal{M}_2, \tau_2,$
$\Sigma_2, \mu_{R_2})$.

Proof. Proof is similar to theorem 4.2.2. ■

Theorem 4.3.3 Let $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$, $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ be Quotient Radon measure manifolds. Let $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $G : (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ be C^∞ measurable homeomorphisms and Radon measure structure - invariant maps. Then if extended countable compactness property holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then the property also holds $\mu_{R_3} - a.e.$, on $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ under the composition mapping $G \circ F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$.

Proof. Proof is similar to theorem 4.2.3. ■

Remark 4.3.4 By using above method, one can show that the countable compactness property remains invariant $\mu_R - a.e.$, on the non-empty set of Quotient Radon measure manifolds $\mathcal{M}_{R_1}, \mathcal{M}_{R_2}, ..., \mathcal{M}_{R_n}$ under the composition of functions $F_1 : \mathcal{M}_{R_1} \longrightarrow \mathcal{M}_{R_2}, F_2 : \mathcal{M}_{R_2} \longrightarrow \mathcal{M}_{R_3}, ..., F_n : \mathcal{M}_{n-1} \longrightarrow \mathcal{M}_{R_n}$ and by using this, one can generate a new **category of countably compact Quotient Radon measure manifold** $(\Pi, (G, \circ))$ where $\Pi = \{M_1 \times M_2 \times ... \times M_n\}$ which is closed under the group action $(G, \circ) = \{F_1, F_2, ..., F_n\}$ of C^∞ measurable homeomorphisms and Radon measure-invariant maps.

Definition 4.3.5 Measurable connected semi-open chart: Let $(\mathcal{M}, \tau, \Sigma, \mu_R)$ be a Quotient Radon measure manifold. A connected measurable chart $(U_i, \phi_i) \in (\mathcal{M}, \tau, \Sigma, \mu_R)$ is said to be semi-open if $(U_i, \phi_i) \subseteq cl(int((U_i, \phi_i))$
$\Rightarrow \mu(U_i, \phi_i) \leq \mu(cl(int((U_i, \phi_i)))$.

4.4. Semi-Compact Quotient Radon measure Manifold

Let $(\mathcal{M}, \tau, \Sigma, \mu_R)$ be a Quotient Radon measure manifold. Let $A^k(\mathcal{M}) \subseteq (\mathcal{M}, \tau, \Sigma, \mu_R)$. For every Borel semi-open cover, say, $\{\cup_{i\in I}\mathcal{A}_i\}$ i.e., $A^k(\mathcal{M}) \subseteq \{\cup_{i\in I}\mathcal{A}_i\}$, if $\exists$ a finite Borel semi-open sub cover, say, $\{\cup_{j=1}^{n}\mathcal{A}_{i_j}\}$ for $j \in J$, $J \subset I$, such that $A^k(\mathcal{M}) \subseteq \{\cup_{j=1}^{n}\mathcal{A}_{i_j}\}$, then $(\mathcal{M}, \tau, \Sigma, \mu_R)$ such that $A^k(\mathcal{M}) \subseteq (\mathcal{M}, \tau, \Sigma, \mu_R)$ satisfying the **Radon measure conditions (1.2.5) and (1.2.6)** is called a **Semi-compact Quotient Radon measure manifold**.

Note 4.4.1 Let $A = \{(U_i, \phi_i)$: $\forall\, i \in I$: $(U_i, \phi_i) \subset \mathcal{A}_i \subset (M, \tau, \Sigma, \mu_R)$ are measurable semi-compact Quotient charts$\}$ where $\mu_R(A) > 0$. If $\mu_R(A) = 0$ then A represents the **dark region** of $A^k(M) \subseteq (\mathcal{M}, \tau, \Sigma, \mu_R)$.

According to theorem 2.4.5, it has been shown that extended semi-compactness property is invariant under measurable homeomorphism and Radon measure structure - invariant map $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Radon measure manifolds.

Now, we prove that extended semi-compactness property which remains invariant under C^∞ measurable homeomorphism and Radon measure structure - invariant map $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Quotient Radon measure manifolds.

Theorem 4.4.2 Let $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Quotient Radon measure manifolds. Then if extended semi-compactness property holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then the property also holds $\mu_{R_2} - a.e.$, on $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ under measurable homeomorphism and Radon measure structure - invariant map
$F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Proof. Proof is similar to theorem 4.2.2. ■

Theorem 4.4.3 Let $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$, $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ be Quotient Radon measure manifolds. Let $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $G : (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ be C^∞ measurable homeomorphisms and Radon measure structure - invariant maps. Then if extended semi-compactness property holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then the property also holds $\mu_{R_3} - a.e.$, on $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ under the composition mapping $G \circ F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$.

Proof. Proof is similar to theorem 4.2.3. ■

Remark 4.4.4 By using above method, one can show that the extended semi-compact property remains invariant $\mu_R - a.e.$, on the non-empty set of metrizable Quotient Radon measure manifold $\mathcal{M}_{R_1}, \mathcal{M}_{R_2}, ..., \mathcal{M}_{R_n}$ under the composition of functions $F_1 : \mathcal{M}_{R_1} \longrightarrow \mathcal{M}_{R_2}$, $F_2 : \mathcal{M}_{R_2} \longrightarrow \mathcal{M}_{R_3}, ..., F_n : \mathcal{M}_{R_{n-1}} \longrightarrow \mathcal{M}_{R_n}$ and by using this, one can generate a new **category of semi-compact Quotient Radon measure manifold**

$(\Pi, (G, \circ))$ where $\Pi = \{M_1 \times M_2 \times ... \times M_n\}$ which is closed under the group action $(G, \circ) = \{F_1, F_2, ..., F_n\}$ of C^∞ measurable homeomorphisms and Radon measure structure-invariant maps.

4.5. Semi-Lindelof Quotient Radon measure Manifold

Let $(\mathcal{M}, \tau, \Sigma, \mu_R)$ be a Quotient Radon measure manifold. Let $A^k(\mathcal{M}) \subseteq (\mathcal{M}, \tau, \Sigma, \mu_R)$. For every Borel semi-open cover, say, $\{\cup_{i \in I} \mathcal{A}_i\}$ i.e., $A^k(\mathcal{M}) \subseteq \{\cup_{i \in I} \mathcal{A}_i\}$, if $\exists$ a countable Borel semi-open sub cover, say, $\{\cup_{j=1}^{\infty} \mathcal{A}_{i_j}\}$ such that $A^k(\mathcal{M}) \subseteq \{\cup_{j=1}^{\infty} \mathcal{A}_{i_j}\}$ for $j \in J$, $J \subset I$, then $(\mathcal{M}, \tau, \Sigma, \mu_R)$ such that $A^k(\mathcal{M}) \subseteq (\mathcal{M}, \tau, \Sigma, \mu_R)$ satisfying the **Radon measure conditions (1.2.5) and (1.2.6)** is called a **Semi-Lindelof Quotient Radon measure manifold**.

Note 4.5.1 Let $A = \{(U_i, \phi_i): \forall\ i \in I: (U_i, \phi_i) \subset \mathcal{A}_i \subset (M, \tau, \Sigma, \mu_R)$ are measurable semi-Lindelof Quotient charts$\}$ where $\mu_R(A) > 0$. If $\mu_R(A) = 0$ then A represents the **dark region** of $(\mathcal{M}, \tau, \Sigma, \mu_R)$.

According to theorem 2.5.6, it has been shown that extended semi-Lindelof property is invariant under measurable homeomorphism and Radon measure structure - invariant map $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Radon measure manifolds.

Now, we prove that extended semi-Lindelof property remains invariant under measurable homeomorphism and Radon measure structure - invariant map $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Quotient Radon measure manifolds.

Theorem 4.5.2 Let $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Quotient Radon measure manifolds. Then if extended semi-Lindelof property holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then the property also holds $\mu_{R_2} - a.e.$, on $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ under measurable homeomorphism and Radon measure structure - invariant map $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Proof. Proof is similar to theorem 4.2.2. ■

Theorem 4.5.3 Let $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$, $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ be Quotient Radon measure manifolds. Let $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $G : (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ be C^∞ measurable homeomorphisms and Radon measure structure - invariant maps. Then if extended semi-Lindelof property holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then the property also holds $\mu_{R_3} - a.e.$, on $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ under the composition mapping $G \circ F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$.

Proof. Proof is similar to theorem 4.2.3. ■

Remark 4.5.4 By using above method, one can show that the semi-Lindelof property remains invariant $\mu_R - a.e.$, on the non-empty set of metrizable Quotient Radon measure manifold $\mathcal{M}_{R_1}, \mathcal{M}_{R_2}, ..., \mathcal{M}_{R_n}$ under the composition of functions $F_1 : \mathcal{M}_{R_1} \longrightarrow \mathcal{M}_{R_2}, F_2 : \mathcal{M}_{R_2} \longrightarrow \mathcal{M}_{R_3}, ..., F_n : \mathcal{M}_{n-1} \longrightarrow \mathcal{M}_{R_n}$ and by using this one can generate a new **category of Semi-Lindelof Quotient Radon measure manifolds** under the group action $G = \{F_1, ..., F_n\}$ of C^∞ measurable homeomorphisms and Radon measure structure - invariant maps.

4.6. Semi-countably compact Quotient Radon measure Manifold

Let $(\mathcal{M}, \tau, \Sigma, \mu_R)$ be a Quotient Radon measure manifold. Let $A^k(\mathcal{M}) \subseteq (\mathcal{M}, \tau, \Sigma, \mu_R)$. For every countable Borel semi-open cover, say, $\{\cup_{i\in I}\mathcal{A}_i\}$ i.e., $A^k(\mathcal{M}) \subseteq \{\cup_{i\in I}\mathcal{A}_i\}$, if $\exists$ a finite Borel semi-open sub cover, say, $\{\cup_{j=1}^n \mathcal{A}_{i_j}\}$ for $j \in J$, $J \subset I$ such that $A^k(\mathcal{M}) \subseteq \{\cup_{j=1}^n \mathcal{A}_{i_j}\}$, then $(\mathcal{M}, \tau, \Sigma, \mu_R)$ such that $A^k(\mathcal{M}) \subseteq (\mathcal{M}, \tau, \Sigma, \mu_R)$ satisfying the **Radon measure conditions (1.2.5) and (1.2.6)** is called a **Semi-countably compact Quotient Radon measure manifold**.

Note 4.6.1 Let $A = \{(U_i, \phi_i) : \forall\ i \in I : (U_i, \phi_i) \subset \mathcal{A}_i \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ are measurable semi-countably compact Quotient charts$\}$ where $\mu_R(A) > 0$. If $\mu_R(A) = 0$ then A represents the **dark region** of $A^k(M) \subseteq (\mathcal{M}, \tau, \Sigma, \mu_R)$.

According to theorem 2.6.5, it has been shown that extended semi-countable compactness property is invariant under measurable homeomorphism and Radon measure structure - invariant map $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Radon measure manifolds.

Now, we prove that extended semi-countable compactness property remains invariant under measurable homeomorphism and Radon measure structure - invariant map $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Quotient Radon measure manifolds.

Theorem 4.6.2 Let $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Quotient Radon measure manifolds. Then if extended semi-countable compactness property holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then the property also holds $\mu_{R_2} - a.e.$, on $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ under measurable homeomorphism and Radon measure structure - invariant map $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Proof. Proof is similar to theorem 4.2.2. ■

Theorem 4.6.3 Let $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$, $(\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ be Quotient Radon measure manifolds. Let $F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $G : (\mathcal{M}_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ be C^∞ measurable homeomorphisms and Radon measure structure - invariant maps. Then if extended semi-countable compactness property holds $\mu_{R_1} - a.e.$, on $(\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1})$ then the property also holds $\mu_{R_3} - a.e.$, on $(\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$ under the composition mapping $G \circ F : (\mathcal{M}_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathcal{M}_3, \tau_3, \Sigma_3, \mu_{R_3})$.

Proof. Proof is similar to theorem 4.2.3. ■

Remark 4.6.4 By using above method, one can show that the semi-countable compactness property remains invariant $\mu_R - a.e.$, on the non-empty set of metrizable Quotient Radon measure manifold $\mathcal{M}_{R_1}, \mathcal{M}_{R_2}, ..., \mathcal{M}_{R_n}$ under the composition of functions $F_1 : \mathcal{M}_{R_1} \longrightarrow \mathcal{M}_{R_2}, F_2 : \mathcal{M}_{R_2} \longrightarrow \mathcal{M}_{R_3}, ..., F_n : \mathcal{M}_{n-1} \longrightarrow \mathcal{M}_{R_n}$ and by using this, one can generate a new **category of semi-countably compact Quotient Radon measure manifold** $(\Pi, (G, \circ))$ where $\Pi = \{M_1 \times M_2 \times ... \times M_n\}$ which is closed under the group action $(G, \circ) = \{F_1, F_2, ..., F_n\}$ of C^∞ measurable homeomorphisms and Radon measure structure-invariant maps.

In the following we examplify the compact Quotient Radon measure manifold as follows:

Example 4.6.5 A collection of $k - cells \in (R^k, \tau, \Sigma, \mu_R)$ forms a compact connected Lebesgue measure manifold.

Solution: Let $(R^k, \tau, \Sigma, \mu_R)$ be a compact connected Radon measure manifold, then $[a_k, b_k]$, a compact and connected subset of $(R^k, \tau, \Sigma, \mu_R)$ is a $k-cell$ in $(R^k, \tau, \Sigma, \mu_R)$.
Case (i): For $k = 1$, $[a_1, b_1]$ a compact connected $1-cell$ in $(R^1, \tau, \Sigma, \mu_R)$, is Lebesgue measurable where the Lebesgue measure λ of $[a_1, b_1]$ is $(b_1 - a_1)$ - the length of $[a_1, b_1]$.

Case (ii): For $k = 2$, $[a_2, b_2]$ a connected compact $2-cell$ in $(R^2, \tau, \Sigma, \mu_R)$ is Lebesgue measurable, where the Lebesgue measure λ of $[a_2, b_2]$ is $\lambda([a_1, b_1] \times [a_2, b_2])$,
i.e., $([b_1 - a_1] \times [b_2 - a_2])$ = area of a $2-cell$ in $(R^2, \tau, \Sigma, \mu_R)$ is Lebesgue measurable.

Case (iii): For $k = 3$, $[a_3, b_3]$ a compact connected $3 - cell$ in $(R^3, \tau, \Sigma, \mu_R)$ is Lebesgue measurable, where the Lebesgue measure of $[a_3, b_3]$ in $(R^3, \tau, \Sigma, \mu_R)$ is $\lambda([a_1, b_1] \times [a_2, b_2] \times [a_3, b_3])$ = volume of a 3-cell, *i.e.*, $([b_1 - a_1] \times [b_2 - a_2] \times [b_3 - a_3])$ = volume of a $3 - cell$ in $(R^3, \tau, \Sigma, \mu_R)$ is Lebesgue measurable.

Continuing for $k = 1, 2, ..., n$, we show that any $k - cell$ in $(R^k, \tau, \Sigma, \mu_R)$ is compact and connected Lebesgue measure manifold. Since $(R^k, \tau, \Sigma, \mu_R)$ is maximally path connected and is an equivalence relation that partitions the entire $(R^k, \tau, \Sigma, \mu_R)$ into disjoint $k - cells$ such that $(R^k, \tau, \Sigma, \mu_R)$ is covered by countable number of $k - cells$.
i.e., $(R^k, \tau, \Sigma, \mu_R) \subseteq \{\cup_{k \in I}[a_k, b_k]\}, \forall\, k = 1, 2, ..., n$.
If $(R^k, \tau, \Sigma, \mu_R)$ has a finite Borel sub cover, say, $\{\cup_{j=1}^n [a_{k_j}, b_{k_j}]\}$, *i.e.*, $(R^k, \tau, \Sigma, \mu_R) \subseteq \{\cup_{j=1}^n [a_{k_j}, b_{k_j}]\}$ then $(R^k, \tau, \Sigma, \mu_R)$ is compact where each $[a_{k_j}, b_{k_j}]$ is Lebesgue measurable for all values of k, such that

$$\lambda(\cup_{j=1}^n [a_{k_j}, b_{k_j}]) \leq \sum_{j=1}^{n} \lambda[a_{k_j}, b_{k_j}]. \qquad \text{(finite sub additivity property)}$$

Since $(R^k, \tau, \Sigma, \mu_R)$ is a compact connected Radon measure manifold, Radon measure is approximated to Lebesgue measure.
If (M, τ, Σ, μ) is modeled on $(R^k, \tau, \Sigma, \mu_R)$ by using measurable homeomorphism $\phi : (M, \tau, \Sigma, \mu) \longrightarrow (R^k, \tau, \Sigma, \mu)$ such that $\forall\, k - cells\ [a_k, b_k] \in (R^k, \tau, \Sigma, \mu_R)\ \exists$ $k - cells$ called $k - charts\ \phi^{-1}([a_k, b_k]) \in (M, \tau, \Sigma, \mu)$ which are compact connected charts in (M, τ, Σ, μ) such that

(**i**) $\cup_{k\in I}(\phi^{-1}[a_k, b_k]) = (M, \tau, \Sigma, \mu)$. That is, the countable union of all compact connected charts cover (M, τ, Σ, μ),
(**ii**) for any pair of compact connected charts $\phi^{-1}[a_i, b_i] = (U_i, \phi_i)$ and $\phi^{-1}[a_j, b_j] = (U_j, \phi_j)$, the transition maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are:
(a) differentiable,
(b) Lebesgue measurable if, any compact connected $k - cell$ $[a_k, b_k] \subseteq \phi_i(U_i \cap U_j)$ is Lebesgue measurable in $(R^k, \tau, \Sigma, \mu_R)$, then $(\phi_i \circ \phi_j^{-1})^{-1}([a_k, b_k]) \in \phi_j(U_i \cap U_j)$ is also Lebesgue measurable. That is, $\lambda(\phi^{-1}[a_k, b_k]) > 0$ implies $\lambda(\phi^{-1}(b_k) - \phi^{-1}(a_k)) > 0$ where λ is Lebesgue measure.
(**iii**) for every Borel cover $\{\cup_{k\in I}(\phi^{-1}([a_k, b_k]))\}$ for (M, τ, Σ, μ), *i.e.*, $(M, \tau, \Sigma, \mu) \subseteq \{\cup_{k\in I}(\phi^{-1}([a_k, b_k]))\}$ if there exists a finite Borel sub cover, say, $\{\cup_{j=1}^{n}\phi^{-1}[a_{k_j}, b_{k_j}]\}$ such that $(M, \tau, \Sigma, \mu) \subseteq \{\cup_{j=1}^{n}\phi^{-1}[a_{k_j}, b_{k_j}]\}$, then (M, τ, Σ, μ) is compact connected Lebesgue measure manifold satisfying the finite sub - additivity property:

$\lambda(\cup_{j=1}^{n}\phi^{-1}[a_k, b_k]) \leq \sum_{j=1}^{n}\lambda(\phi^{-1}[a_{k_j}, b_{k_j}])$. (finite sub additivity property). Hence, the countable collection of all compact connected Lebesgue measure $k - cells$ called $k - charts$ $\phi^{-1}[a_k, b_k]$ form a compact connected Lebesgue measure manifold (M, τ, Σ, μ).

5. Micro analysis on Compact Quotient Radon measure Manifold

In this section, the first author develops a micro analysis on one of the categories of Quotient Radon measure manifolds namely **compact Quotient Radon measure manifold**. Maximal path connectedness equivalence relation partitions $A^k(M)$ into compact Quotient Radon measure atlases $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n \in A^k(\mathcal{M})$, internal path connectedness equivalence relation partitions $\mathcal{A}_i \in A^k(\mathcal{M})$ into compact Quotient Radon measurable domain $[(U_i, \phi_i) \cup (U_j, \phi_j)]$ and local path connectedness equivalence relation partitions $(U_i, \phi_i) \in \mathcal{A}_i \in A^k(\mathcal{M})$ into compact Quotient measurable/Borel subsets $S_i \subset (U_i, \phi_i)$ such that $A^k(\mathcal{M}) \subseteq \{\cup_{i\in I}\mathcal{A}_i\}$, $\mathcal{A}_i \subseteq \{\cup_{i\in I}[(U_i, \phi_i) \cup (U_j, \phi_i)]\}$, $(U_i, \phi_i) \subseteq \{\cup_{i\in I}S_i\}$.

We first focus on compact Quotient Radon measure chart $(U_i, \phi_i) \subseteq \{\cup_{i\in I}S_i\}$ where,
$S_i = \{p_i \in N_{\delta_i}(p_i), p_j \in N_{\delta_j}(p_j) \,\exists\, \gamma_i \in G/p_i \sim p_j; \forall\, i, j \in I\}, \mu_R(S_i) > 0.$
for $\delta_i, \delta_j > 0$ are radii of compact Borel neighbourhoods $N_{\delta_i}(p_i)$ and $N_{\delta_j}(p_j)$ respectively: $N_{\delta_i}(p_i) \cap N_{\delta_j}(p_j) = \phi$, where,
$G = \{\gamma_1, \gamma_2, ..., \gamma_i, \gamma_j, \gamma_i^{-1}, \gamma_j^{-1}, \gamma_i \circ \gamma_j, \gamma_j \circ \gamma_i, ...\}$ is a non-empty collection of measurable C^∞ paths.

Further each compact Quotient measurable/Borel subset $S_i \subset (U_i, \phi_i)$ is also locally path connected and local path connectedness is an equivalence relation and hence the equivalence relation partitions S_i further into disjoint equivalence classes of compact Quotient Borel neighbourhoods $N_{\delta_i}(p_i) \subset S_i \subset (U_i, \phi_i)$ such that $S_i \subseteq \{\cup_{i\in I}N_{\delta_i}(p_i)\}$ where,
$N_{\delta_i}(p_i) = \{p_i \in N_{\epsilon_i}(p_i), p_j \in N_{\epsilon_j}(p_j) \,\exists\, \gamma_i \in G/p_i \sim_{\gamma_i} p_j; \forall\, i, j \in I\}, \mu_R(N_{\delta_i}(p_i)) >$

0 for $\epsilon_i, \epsilon_j > 0$ are the smaller radii of Borel neighbourhoods $N_{\epsilon_i}(p_i)$ and $N_{\epsilon_j}(p_j)$ and $N_{\epsilon_i}(p_i) \cap N_{\epsilon_j}(p_j) = \phi$, $\mu_R(N_\epsilon(p)) > 0$.
Continuing the same process until we get the smallest possible Borel neighbourhood namely $N_\epsilon(p)$, $\mu_R(N_\epsilon(p)) > 0$ where, $\epsilon = inf\{\delta_1, \delta_1, ..., \delta_n; \forall n \in I\}$.
Since $N_\epsilon(p)$ is also locally path connected, then there are two possibilities:

(i) If $p_i \neq p_j \in N_\epsilon(p)$ then p_i is locally path connected to p_j by measurable C^∞ path $\gamma_i \in G, \forall\, i, j \in I$,

(ii) If $p_i = p_j \in N_\epsilon(p)$ where $\mu_R(N_\epsilon(p)) > 0$ then p_i is locally path connected to p_i itself by a measurable C^∞ path $\gamma_i \in G; \forall\, i, j \in I$. Such γ_i is called as a **self-connected path** and $N_\epsilon(p)$ is called as **self-connected domain** that does not have partition further. This property is generated by reflexive property of equivalence relation on $N_\epsilon(p) \subset S_i \subset (\mathcal{M}, \tau, \Sigma, \mu_R)$.

This exhibits a new property of compact Quotient Radon Measure Manifold at the most micro level. The **self connected domains** are the fundamental features of micro structures of compact Quotient Radon Measure Manifold. When these fundamental structures are locally, internally and maximally path connected and evolve into micro and macro structures, then the compact Quotient Radon Measure Manifold evolves into a well defined Network Radon Measure Manifold which provides a new explanation for the micro structure of Quantum Field and macro structures of Gravitational Field in terms of fundamental features of Quotient Radon measure manifold. Thus Radon measure manifolds and Quotient Radon measure manifolds provide a unified mathematical framework to describe evolving micro and macro structures of the universe in terms of different types of path connectedness and their corresponding equivalence relations.

Remark 5.1 On the similar line, one can develop a micro analysis on different categories of Quotient Radon measure manifolds like Lindelof Quotient Radon measure manifolds, countably compact Quotient Radon measure manifolds, semi-compact Quotient Radon measure manifolds, semi-Lindelof Quotient Radon measure manifolds and semi-countably compact Quotient Radon measure manifolds.

6. Proposed method to construct Network Radon measure Manifold

In this section we study the existence of two measurable group structures on one of the categories of Quotient Radon measure Manifolds namely **compact Quotient Radon measure manifold** $(\mathcal{M}, \tau, \Sigma, \mu_R)$. Since $(\mathcal{M}, \tau, \Sigma, \mu_R)$ is locally, internally and maximally path connected on three different measurable domains by the set of measurable C^∞ paths $\gamma_i's \in G$ respectively, one can prove that a non-empty collection $G = \{\gamma_1, \gamma_2, ..., \gamma_i, \gamma_j, \gamma_i^{-1}, \gamma_j^{-1}, \gamma_i \circ \gamma_j, \gamma_j \circ \gamma_i, ...\}$ of all locally, internally and maximally path connected measurable C^∞ paths $\gamma_i \in G$ containing identity and inverse forms an abelian group as shown in the following results:

Theorem 6.1 If $(\mathcal{M}, \tau, \Sigma, \mu_R)$ be a compact Quotient Radon measure manifold then a non-empty collection $(G, \circ) = \{\gamma_1, \gamma_2, ..., \gamma_i, \gamma_j, \gamma_i^{-1}, \gamma_j^{-1}, id, \gamma_i \circ \gamma_j, \gamma_j \circ \gamma_i, ...\}$ of all measurable C^∞ paths forms an abelian group on $(\mathcal{M}, \tau, \Sigma, \mu_R)$.

Proof. Let $(\mathcal{M}, \tau, \Sigma, \mu_R)$ be a compact Quotient Radon measure manifold. Let $G = \{\gamma_1, \gamma_2, ...,$
$\gamma_i, \gamma_j, \gamma_i^{-1}, \gamma_j^{-1}, id, \gamma_i \circ \gamma_j, \gamma_j \circ \gamma_i, ...\}$ be a non-empty collection of measurable C^∞ paths on $(\mathcal{M}, \tau_2, \Sigma_2, \mu_R)$ and $P = \{p_i \in (\mathcal{M}, \tau, \Sigma, \mu_R); \forall\, i, j \in I\}$ be the collection of all points on $(\mathcal{M}, \tau, \Sigma, \mu_R)$.

To show that G forms an abelian group under the composition of these paths.

(i) If $\gamma_i, \gamma_j \in G$ then to show that $\gamma_j \circ \gamma_i \in G$. By condition (iii) of theorem 3.2, there exists measurable C^∞ paths
$\gamma_i : [0, 1] \longrightarrow (U_i, \phi_i) \cup (U_j, \phi_j) \in \mathcal{A}_i \cup \mathcal{A}_j \in A^k(\mathcal{M})$ such that
$\gamma_i(0) = p_i \in (U_i, \phi_i) \in \mathcal{A}_i$ for which $\mu_R(U_i) > 0$ and $\mu_R(\mathcal{A}_i) > 0$, and
$\gamma_i(\frac{1}{2}) = p_j \in (U_j, \phi_j) \in \mathcal{A}_j$ for which $\mu_R(U_j) > 0$ and $\mu_R(\mathcal{A}_j) > 0$ and
$\gamma_j : [0, 1] \longrightarrow (U_j, \phi_j) \cup (U_k, \phi_k) \in \mathcal{A}_j \cup \mathcal{A}_k \in A^k(\mathcal{M})$ such that
$\gamma_j(\frac{1}{2}) = p_j \in (U_j, \phi_j) \in \mathcal{A}_j$ for which $\mu_R(U_j) > 0$, $\mu_R(\mathcal{A}_j) > 0$ and
$\gamma_j(1) = p_k \in U_k, \phi_k) \in \mathcal{A}_k$ for which $\mu_R(U_k) > 0$, $\mu_R(\mathcal{A}_k) > 0$.
Then by transitivity relation of theorem 3.1.2, there exists a measurable C^∞ path $\gamma_j \circ \gamma_i : [0, 1] \longrightarrow (U_i, \phi_i) \cup (U_k, \phi_k) \in \mathcal{A}_i \cup \mathcal{A}_k$ from p_i to p_k such that
$\gamma_j \circ \gamma_i(0) = p_i \in (U_i, \phi_i) \in \mathcal{A}_i$ and $\gamma_j \circ \gamma_i(1) = p_k \in (U_i, \phi_i) \in \mathcal{A}_j$.
This shows that $\gamma_j \circ \gamma_i \in G$.
Since by symmetric relation of theorem 3.1.2, $\gamma_j \circ \gamma_i : [0, 1] \longrightarrow (U_k, \phi_k) \cup (U_i, \phi_i) \in \mathcal{A}_k \cup \mathcal{A}_i$ is a path from p_k to p_i such that
$\gamma_j \circ \gamma_i(0) = p_k \in (U_k, \phi_k) \in \mathcal{A}_k$ and $\gamma_j \circ \gamma_i(1) = p_i \in (U_i, \phi_i) \in \mathcal{A}_i$. Hence, $\gamma_i \circ \gamma_j \in G$.

(ii) If $\gamma_i, \gamma_j, \gamma_k \in G$ then to show that $(\gamma_i \circ \gamma_j) \circ \gamma_k = \gamma_i \circ (\gamma_j \circ \gamma_k)$
Let $\gamma_i \circ \gamma_j = \gamma_R$ and $\gamma_j \circ \gamma_k = \gamma_S$
Then $\gamma_R \circ \gamma_k = \gamma_i \circ \gamma_S$ [By theorem 3.1.2]
Therefore, $(\gamma_i \circ \gamma_j) \circ \gamma_k = \gamma_i \circ (\gamma_j \circ \gamma_k)$.

(iii) By definition 1.2.9 [4], for every $\gamma_i \in G$ there exists an inverse of γ_i i.e. $\gamma_i^{-1} : [0, 1] \longrightarrow (U_i, \phi_i)$ defined by $\gamma_i^{-1}(p) = \gamma_i(1 - p)$ and $\gamma_i^{-1}(q) = \gamma_i(1 - q)$ for each $p, q \in [0, 1]$ such that $\gamma_i \circ \gamma_i^{-1} = \gamma_i^{-1} \circ \gamma_i = id$ holds.

(iv) For any $\gamma_i \in G$, $\gamma_i \circ \gamma_i^{-1} = \gamma_i^{-1} \circ \gamma_i = id$ holds, where $id : [0, 1] \longrightarrow [0, 1]$ is the identity map in G. [By theorem 3.1.2]

(v) We know that by (i) $\gamma_j \circ \gamma_i \in G$ and by symmetric relation of theorem 3.1.2, $\gamma_i \circ \gamma_j \in G$

implies $\gamma_j \circ \gamma_i = \gamma_i \circ \gamma_j \in G$. Therefore, $(G, \circ)$ forms an abelian group under the composition of these paths in G on $(\mathcal{M}, \tau, \Sigma, \mu_R)$. ■

Hence, $(\mathcal{M}, \tau, \Sigma, \mu_R)$ admitting the measurable group structure $(G, \circ)$ defines the Network structure on $(\mathcal{M}, \tau, \Sigma, \mu_R)$ and denoted by $((\mathcal{M}, \tau, \Sigma, \mu_R), (G, \circ))$.

Further the first author develops an analysis on $(G, \circ)$ by defining a relation '$\sim$' on G which is an equivalence relation on G and partitions G into disjoint set of equivalence classes.

Definition 6.2 Let $G = \{\gamma_1, \gamma_2, ..., \gamma_i, \gamma_j, \gamma_i^{-1}, \gamma_j^{-1}, id, \gamma_i \circ \gamma_j, \gamma_j \circ \gamma_i, ...\}$ be an abelian group of all measurable C^∞ paths. Now a relation '$\sim$' on G is defined as:
A measurable C^∞ path γ_i is related to a measurable C^∞ path γ_j if $\gamma_j \circ \gamma_i \in G$.

This relation forms an equivalence relation on G as follows:
(i) '$\sim$' is reflexive: Let $\gamma_i, \gamma_j \in G$ and if $\gamma_i \sim \gamma_j$ then $\gamma_i \circ \gamma_j \in G$.
Suppose $i = j$,$\gamma_i \sim \gamma_i$ then $\gamma_i \circ \gamma_i \in G$.
Therefore, '$\sim$' is reflexive.
(ii) '$\sim$' is symmetric: If $\gamma_i \sim \gamma_j$ then $\gamma_j \circ \gamma_i \in G$ and since $(G, \circ)$ is an abelian group $\exists$ an inverse relation $\gamma_j \sim \gamma_i$ such that $\gamma_i \circ \gamma_j \in G$.
Therefore, if $\gamma_i \sim \gamma_j$ then $\gamma_j \circ \gamma_i \in G$.
Therefore, '$\sim$' is symmetric.
(iii) '$\sim$' is transitive: Let $\gamma_i \sim \gamma_j, \gamma_j^{-1} \sim \gamma_k$ and since $(G, \circ)$ is an abelian group $\exists$ associative property such that
$(\gamma_i \circ \gamma_j) \circ (\gamma_j^{-1} \circ \gamma_k)$
$= \gamma_i \circ id_{\mathcal{M}} \circ \gamma_k = \gamma_i \circ \gamma_k \in G$,
then $\gamma_j \sim \gamma_i$. Therefore, '$\sim$' is transitive. Hence, $(G, \circ)$ is an equivalence relation on $(\mathcal{M}, \tau, \Sigma, \mu_R)$.

Note 6.3 This equivalence relation partitions the set G into disjoint equivalence classes denoted by $[G]$ in terms of locally path connected γ_i i.e., $G_{(U_i,\phi_i)} = G \in [G]$, internally path connected γ_i i.e., $G_{(U_i,\phi_i)\cup(U_j,\phi_j)} = G \in [G]$ and maximally path connected γ_i i.e., $G_{\mathcal{A}_i\cup\mathcal{A}_j\cup\mathcal{A}_k} = G \in [G]$.

It is observed that the Network structure $((\mathcal{M}, \tau, \Sigma, \mu_R), (G, \circ))$ admits one more measurable group structure formed by the non-empty collection $\mathcal{G}$ of all measurable homeomorphisms of transition maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ *i.e.*, $\mathcal{G} = \{\phi_i \circ \phi_j^{-1} : 1 \leq i, j \leq n\}$ with identity and inverse transition maps forms a group $(\mathcal{G}, \circ)$ under the composition of transition maps on $((\mathcal{M}, \tau, \Sigma, \mu_R), (G, \circ))$.

A relation '$\sim$' on $\mathcal{G}$ is defined as follows:

Definition 6.4 Let $\mathcal{G} = \{\phi_i \circ \phi_j^{-1} : 1 \leq i, j \leq n\}$ be a group of measurable homeomorphisms. Then the relation '$\sim$' on $\mathcal{G}$ is defined as :
Suppose $\phi_i \circ \phi_j^{-1}, \phi_j \circ \phi_k^{-1} \in (\mathcal{G}, \circ)$ for any $p \in (U_i \cup U_j \cup U_k)$ we say that $\phi_i \circ \phi_j^{-1}$ is related to $\phi_j \circ \phi_k^{-1}$, *i.e.*, $\phi_i \circ \phi_j^{-1} \sim \phi_j \circ \phi_k^{-1}$.

It can be shown that the above relation '$\sim$' forms an equivalence relation on $(\mathcal{G}, \circ)$.

This relation partitions $(\mathcal{G}, \circ)$ into disjoint equivalence classes denoted by $\mathcal{H}(\mathcal{M})$.

Definition 6.5 Compact Network Radon Measure Manifold
A compact Quotient Radon Measure Manifold $(\mathcal{M}, \tau, \Sigma, \mu_R)$ admitting two measurable group structures $(G, \circ)$ and $(\mathcal{G}, \circ))$ defines the compact Network Radon Measure Manifold denoted by $((\mathcal{M}, \tau, \Sigma, \mu_R, (G, \circ), (\mathcal{G}, \circ))$.

The analysis developed indicates that, on one of the categories of Network Radon Measure Manifolds namely **compact Network Radon Measure Manifolds** $((\mathcal{M}, \tau, \Sigma, \mu_R)$ $, (G, \circ), (\mathcal{G}, \circ))$ if there exists an operation $G \times G \longrightarrow G$ such that $(\gamma_i, \gamma_j) \longrightarrow \gamma_i \gamma_j^{-1}$ is continuous and measurable then G forms a **measurable topological group** on $((\mathcal{M}, \tau, \Sigma, \mu_R)$ $, (G, \circ), (\mathcal{G}, \circ))$.

If this measurable map is differentiable of class C^∞ then G forms a **measurable Lie group** and also $(\mathcal{G}, \circ)$ forms a measurable Lie group structure on $((\mathcal{M}, \tau, \Sigma, \mu_R), (G, \circ), (\mathcal{G}, \circ))$ (admitting Haar measure which is a special case of Radon measure) that generates the measurable fibre bundle over $((\mathcal{M}, \tau, \Sigma, \mu_R), (G, \circ), (\mathcal{G}, \circ))$, if it is a Base space B, Total space E and the Fibre F that is closed under a continuous measurable surjective projection map $\Pi : E \longrightarrow B$, and measurable diffeomorphism $\Phi : B \longrightarrow E$ with the Lie group action then the Network Radon measure manifold generates a **measurable fibre bundle** $(E, B, \Pi, F, \Phi, G, \mathcal{G})$ over compact Network Radon Measure Manifold under the two group actions $(G, \circ)$ and $(\mathcal{G}, \circ)$.

Further the results of A. Fathi [15] are extended on our compact Network Radon Measure Manifold $((\mathcal{M}, \tau, \Sigma, \mu_R), (G, \circ), (\mathcal{G}, \circ))$. A. Fathi have made a comprehensive study on topological and algebraic properties of groups of measure - preserving homeomorphisms of compact manifolds of dimension n. In the following we use the analysis developed by A. Fathi on the compact Network Radon Measure Manifold which carries two measurable group structures is considered.

Let μ_R be a Radon measure on the compact Network Measure Manifold $((\mathcal{M}, \tau, \Sigma, \mu_R)$, $(G, \circ), (\mathcal{G}, \circ))$ which is locally positive measure defined on the σ-algebra of all measurable/Borel charts of $((\mathcal{M}, \tau, \Sigma, \mu_R), (G, \circ), (\mathcal{G}, \circ))$. Let $\mathcal{H}(\mathcal{M})$ be the group of all measurable homeomorphisms of transition maps on $((\mathcal{M}, \tau, \Sigma, \mu_R), (G, \circ), (\mathcal{G}, \circ))$ and let $\mathfrak{M}_R(\mathcal{M})$ be the set of all Radon measures on $((\mathcal{M}, \tau, \Sigma, \mu_R), (G, \circ), (\mathcal{G}, \circ))$.

Given a Radon measure μ_R on $((\mathcal{M}, \tau, \Sigma, \mu_R), (G, \circ), (\mathcal{G}, \circ))$ which preserves μ_R: $(\mathcal{H}(\mathcal{M}), \mu_R) = \{\phi_i \circ \phi_j^{-1} = h \in \mathcal{H}(\mathcal{M}) / h_{\star\mu_R} = \mu_R \,; \forall\, i, j \in I\}$ where $h = \phi_i \circ \phi_j^{-1}$: $\phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$ is a measurable homeomorphism from $\phi_j(U_i \cap U_j)$ to $\phi_i(U_i \cap U_j)$, where $(U_i \cap U_j) \in ((\mathcal{M}, \tau, \Sigma, \mu_R, (G, \circ), (\mathcal{G}, \circ))$.

If $\mathfrak{M}_R(\mathcal{M})$ is the set of all Radon measures on $((\mathcal{M}, \tau, \Sigma, \mu_R), (G, \circ), (\mathcal{G}, \circ))$, then there is an action:
$(h, \mu_R) \longrightarrow h_{\star\mu_R}$ where $h_{\star\mu_R}$ is defined by $h_{\star\mu_R}(U_i \cap U_j) = \mu_R(h^{-1}(U_i \cap U_j))$, for each compact and Quotient measurable subset $(U_i \cap U_j) \in ((\mathcal{M}, \tau, \Sigma, \mu_R), (G, \circ), (\mathcal{G}, \circ))$ which is Radon measurable in $((\mathcal{M}, \tau, \Sigma, \mu_R), (G, \circ), (\mathcal{G}, \circ))$. The above map defines an action of the group $\mathcal{H}(\mathcal{M})$ on $\mathfrak{M}_R(\mathcal{M})$.

Remark 6.6 On the similar line, one can construct different categories of Network Radon measure manifolds like Lindelof Network Radon measure manifolds, countably compact Network Radon measure manifolds, semi-compact Network Radon measure manifolds, semi-Lindelof Network Radon measure manifolds and semi-countably compact Network Radon measure manifolds.

7. Conclusion

In this paper, different categories of Radon measure manifolds, Quotient Radon measure manifolds and Network Radon measure manifolds are constructed. These manifolds provide a unified mathematical framework to describe evolving micro and macro structures of the physical universe in terms of different types of path connectedness and their corresponding equivalence relations.

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