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A Common Fixed Point Result in Complex Valued b-Metric Spaces under Contractive Condition

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Abstract

In this paper we prove a common fixed point theorem for two self-mappings in complex valued b-metric spaces under contractive condition. Our result generalizes the result of S. Ali [3].

Keywords and phrases: Complex Valued b-Metric Space, Common Fixed Point, Contractive Type Mapping.

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1 INTRODUCTION AND PRELIMINARIES

The notion of complex valued metric space was introduced by A. Azam, B. Fisher and M. Khan [4] in 2011. The concept of b-metric space was introduced by Bakhtin [5] in 1989. Rao et al. [9] introduced complex valued b-metric space which is more general than well-known complex valued metric space. There are many fixed point results in complex valued metric spaces [see [2], [6], [7], [8], [10], [11]] also in complex valued b-metric spaces [see [1], [9]]. In this paper we present a common fixed point result for two self-mappings satisfying a contractive condition in complex valued b-metric spaces. This result generalizes the result obtained by S. Ali [3].

Let $\mathbb{C}$ be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order relation $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

Thus $z_1 \precsim z_2$ if one of the followings holds:

(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$ and

(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e., one of (2), (3) and (4) is satisfied and we will write $z_1 \prec z_2$ if only (4) is satisfied.

Remark 1: We can easily check the followings:

(i) $a, b \in \mathbb{R}, a \leq b \Rightarrow az \precsim bz,\ \forall\, z \in \mathbb{C}$.

(ii) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$.

(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Azam et al. [4] defined the complex valued metric space in the following way:

Definition 1 ([4]): Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:

(C1) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(C2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(C3) $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 1([7]): Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = i|z_1 - z_2| \ \forall\ z_1, z_2 \in \mathbb{C}.$$

One can easily verify that $(\mathbb{R}, d)$ is a complex valued metric space.

Definition 2([9]): Let X be a nonempty set and let $s \geq 1$ be given real number. A function $d: X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $0 \precsim d(x,y)$, for all x, y∈X and d(x,y)=0 if and only if x=y;

(2) d(x,y)=d(y,x) for all x,y∈X

(3) d(x,y)≾s[d(x,z)+d(z,y)] for all x,y,z∈X.

The pair (X,d) is called complex valued b-metric space.

Example 2([9]): Let $X=[0,1]$. Define the mapping $d: X\times X\to\mathbb{C}$ by

$$d(x,y)=|x-y|^2+i|x-y|^2, \text{ for all } x,y\in X.$$

Then (X,d) is a complex valued b-metric space with $s=2$.

Definition 3([9])): Let (X,d) be a complex valued b-metric space. Then

(i) A point $x\in X$ is called an interior point of a set $A\subseteq X$ if there exists $0\prec r\in\mathbb{C}$ such that

$$B(x,r)=\{y\in X: d(x,y)\prec r\}\subseteq A.$$

A subset $A\subseteq X$ is called open if each element of A is an interior point of A.

(ii) A point $x\in X$ is called a limit point of $A\subseteq X$ if for every $0\prec r\in\mathbb{C}$,

$$B(x,r)\cap(A-\{x\})\neq\phi.$$

A subset $A\subseteq X$ is called closed if each element of $X-A$ is not a limit point of A.

(iii) The family

$$F=\{B(x,r): x\in X, 0\prec r\}$$

is a sub-basis for a Hausdorff topology τ on X.

Definition 4([9]): Let (X,d) be a complex valued b-metric space. Then

(i) A sequence $\{x_n\}$ in X is said to converge to $x\in X$ if for every $0\prec r\in\mathbb{C}$ there exists $N\in\mathbb{N}$ such that $d(x_n,x)\prec r,\ \forall\ n>N$. We denote this by $\lim_{n\to\infty} x_n=x$ or $x_n\to x$ as $n\to\infty$.

(ii) If for every $0\prec r\in\mathbb{C}$ there exists $N\in\mathbb{N}$ such that $d(x_n, x_{n+m})\prec r$ for all $n>N, m\in\mathbb{N}$, then $\{x_n\}$ is called a Cauchy sequence in (X,d).

(iii) If every Cauchy sequence in X is convergent in X then (X,d) is called a complete complex valued b-metric space.

Lemma 1 ([9]): Let (X,d) be a complex valued b-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to $x\in X$ if and only if $|d(x_n,x)|\to 0$ as $n\to\infty$.

Lemma 2 ([9]): Let (X, d) be a complex valued b-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$ where $m \in \mathbb{N}$.

Definition 5 ([11]): The 'max' function for the partial order $\precsim$ is defined as follows:

(1) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$.

(2) $z_1 \precsim \max\{z_2, z_3\} \Rightarrow z_1 \precsim z_2$ or $z_1 \precsim z_3$.

(3) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$ or $|z_1| \leq |z_2|$.

2. MAIN THEOREM

In this section we present the main result of the paper.

Theorem 1: Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and $f, g: X \to X$ be self-maps satisfying the following condition:

$$d(fx, gy) \precsim \alpha. \max\left\{d(x, y), \frac{d(x, fx)d(y, gy)}{1 + d(fx, gy)}\right\}$$

for all $x, y \in X$, where α is a real with $0 < \alpha < 1$. Then f and g have a unique common fixed point.

Proof : Let $x_0 \in X$ be arbitrary.

We define a sequence $\{x_n\}$ in X as

$$x_{2k+1} = fx_{2k}$$

$$x_{2k+2} = gx_{2k+1}, \quad k = 0,1,2, \ldots$$

Then

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d(fx_{2k}, gx_{2k+1}) \\ &\precsim \alpha. \max\left\{d(x_{2k}, x_{2k+1}), \frac{d(x_{2k}, fx_{2k})d(x_{2k+1}, gx_{2k+1})}{1 + d(fx_{2k}, gx_{2k+1})}\right\} \\ &\precsim \alpha. \max\left\{d(x_{2k}, x_{2k+1}), \frac{d(x_{2k}, x_{2k+1})d(x_{2k+1}, x_{2k+2})}{1 + d(x_{2k+1}, x_{2k+2})}\right\} \\ &\precsim \alpha. d(x_{2k}, x_{2k+1}). \end{aligned}$$

Thus

$$d(x_{2k+1}, x_{2k+2}) \precsim \alpha. d(x_{2k}, x_{2k+1}). \tag{1}$$

Similarly

$$d(x_{2k+2}, x_{2k+3}) = d(fx_{2k+2}, gx_{2k+1})$$

$$\precsim \alpha.\max\left\{d(x_{2k+2}, x_{2k+1}), \frac{d(x_{2k+2}, fx_{2k+2})d(x_{2k+1}, gx_{2k+1})}{1+d(fx_{2k+2}, g\,x_{2k+1})}\right\}$$

$$\precsim \alpha.\max\left\{d(x_{2k+2}, x_{2k+1}), \frac{d(x_{2k+2}, x_{2k+3})d(x_{2k+1}, x_{2k+2})}{1+d(x_{2k+3}, x_{2k+2})}\right\}$$

$$= \alpha.d(x_{2k+1}, x_{2k+2}).$$

Hence

$$d(x_{2k+2}, x_{2k+3}) \precsim \alpha.d(x_{2k+1}, x_{2k+2}). \tag{2}$$

Therefore from (1) and (2) for $n \in \mathbb{N}$ we have

$$d(x_{n+1}, x_{n+2}) \precsim \alpha d(x_n, x_{n+1}) \precsim \alpha^2 d(x_{n-1}, x_n) \precsim \cdots \precsim \alpha^{n+1} d(x_0, x_1).$$

So for m, n∈ℕ,

$$d(x_n, x_{m+n})$$

$$\precsim s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+n})]$$

$$\precsim sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{m+n})]$$

$$\precsim sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{m-1} d(x_{m+n-2}, x_{m+n-1}) + s^{m-1} d(x_{m+n-1}, x_{m+n})$$

$$\precsim sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{m-1} d(x_{m+n-2}, x_{m+n-1}) + s^m d(x_{m+n-1}, x_{m+n})$$

$$\precsim s\alpha^n d(x_0, x_1) + s^2 \alpha^{n+1} d(x_0, x_1) + \cdots + s^m \alpha^{m+n-1} d(x_0, x_1)$$

$$\precsim s\alpha^n (1 + s\alpha + (s\alpha)^2 + \cdots + (s\alpha)^{m-1}) d(x_0, x_1)$$

$$\to 0 \text{ as } n \to \infty \text{ where } m \in \mathbb{N}.$$

Therefore from Lemma 2, we see that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete $\exists\, u \in X$ such that $x_n \to u$ as $n \to \infty$.

Thus

$$\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} gx_{2n+1} = u. \tag{3}$$

Now from the given condition we have

$$d(fu,u) \precsim s[d(fu,gx_{2n+1}) + d(gx_{2n+1},u)]$$

$$\precsim s\alpha.\max\left\{d(u,x_{2n+1}), \frac{d(u,fu)d(x_{2n+1},gx_{2n+1})}{1+d(fu,gx_{2n+1})}\right\} + s.d(gx_{2n+1},u)$$

$$\precsim s\,\alpha.\max\left\{d(u,x_{2n+1}), \frac{d(u,fu)d(x_{2n+1},x_{2n+2})}{1+d(fu,x_{2n+2})}\right\} + s.d(x_{2n+2},u)$$

$$\to 0 \quad \text{as } n \to \infty$$

Thus

$$d(fu,u) \precsim 0.$$

Thus $d(fu,u) = 0$ and hence $fu = u$.

Again

$$d(u,gu) \precsim d(fu,gu)$$

$$\precsim \alpha.\max\left\{d(u,u), \frac{d(u,fu)d(u,gu)}{1+d(fu,gu)}\right\}$$

$$= 0.$$

Hence $gu = u$.

Therefore u is a common fixed point of f and g.

Now for the uniqueness part, let us suppose that $fu^* = gu^* = u^*$ for some $u^* \in X$.

Then

$$d(u,u^*) = d(fu,gu^*)$$

$$\precsim \alpha.\max\left\{d(u,u^*), \frac{d(u,fu)d(u^*,gu^*)}{1+d(fu,gu^*)}\right\}$$

$$= \alpha d(u,u^*).$$

This implies $(1-\alpha)|d(u,u^*)| \leq 0$.

Since $0 < \alpha < 1$, we must have $u = u^*$ and this completes the proof. ■

By setting $f = g$ we get the following corollary.

Corollary 1: Let (X,d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and $f : X \to X$ be a self-map satisfying the following condition:

$$d(fx,fy) \precsim \alpha.\max\left\{d(x,y), \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}\right\}$$

for all $x, y \in X$, where α is a real with $0 < \alpha < 1$. Then f has a unique fixed point.

By setting $s = 1$ we get the following corollary.

Corollary 2(Theorem 1, [3]): Let (X, d) be a complete complex valued metric space and

$f, g : X \to X$ be self-maps satisfying the following condition:

$$d(fx, gy) \precsim \alpha.\max\left\{d(x, y), \frac{d(x,fx)d(y,gy)}{1+d(fx,gy)}\right\}$$

for all $x, y \in X$, where α is a real with $0 < \alpha < 1$. Then f and g have a unique common fixed point.

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