Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 9 (2017), pp. 6973–6978

http://www.ripublication.com/gjpam.htm

A general voltera-type integral equation associated with an integral operator with the $\overline{H}$-function in the kernel

P. Harjule, M. K. Bansal[1] and R. Jain

Department of Mathematics,
Malaviya National Institute of Technology,
Jaipur 302017, Rajasthan, India.

Abstract

In this paper, we solve a general Volterra-type fractional equation associated with an integral operator with the $\overline{H}$-function in its Kernel. We make use of convolution technique to solve the main equation. Since $\overline{H}$-function occuring in the fractional operator herein is general in nature we can obtain a number of special cases from our main findings, by reducing $\overline{H}$-function to its many special cases. We record here two such special cases which involve generalized Riemann-Zeta function and polylogarithmic function respectively. Results obtained by Srivastava and Bushman [4, 5], Rashmi Jain [9] follow as special cases of our main findings.

AMS subject classification: 45E10, 33C60, 44A10.
Keywords: Convolution Integral Equation; $\overline{H}$-function; Laplace transform.

1. Introduction

The $\overline{H}$-function occurring in the present work will be defined and represented here in the following manner [3]

$$\overline{H}^{m,n}_{p,q}[z] = \overline{H}^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (a_j, \alpha_j; A_j)_{1,n}, & (a_j, \alpha_j)_{n+1,p} \\ (b_j, \beta_j)_{1,m}, & (b_j, \beta_j; B_j)_{m+1,q} \end{matrix} \right]$$
$$:= \frac{1}{2\pi\omega} \int_{\mathfrak{L}} \overline{\Theta}(\xi) z^{\xi} d\xi \tag{1.1}$$

[1]Corresponding author: manish.math12@gmail.com

where, $\omega = \sqrt{-1}$, $z \in \mathbb{C} \setminus \{0\}$, $\mathbb{C}$ being the set of complex numbers,

$$\overline{\Theta}(\xi) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{n} \left\{\Gamma(1 - a_j + \alpha_j \xi)\right\}^{A_j}}{\prod_{j=m+1}^{q} \left\{\Gamma(1 - b_j + \beta_j \xi)\right\}^{B_j} \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j \xi)} \tag{1.2}$$

The sufficient condition for the absolute convergence of the integral have been established by Bushman and Srivastava [8, p. 4708] The series representation for the $\overline{H}$−Function is as follows:

$$\overline{H}^{m,n}_{p,q}\left[z \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1,n}, & (a_j, \alpha_j)_{n+1,p} \\ (b_j, \beta_j)_{1,m}, & (b_j, \beta_j; B_j)_{m+1,q} \end{matrix} \right. \right] = \sum_{t=0}^{\infty} \sum_{h=1}^{m} \overline{\Theta} \mathfrak{s}_{t,h} z^{\mathfrak{s}_{t,h}} \tag{1.3}$$

where,

$$\overline{\Theta}(\mathfrak{s}_{t,h}) = \frac{\prod_{j=1, j\neq h}^{m} \Gamma(b_j - \beta_j \mathfrak{s}_{t,h}) \prod_{j=1}^{n} \left\{\Gamma(1 - a_j + \alpha_j \mathfrak{s}_{t,h})\right\}^{A_j}}{\prod_{j=m+1}^{q} \left\{\Gamma(1 - b_j + \beta_j \mathfrak{s}_{t,h})\right\}^{B_j} \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j \mathfrak{s}_{t,h})} \tag{1.4}$$

2. An integral operator involving $\overline{H}$-function

In our present investigation, we make use of the following fractional integral operator with $\overline{H}$-function in its kernel

$$\left(\overline{\mathcal{H}}^{1,n;\sigma}_{a+;p,q;\rho}\, \varphi\right)(x) := \int_a^x (x-t)^{\rho-1}\, \overline{H}^{1,n}_{p,q}[(x-t)^{\sigma}]\varphi(t)dt \qquad \Re(\rho) > 0 \tag{2.1}$$

The following property of Laplace transform [1]

$$L(f^{(n)}(x); s) = s^n F(s)$$

holds provided that $f^{(i)}(0) = 0$, $i = 0, 1, 2, \ldots, n-1$, n being a positive integer, where

$$L(f(x); s) = \int_0^x e^{-sx} f(x)dx = F(s)$$

The well-known convolution theorem for Laplace transform

$$L\left(\int_0^x f(x-u)g(u)du; s\right) = L(f(x); s)L(g(x); s) \tag{2.2}$$

holds provided that the various Laplace transforms occuring in (2.2) exists. For $a = 0$, by using the *Convolution Theorem* for the Laplace Transform, we find from the definition (2.1) that

$$\begin{aligned}
&\mathcal{L}\left[\left(\overline{\mathcal{H}}^{1,n;\sigma}_{0+;p,q;\rho}\,\varphi\right)(x)\right](s) \\
&\quad = \mathcal{L}\left[x^{\rho-1}\,\overline{H}^{1,n}_{p,q}[x^{\sigma}]\right](s)\cdot\mathcal{L}[\varphi(x)](s) \\
&\quad = s^{-\rho}\overline{H}^{1,n+1}_{p+1,q}\left[s^{-\sigma}\left|\begin{matrix}(1-\rho,\sigma;1),(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\ \\ (0,1),(b_j,\beta_j;B_j)_{2,q}\end{matrix}\right.\right]\Phi(s)
\end{aligned} \tag{2.3}$$

where

$$\Re(s,\rho,\sigma) > 0$$

3. Main Result

A general Volterra-type integral equation associated with an integral operator with the $\overline{H}$-function in its kernel is given by

$$\left(\overline{\mathcal{H}}^{1,n;\sigma}_{0+;p,q;\rho}\,y\right)(x) + \frac{a}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}y(t)dt := g(x) \tag{3.1}$$

$$\Re(\rho,\sigma,\eta) > 0;\, 0 \le n \le p$$

has the solution

$$y(x) = \int_0^x (x-t)^{l-\sigma k-\rho-1}\sum_{\lambda=0}^{\infty}\frac{C_\lambda (x-t)^{\sigma\lambda}}{\Gamma(l-\sigma k+\sigma\lambda-\rho)}D_t^l\{g(t)\}dt, \tag{3.2}$$

where l is a positive integer such that $Re(l-\sigma k-\rho) > 0$, where k denotes the least ν for which $C_v^{'} \neq 0$, where

$$C_v^{'} = \frac{\Gamma(\rho+\sigma\nu)\prod_{j=1}^{n}\{\Gamma(1-a_j+\alpha_j\nu)\}^{A_j}}{\prod_{j=2}^{q}\{\Gamma(1-b_j+\beta_j\nu)\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\nu)\nu!}(-1)^{\nu} \tag{3.3}$$

g is prescribed such that $g^{(u)}(0) = 0$ for $0 \leq u \leq l-1$
C_λ are given by

$$C_\lambda = (-1)^\lambda (C_k^{'})^{-\lambda-1} \det \begin{bmatrix} C_{k+1}^{'} & C_k^{'} & \cdots & 0 & \cdots & 0 \\ C_{k+2}^{'} & C_{k+1}^{'} & \cdots & \cdots & \cdots & 0 \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \left(C_{k+\frac{\eta-\rho}{\sigma}} + a\right) & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ C_{k+\lambda}^{'} & C_{k+\lambda-1}^{'} & \cdots & \left(C_{k+\frac{\eta-\rho}{\sigma}} + a\right) & \cdots & C_{k+1}^{'} \end{bmatrix} \tag{3.4}$$

Proof. To solve Eq. (3.1), we first take the Laplace transform of its both sides with the help of (2.3), we get

$$s^{-\rho}\overline{H}_{p+1,q}^{1,n+1}\left[s^{-\sigma}\left|\begin{matrix}(1-\rho,\sigma;1),(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\ \\(0,1),(b_j,\beta_j;B_j)_{2,q}\end{matrix}\right.\right]Y(s)+aS^{-\eta}Y(s)=G(s) \tag{3.5}$$

Now, expressing $\overline{H}$-function involved in the above equation in terms of series with the help of (1.3) we have

$$s^{-\rho}\left[\sum_{\nu=0}^{\infty} C_\nu^{'} s^{-\sigma\nu} + as^{-\eta+\rho}\right] Y(s) = G(s) \tag{3.6}$$

where $C_\nu^{'}$ is given by (3.4). The other details of the proof would run parallel to those given already in [7] so we omit them here. ■

4. Special Cases

If we take $\sigma = 1$, $A_j (j = 1, 2, \ldots, n) = B_j (j = 2, \ldots q) = 1$ and $a = 0$ in our main result we arrive at the result derived by [4, 5].

If we take $a = 0$ in the main result we arrive at the result obtained by [9]

1. If we reduce the $\overline{H}$-function involved in (3.1) to the generalized Riemann Zeta function, $\phi\left((x-t)^\sigma, \mu, \xi\right)$, [2, 6], we arrive at the following interesting result:

$$\int_0^x (x-t)^{\rho-1}\phi\left((x-t)^\sigma,\mu,\xi\right)y(t)dt + \frac{a}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}y(t)dt := g(x) \tag{4.1}$$

has the solution given by

$$y(x) = \int_0^x (x-t)^{l-\sigma k-\rho-1} \sum_{\lambda=0}^{\infty} \frac{C_\lambda (x-t)^{\sigma\lambda}}{\Gamma(l-\sigma k+\sigma\lambda-\rho)} D_t^l\{g(t)\}dt \tag{4.2}$$

provided that $\min\{\Re(\rho), \Re(\sigma), \Re(l-\rho-\sigma)\} > 0$, l is a positive integer and C_λ is given by (3.4) where

$$C'_\nu = \frac{\Gamma(\rho+\sigma\nu)}{(\xi+\nu)^\mu}, \nu = 0, 1, 2, ... \tag{4.3}$$

Also $g^{(u)}(0) = 0$ for $0 \le u \le l-1$.

2. If we reduce the $\overline{H}$-function involved in (3.1) to the Polylogarithm function $F(t, \mu)$ of order μ [2, p.30,p.315][6], we get the following interesting result:

$$\int_0^x (x-t)^{\rho-1} F\left((x-t)^\sigma, \mu\right) y(t)dt + \frac{a}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} y(t)dt := g(x) \tag{4.4}$$

has the solution given by

$$y(x) = \int_0^x (x-t)^{l-\sigma k-\rho-1} \sum_{\lambda=0}^{\infty} \frac{C_\lambda (x-t)^{\sigma\lambda}}{\Gamma(l-\sigma k+\sigma\lambda-\rho)} D_t^l\{g(t)\}dt \tag{4.5}$$

provided that $\min\{\Re(\rho), \Re(\sigma), \Re(l-\rho-\sigma)\} > 0$, l is a positive integer and C_λ is given by (3.4) where

$$C'_\nu = \frac{\Gamma(\rho+\sigma+\sigma\nu)}{(1+\nu)^\mu}, \nu = 0, 1, 2, ... \tag{4.6}$$

Also $g^{(u)}(0) = 0$ for $0 \le u \le l-1$.

References

[1] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. **I**, McGraw-Hill Book Company, New York, Toronto and London, 1953.

[2] A. Erdélyi, *Higher Transcendental Functions*, Vol. I, McGraw Hill Book Co., New York, (1953).

[3] A.A. Inayat-Hussain, New properties of hypergeometric series derivable from Feynman Integrals II. A generalization of the H-function, *J. Phys. A: Math. Gen.*, **20**, 4119–4128, (1987).

[4] H.M. Srivastava, and R.G. Bushman, Some convolution integral equations, *Nederl. Akad. Wetensch. Proc. Ser. A77, Indag. Math.*, **36**, 211–216, (1974).

[5] H.M. Srivastava, and R.G. Bushman, *Theory and Applications of Convolution Integral Equations*, Kluwer Academic Publishers, Dordrecht, Boston, London (1992).

[6] M. Garg and R. Mishra, On product of hypergeometric functions, general class of multivariable polynomials and a generalized hypergeometric series associated with Feynman Integrals, *Bull. Calc. Math. Soc.*, **95**, 313–324, (2003).

[7] Priyanka Harjule and R. Jain, A General Volterra type Fractional Equation Associated with an Integral operator with the H-function in the Kernel, Journal of Rajasthan Academy of Physical Sciences, 14(3) (2015), 289–294.

[8] R.G. Bushman, and H.M. Srivastava, The $\overline{H}$-function associated with a certain class of Feynman integrals, *J. Phys. A: Math. Gen.*, **23**, 4707–4710, (1990).

[9] Rashmi Jain, Convolution Integral Equations Involving the $\overline{H}$-function, *Tamsui Oxford Journal of Mathematical Sciences*, **23**, 317–324, (2007).