

# New Method to Solve Partial Fractional Differential Equations

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## Abstract

In this paper, we establish a modified reduced differential transform method, which are successfully applied to obtain the analytical solutions of the time-space fractional Navier-Stokes equations. The fractional derivative is taken in Caputo sense. The obtained results show that the proposed techniques are simple, efficient, and easy to implement for fractional differential equations. We make the Figures to compare between the approximate solutions. We compare between the approximate solutions and the exact solutions for the partial fractional differential equations when  $\alpha, \beta \rightarrow 1$ .

## 1. INTRODUCTION

Nonlinear differential equations describe many physical phenomena, and analytic solution to these equations is important because they usually contain global information on the solution structures of these nonlinear equations [1, 2]. Many analytic approaches for solving nonlinear differential equations have been proposed and the most outstanding one is the homotopy analysis method (HAM). In recent

years, many authors have paid attention to studying the solutions of nonlinear partial differential equations by various methods [3-14].

In this paper, we consider the unsteady flow of a viscous fluid in a tube, the velocity field is a function of only one space coordinate, and the time is a dependent variable. This kind of time-space fractional Navier-Stokes equation has been studied by Momani and Odibat [15], Kumar *et al.* [16, 17], and Khan [18] by using the Adomian decomposition method (ADM), the homotopy perturbation transform method (HPTM), the modified Laplace decomposition method (MLDM), the variational iteration method (VIM), and the homotopy perturbation method (HPM), respectively. In 2006, Daftardar-Gejji and Jafari [19] were first to propose the Gejji-Jafari iteration method for solving a linear and nonlinear fractional differential equation. The Gejji-Jafari iteration method is easy to implement and obtains a highly accurate result. The reduced differential transform method (RDTM) was first proposed by Keskin and Oturanc [20, 21]. The RDTM was also applied by many researchers to handle nonlinear equations arising in science and engineering. In recent years, Kumar *et al.* [22–28] used various methods to study the solutions of linear and nonlinear fractional differential equation combined with a Laplace transform.

The main aim of this article is to present approximate analytical solutions of time-space fractional Navier–Stokes equation by using the reduced differential transform method (MRTM). The Navier–Stokes equation (NSE) with time-space fractional derivatives is written in operator form as:

$$D_t^\alpha u(r,t) = p + v(D_r^{2\beta} u(r,t) + \frac{1}{r} D_r^\beta u(r,t)), \quad 0 < \alpha, \beta < 1 \quad (1)$$

Where  $\alpha > 0$  is parameter describing the order of the time fractional derivatives. In

the case of  $\alpha > 0$ , the fractional equation reduces to the standard Navier–Stokes

equation.

## 2. BASIC DEFINITION OF FRACTIONAL CALCULUS

In this section, we give some basic definitions and properties of fractional calculus theory which shall be used in this paper:

**Definition 2.1.** The fractional derivative ( $D^\alpha$ ) of  $f(x)$  in the Caputo's sense is defined as [29]

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi, \text{ for } n-1 < \alpha \leq n, n \in \mathbb{N} \tag{2}$$

**Definition 2.2.** For  $\alpha > 0$  the Caputo fractional derivative of order  $\alpha$  on the whole space, denoted by  ${}^c D_+^\alpha$ , is defined by [30]

$${}^c D_+^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^x (x-\xi)^{n-\alpha-1} D^n f(\xi) d\xi \tag{3}$$

**Property.** Some useful formula and important properties for the modified Riemann-Liouville derivative as follows [31-34]:

$$\begin{aligned} D_t^\alpha t^r &= \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, r > 0 \\ D_t^\alpha [f(t)g(t)] &= f(t)D_t^\alpha g(t) + g(t)D_t^\alpha f(t) \\ D_t^\alpha [f(g(t))] &= f'_g(g(t))D_t^\alpha g(t) \\ D_t^\alpha [f(g(t))] &= D_g^\alpha f(g(t))[g'(t)]^\alpha \end{aligned} \tag{4}$$

### 3. Reduced differential transform method (RDTM)

In this section, we introduce the basic definitions of the reduced differential transformations. Consider a function of three variables  $w(x,y,t)$ , and assume that it can be represented as a product [20-21]

$$w(x,y,t) = F(x,y)G(t). \tag{5}$$

Based on the properties of one dimensional differential transform, the function  $w(x,y,t)$  can be represented as

$$w(x,y,t) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} F(i_1,i_2) x^{i_1} y^{i_2} \sum_{j=0}^{\infty} G(j) t^j = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \sum_{j=0}^{\infty} W(i_1,i_2) x^{i_1} y^{i_2} t^j. \tag{6}$$

where  $W(i_1,i_2) = F(i_1,i_2)G(j)$  is called the spectrum of  $w(x,y,t)$ . Let  $R_D$  denotes the reduced differential transform operator and  $R_D^{-1}$  the inverse reduced differential transform operator. The basic definition and operation of the RDTM method is described below.

**Definition 3.1 .** If  $w(x,y,t)$  is analytic and continuously differentiable with respect to space variables  $x,y$  and time variable  $t$  in the domain of interest, then the spectrum function,

$$R_D[w(x, y, t)] = W_k(x, y) = \frac{1}{\Gamma(k\alpha + 1)} \left[ \frac{\partial^k}{\partial t^k} w(x, y, t) \right]_{t=t_0}. \quad (7)$$

is the reduced transformed function of  $w(x, y, t)$ . The differential inverse reduced transform of  $W_k(x, y)$  is defined as

$$R_D^{-1}[W_k(x, y)] = w(x, y) = \sum_{k=0}^{\infty} W_k(x, y)(t - t_0)^{k\alpha}. \quad (8)$$

Combining Eqs. (7) and (8), we get

$$w(x, y, t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha + 1)} \left[ \frac{\partial^k}{\partial t^k} w(x, y, t) \right]_{t=t_0} (t - t_0)^{k\alpha}. \quad (9)$$

When  $t = 0$ , Eq. (9) reduces to

$$w(x, y, t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha + 1)} \left[ \frac{\partial^k}{\partial t^k} w(x, y, t) \right]_{t=t_0} t^{k\alpha}. \quad (10)$$

From the Eq. (8), it can be seen that the concept of the reduced differential transform is derived from the power series expansion of the function.

**Definition 3.2.** If  $u(x, y, t) = R_D^{-1}[U_k(x, y)]$ ,  $v(x, y, t) = R_D^{-1}[V_k(x, y)]$ , and the convolution  $\otimes$  denotes the reduced differential transform version of the multiplication, then the fundamental operations of the reduced differential transform are shown in the Table 1

Table 1 - Fundamental operations of the reduced differential transform method.

Original function	reduced differential transform function
$R_D[u(x, y, t)v(x, y, t)]$	$U_k(x, y) \otimes V_k(x, y) = \sum_{r=0}^k U_r(x, y)V_{k-r}(x, y)$
$R_D[\alpha u(x, y, t) \pm \beta v(x, y, t)]$	$\alpha U_k(x, y) \pm \beta V_k(x, y)$
$R_D\left[\frac{\partial^{N\alpha}}{\partial t^{N\alpha}} u(x, y, t)\right]$	$\frac{\Gamma(k\alpha + N\alpha + 1)}{\Gamma(k\alpha + 1)} U_{k+N}(x, y)$

$R_D[\frac{\partial^{m+n+s}}{\partial x^m \partial y^n \partial t^s} u(x, y, t)]$	$\frac{(k+s)!}{k!} \frac{\partial^{m+n}}{\partial x^m \partial y^n} U_{k+s}(x, y)$
$R_D[e^{\lambda t}]$	$\frac{\lambda^k}{k!}$

**4. solving The time and space fractional of partial differential equations by reduced differential transform method (RDTM)**

**Example1. Consider the following time-space fractional Navier-Stokes equation:**

$$D_t^\alpha u(r, t) = p + D_r^{2\beta} u(r, t) + \frac{1}{r} D_r^\beta u(r, t), \quad 0 < \alpha, \beta < 1 \tag{11}$$

subject to the initial condition

$$u(r, 0) = 1 - r^2. \tag{12}$$

Applying the RDTM to Eqs. (11), we obtain the following recurrence relations

$$U_{k+1} = \frac{\Gamma(k\alpha + 1)}{\Gamma[\alpha(k + 1) + 1]} [D_r^{2\beta} U_k + \frac{1}{r} D_r^\beta U_k + P\delta(k - 0)], \tag{13}$$

Using the RDTM to the initial conditions (12), we get

$$U_0(r, t) = 1 - r^2, \tag{14}$$

Applying the initial conditions (14) into Eqs.(13) at  $\beta \rightarrow 1$ , we have

$$\begin{aligned} U_0 &= 1 - r^2, \\ U_1 &= \frac{(p - 4)}{\Gamma(\alpha + 1)}, \\ U_3 &= 0, \\ U_4 &= 0, \\ &\dots \\ U_n &= 0. \end{aligned} \tag{15}$$

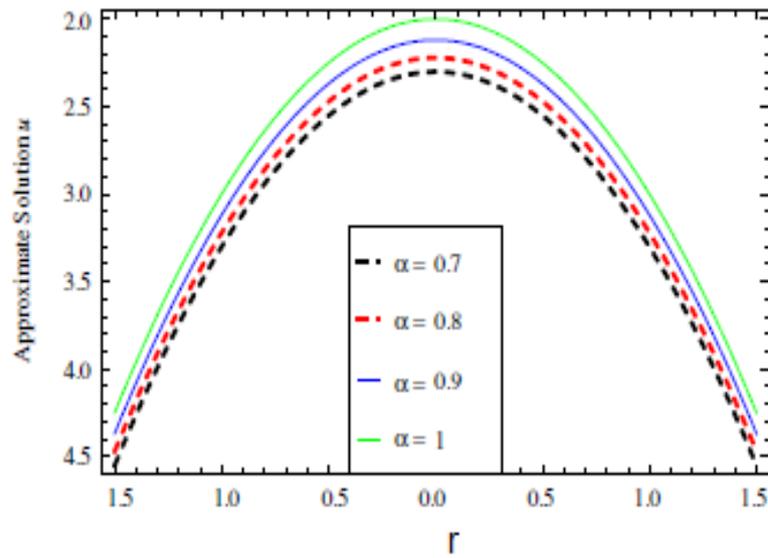
So, the general solutions of Eqs.(11) are :

$$u(r, t) = \sum_{k=0}^{\infty} U_k(r, t) t^{k\alpha}. \quad \text{at } t_0 = 0 \tag{16}$$

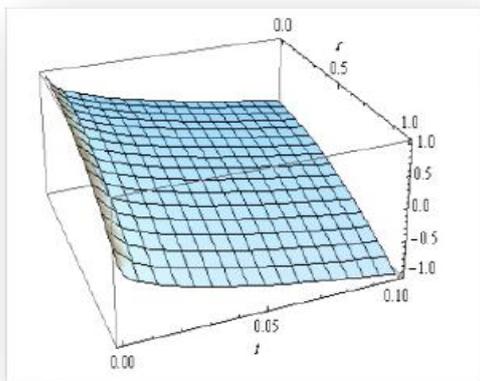
So, the solution of equation (11) at  $\beta \rightarrow 1$  is given as:

$$u(r,t) = 1 - r^2 + \frac{1}{\Gamma(\alpha + 1)}(P - 4)t^\alpha. \tag{17}$$

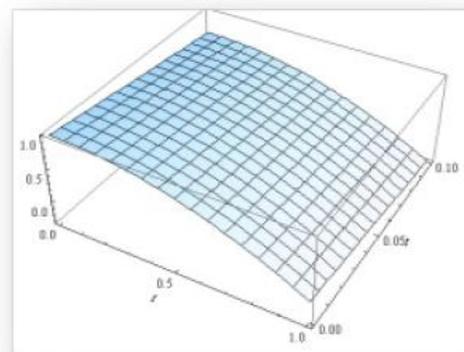
The result is the same as ADM, HPTM, HPM, VIM, and HAM by [15], Kumar et al. [16, 17], [18] and [35].



**Figure 1:** The behavior of the solutions for different value of  $\alpha$ ,  $\beta \rightarrow 1$  at  $p = 1$  and  $t = 1$



(a)



(b)

**Figure 2:** The behavior of the solutions for (a):  $\alpha = 1$ ,  $\beta \rightarrow 1$  and (b):  $\alpha = 0.5$ ,  $\beta \rightarrow 1$  at  $p = 1$

Example2. Consider the following time-space fractional Navier-Stokes equation:

$$D_t^\alpha u(r,t) = D_r^{2\beta} u(r,t) + \frac{1}{r} D_r^\beta u(r,t), \quad 0 < \alpha, \beta \leq 1 \tag{18}$$

subject to the initial condition

$$u(r,0) = r. \tag{19}$$

Applying the RDTM to Eqs. (18), we obtain the following recurrence relations

$$U_{k+1} = \frac{\Gamma(k\alpha + 1)}{\Gamma[\alpha(k + 1) + 1]} [D_r^{2\beta} U_k + \frac{1}{r} D_r^\beta U_k], \tag{20}$$

Using the RDTM to the initial conditions (19), we get

$$U_0(r,t) = r. \tag{21}$$

Applying the initial conditions (21) into Eqs. (20), we have

$$\begin{aligned} U_1 &= \frac{1}{\Gamma(\alpha + 1)} \left[ \frac{\Gamma(2)}{\Gamma(2 - 2\beta)} r^{1-2\beta} + \frac{\Gamma(2)}{\Gamma(2 - \beta)} r^{-\beta} \right], \\ U_2 &= \frac{\Gamma(2)}{\Gamma(2\alpha + 1)\Gamma(2 - 4\beta)} r^{1-4\beta} + \frac{\Gamma(2)\Gamma(\alpha + 1)\Gamma(1 - \beta)}{\Gamma(2\alpha + 1)\Gamma(\alpha)\Gamma(2 - \beta)\Gamma(1 - 2\beta)} r^{-1-2\beta} \\ &\quad + \frac{\Gamma(2)}{\Gamma(2\alpha + 1)\Gamma(2 - \beta)} r^{-3\beta} \left( \frac{\Gamma(\alpha + 1)\Gamma(1 - \beta)}{\Gamma(\alpha)\Gamma(1 - 3\beta)} + \frac{\Gamma(2 - 2\beta)}{\Gamma(2 - 3\beta)} \right), \\ &\dots \\ &\dots \end{aligned} \tag{22}$$

The general solutions of Eqs. (18) are :

$$u(r,t) = \sum_{k=0}^{\infty} U_k(r,t) t^{k\alpha}. \quad \text{at } t_0 = 0 \tag{23}$$

So, we have:

$$u(r,t) = r + \frac{1}{\Gamma(\alpha + 1)} \left( \frac{\Gamma(2)}{\Gamma(2 - 2\beta)} r^{1-2\beta} + \frac{1}{r} \frac{\Gamma(2)}{\Gamma(2 - \beta)} r^{1-\beta} \right) + \dots \tag{24}$$

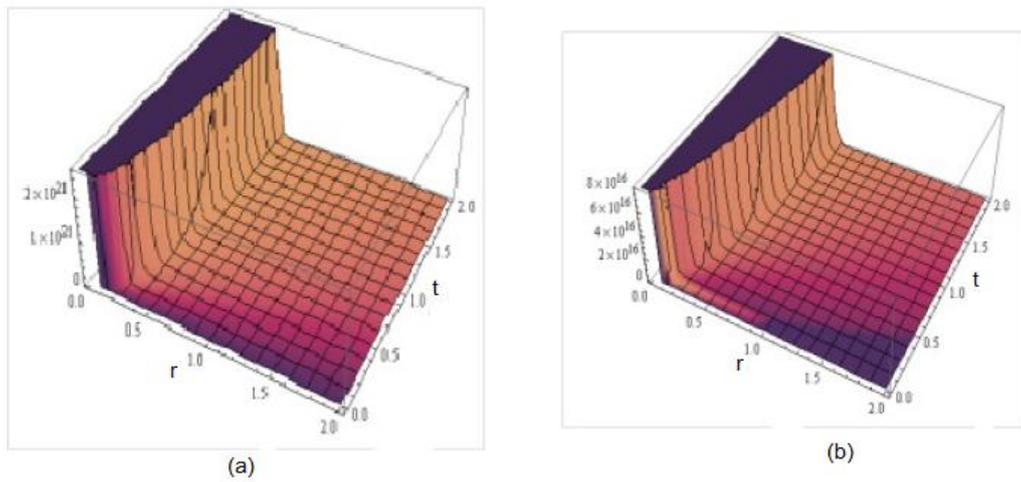
So, the solution of equation (18) at  $\beta \rightarrow 1$  is given as:

$$u(r,t) = r + \sum_{n=1}^{\infty} \frac{1^2 x 3^2 x \dots x (2n-3)^2}{r^{2n-1}} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}. \tag{25}$$

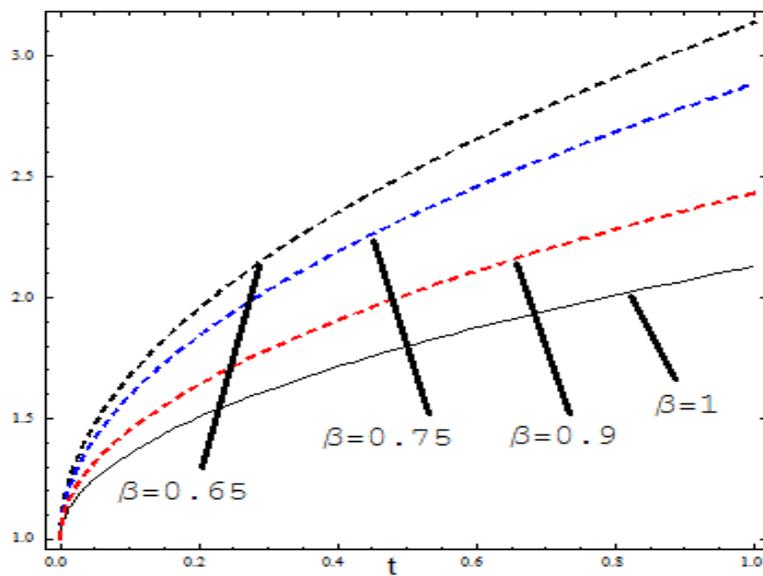
When  $\alpha = 1$  equation (25) is the same as the exact solution of the Navier-Stokes equation [15]

$$u(r,t) = r + \sum_{n=1}^{\infty} \frac{1^2 \times 3^2 \times \dots \times (2n-3)^2}{r^{2n-1}} \frac{t^n}{\Gamma(n+1)}. \tag{26}$$

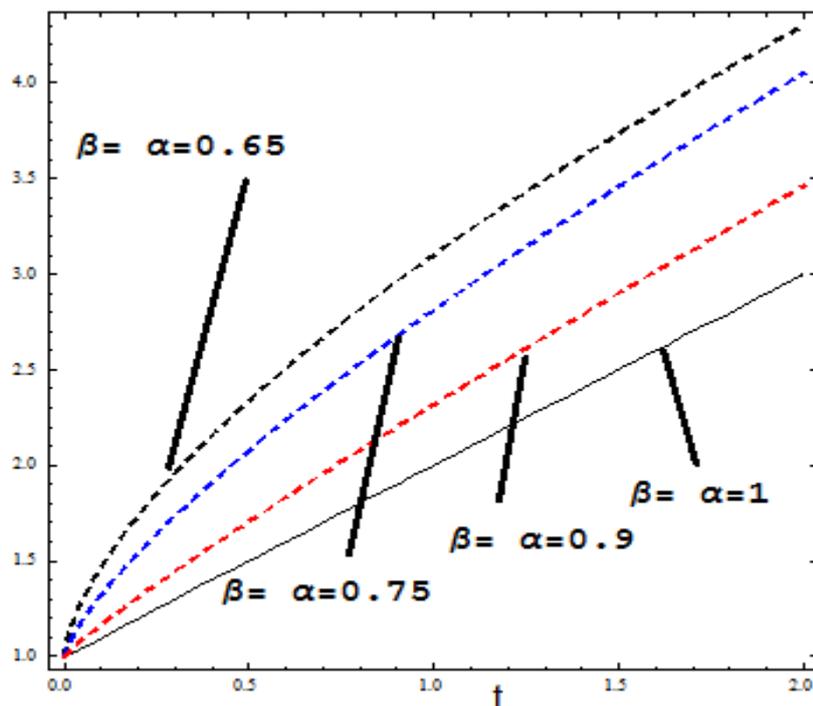
The result is the same as ADM, HPTM, HPM, VIM and HAM by Momani and Odibat [15], Kumar *et al.* [16, 17], Khan [18] and [35].



**Figure 3:** The behavior of the solutions for (a):  $\alpha = 1, \beta \rightarrow 1$  and (b):  $\alpha = 0.5, \beta \rightarrow 1$ .



**Figure 4:** The behavior of the solutions for different values of  $\beta$ ,  $\alpha = 0.5$  at  $r = 1$ .



**Figure 5:** The behavior of the solutions for different values of  $\beta$ ,  $\alpha$  at  $r = 1$ .

## 6. CONCLUSION

In this paper, FRDTM has been implemented for the Caputo time-space fractional order Navier-Stokes equation. The proposed approximated solutions of Navier-Stokes equation with an appropriate initial condition are obtained in terms of a power series, without using any kind of discretization, perturbation, or restrictive conditions, etc. Two examples are illustrated to study the effectiveness and accurateness of FRDTM. It is found that FRDTM solutions are in excellent agreement with those obtained using ADM, HPTM, HPM, VIM, HAM, DTM and FHATM. However, computations show that the FRDTM is very easy to implement and needs small size of computation contrary to ADM, DTM and FHATM. This shows that FRDTM is very effective and efficient powerful mathematical tool, which is easily applicable in finding out the approximate analytic solutions of a wide range of real world problems arising in engineering and allied sciences. Mathematica has been used for computations in this paper.

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