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Using Laplace transform method for obtaining the exact analytic solutions of some ordinary fractional differential equations

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Abstract

In this paper, we applied the Laplace transform to obtain an exact analytic solution of some ordinary fractional differential equations. We used the Cauchy residue theorem and the Jordan Lemma to obtain the inverse Laplace transform for some complicated functions and this implied to obtain an exact analytic solution of some ordinary fractional differential equations. The fractional derivatives would described in the Caputo sense which obtained by Riemann-Liouville fractional integral operator. We showed that the Laplace transform method was a powerful and efficient techniques for obtaining an exact analytic solution of some ordinary fractional differential equations.

Keywords: Fractional-order differential equations; Laplace Transform; Inverse Laplace Transform.

1. INTRODUCTION

In the past two decades, the widely investigated subject of fractional calculus has remark ablygained importance and popularity due to its demonstrated applications in numerous diverse fields of science and engineering. These contributions to the fields of science and engineering are based on the mathematical analysis. It covers the widely known classical fields such as Abel's integral equation and viscoelasticity. Also, including the analysis of feedback amplifiers, capacitor theory, generalized voltage dividers, fractional-order Chua-Hartley systems, electrode-electrolyte interface models, electric conductance of biological systems, fractional-order models of neurons, fitting of experimental data, and the fields of special functions [1-6].

Several methods have been used to solve fractional differential equations, fractional partial differential equations, fractional integro-differential equations and dynamic systems containing fractional derivatives, such as Adomian's decomposition method [7–11], He's variational iteration method [12–16], homotopy perturbation method [17–19], homotopy analysis method [20], spectral methods [21–24], and other methods [25–28].

This paper is organized as follows;we begin by introducing some necessary definitions and mathematical preliminaries of the fractional calculus theory. In section 3, the Laplace transform and the inverse Laplace transform for some functions is demonstrated. In section 4, the proposed method is applied to several examples. Also conclusions given in the last section.

2. PRELIMINARIES AND NOTATIONS

In this section, we give some basic definitions and properties of fractional calculus theory which are further used in this article.

Definition 2.1.*A real function* $f(x), x > 0$ *is said to be in space* $C_{\mu}, \mu \in \mathbb{R}$ *if there exists a real number* $p > \mu$, *such that* $f(t) = t^{p} f_1(t)$, *where* $f_1(t) \in C(0,\infty)$, *and it is said to be in the space* C_{μ}^{n} *if and only if* $f^{n} \in C_{\mu}$, $n \in \mathbb{N}$.

Definition 2.2.*The Riemann-Liouville fractional integral operator of order* $\alpha > 0$, *of a function* $f \in C_{\mu}, \mu \geq -1$, *is defined as*

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0$$
$$J^{0} f(t) = f(t) \tag{1}$$

Some properties of the operator J^{α}, which are needed here, are as follows:

for $f \in C_{\mu}, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma \geq -1$:

$$(1)\ J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t)$$
$$(2)\ J^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\alpha+\gamma} \tag{2}$$

Definition 2.3. *The fractional derivative of $f(t)$ in the Caputo sense is defined as*

$$D^{\alpha} f(t) = J^{m-\alpha} D^{m} f(t) \tag{3}$$

for $m-1 < \alpha \le m$, $m \in \mathbb{N}$, $t > 0$ and $f \in C_{-1}^{m}$.

Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative.Similar to the integer-order integration, the Riemann-Liouville fractional integral operator is a linear operation:

$$J^{\alpha} \left(\sum_{i=1}^{n} c_i f_i(t)\right) = \sum_{i=1}^{n} c_i J^{\alpha} f_i(t) \tag{4}$$

where $\{c_i\}_{i=1}^{n}$ are constants.

In the present paper, the fractional derivatives are considered in the Caputo sense. The reason for adopting the Caputo definition, as pointed by [10], is as follows: to solve differential equations (both classical and fractional), we need to specify additional conditions in order to produce a unique solution. For the case of the Caputo fractional differential equations, these additional conditions are just the traditional conditions,which are akin to those of classical differential equations, and are therefore familiar to us. In contrast, for the Riemann-Liouville fractional differential equations, these additional conditions constitute certain fractional derivatives (and/or integrals) of the unknown solution at the initial point $x = 0$, which are functions of x. These initial conditions are not physical; furthermore, it is not clear how such quantities are to be measured from experiment, say, so that they can be appropriately assigned in an analysis. For more details see [2].

3. LAPLACE OPERATION

The Laplace transform is a powerful tool in applied mathematics and engineering. Virtually every beginning course in differential equations at the undergraduate level introduces this technique for solving linear differential equations. The Laplace transform is indispensable in certain areas of control theory.

3.1. Laplace Transform

Given a function $f(x)$ defined for $0 < x < \infty$, the Laplace transform $F(s)$ is defined as

$$F(s) = L[f(x)] = \int_0^\infty f(x)e^{-sx}dx \tag{5}$$

at least for those s for which the integral converges.

Let $f(x)$ be a continuous function on the interval $[0,\infty)$ which is of exponential order, that is, for some $c \in \mathbb{R}$ and $x > 0$

$$\sup \frac{|f(x)|}{e^{cx}} < \infty .$$

In this case the Laplace transform exists for all $s > c$.

Some of the useful Laplace transforms which are applied in this paper, are as follows:

For $L[f(x)] = F(s)$ and $L[g(x)] = G(s)$

$$\begin{aligned}
&L[f(x)+g(x)] = F(s)+G(s),\\
&L[x^\beta] = \frac{\Gamma(\beta+1)}{s^{\beta+1}}, \qquad \beta > -1,\\
&L[f^{(n)}(x)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'^{(n-1)}(0),\\
&L[x^n\ f(x)] = (-1)^n\ F^{(n)}(s),\\
&L[\int_0^x f(t)dt] = \frac{F(s)}{s},\\
&L[\int_0^x f(x-t)g(t)dt] = F(s)G(s).
\end{aligned} \tag{6}$$

Lemma 3.1.1. *The Laplace transform of Riemann-Liouville fractional integral operator of order* $\alpha > 0$ *can be obtained in the form of:*

$$L[J^\alpha f(x)] = \frac{F(s)}{s^\alpha} \tag{7}$$

Proof. The Laplace transform of Riemann-Liouville fractional integral operator of order $\alpha > 0$ is :

$$L[j^\alpha f(x)] = L[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt] = \frac{1}{\Gamma(\alpha)} F(s)G(s), \tag{8}$$

where

$$G(s) = L[x^{\alpha-1}] = \frac{\Gamma(\alpha)}{s^\alpha} \tag{9}$$

and the lemma can be proved.

Lemma 3.1.2. *The Laplace transform of Caputo fractional derivative for* $m-1<\alpha\le m$, $m\in\mathbb{N}$, *can be obtained in the form of:*

$$L[D^{\alpha}f(x)]=\frac{s^{m}F(s)-s^{m-1}f(0)-s^{m-2}f^{'(m-1)}(0)}{s^{m-\alpha}}. \qquad (10)$$

Proof. The Laplace transform of Caputo fractional derivative of order $\alpha>0$ is :

$$L[D^{\alpha}f(x)]=L[J^{m-\alpha}f^{(m)}(x)]=\frac{L[f^{(m)}(x)]}{s^{m-\alpha}}. \qquad (11)$$

Using Eq.(6), the lemma can be proved.

Now, we can transform fractional differential equations into algebraic equations and then by solving this algebraic equations, we can obtain the unknown Laplace function $F(s)$.

3.2.Inverse Laplace Transform

The function $f(x)$ in ((5)) is called the inverse Laplace transform of $F(s)$ and will be denoted by $f(x)=L^{-1}[F(s)]$ in the paper. In practice when one uses the Laplace transform to, for example, solve a differential equation, one has to at some point invert the Laplace transform by finding the function $f(x)$ which corresponds to some specified $F(s)$.The Inverse Laplace Transform of $F(s)$ is defined as:

$$f(x)=L^{-1}[F(s)]=\frac{1}{2\pi i}\lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT}e^{sx}F(s)\,ds, \qquad (12)$$

where σ is large enough that $F(s)$ is defined for the real part of $s\ge\sigma$. surprisingly, this formula isn't really useful. Therefore, in this section some useful function $f(x)$ is obtained from their Laplace Transform. In the first we define the most important special functions used in fractional calculus the Mittag-Leffler functions and the generalized Mittag-Leffler functions

For $\alpha,\beta>0$ and $z\in\mathbb{C}$

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)},$$
$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)}. \tag{13}$$

Now, we prove some Lemmas which are useful for finding the function $f(x)$ from its Laplace transform.

Lemma 3.2.1. *For* $\alpha, \beta > 0$ *,* $a \in \mathbb{R}$ *and* $s^{\alpha} > |a|$ *we have the following inverse Laplace transform formula*

$$\int_0^{\infty} e^{-sx} x^{\beta-1} E_{\alpha,\beta}\left(ax^{\alpha}\right)dx = \frac{s^{\alpha-\beta}}{s^{\alpha}-a}.$$
$$i.e. \quad L^{-1}\left[\frac{s^{\alpha-\beta}}{s^{\alpha}-a}\right] = x^{\beta-1}E_{\alpha,\beta}(ax^{\alpha}) \tag{14}$$

Proof.

$$\begin{aligned}\int_0^{\infty} e^{-sx} x^{\beta-1} E_{\alpha,\beta}\left(ax^{\alpha}\right)dx &= \sum_{k=0}^{\infty} \frac{a^k}{\Gamma(\alpha k+\beta)} \int_0^{\infty} e^{-st} x^{\beta-1+\alpha k}\, dx \\ &= \sum_{k=0}^{\infty} \frac{a^k}{\Gamma(\alpha k+\beta)} \Gamma(\alpha k+\beta) s^{-\alpha k-\beta} \\ &= s^{-\beta} \sum_{k=0}^{\infty} \left(as^{-\alpha}\right)^k \\ &= \frac{s^{-\beta}}{1-as^{-\alpha}} \\ &= \frac{s^{\alpha-\beta}}{s^{\alpha}-a}.\end{aligned} \tag{15}$$

So the inverse Laplace Transform of above function is

$$x^{\beta-1}E_{\alpha,\beta}(-ax^{\alpha}). \tag{16}$$

The following two lemmas are known see [29], but we include the proof for convenience of the reader.

Lemma 3.2.2. *For* $\alpha \geq \beta > 0$*,* $a \in \mathbb{R}$ *and* $s^{\alpha-\beta} > |a|$ *we have*

$$L^{-1}[\frac{1}{(s^{\alpha}+as^{\beta})^{n+1}}]=x^{\alpha(n+1)-1}\sum_{k=0}^{\infty}\frac{(-a)^{k}\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha)}x^{k(\alpha-\beta)}. \quad (17)$$

Proof. Using the series expansion of $(1+x)^{-n-1}$ of the from

$$\frac{1}{(1+x)^{n+1}}=\sum_{k=0}^{\infty}\binom{n+k}{k}(-x)^{k} \quad (18)$$

we have:

$$\frac{1}{(s^{\alpha}+as^{\beta})^{n+1}}=\frac{1}{(s^{\alpha})^{n+1}}\frac{1}{(1+\frac{a}{s^{\alpha-\beta}})^{n+1}}=\frac{1}{(s^{\alpha})^{n+1}}\sum_{k=0}^{\infty}\binom{n+k}{k}(\frac{-a}{s^{\alpha-\beta}})^{k} \quad (19)$$

Giving the inverse Laplace Transform of above function can prove the Lemma.

Lemma 3.2.3. *For* $\alpha\geq\beta,\alpha>\gamma,\ a\in\mathbb{R},\ s^{\alpha-\beta}>|a|$ *and* $|s^{\alpha}+as^{\beta}|>|b|$ *we have*

$$L^{-1}[\frac{s^{\gamma}}{s^{\alpha}+as^{\beta}+b}]=x^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^{n}(-a)^{k}\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha-\gamma)}x^{k(\alpha-\beta)+n\alpha}. \quad (20)$$

Proof. $s^{\gamma}/(s^{\alpha}+as^{\beta}+b)$ by using the series expansion can be rewritten as

$$\frac{s^{\gamma}}{s^{\alpha}+as^{\beta}+b}=\frac{s^{\gamma}}{s^{\alpha}+as^{\beta}}\frac{1}{1+\frac{b}{s^{\alpha}+as^{\beta}}}=\sum_{n=0}^{\infty}\frac{s^{\gamma}(-b)^{n}}{(s^{\alpha}+as^{\beta})^{n+1}} \quad (21)$$

Now by using **Lemma 3.2.2** the **Lemma3.2.3** can be proved.

Titchmarsh Theorem [28]: Let $F(p)$ be an analytic function having no singularities in the cut plane $\mathbb{C}/\mathbb{R}$. Assuming that $\overline{F(p)}=F(\bar{p})$ and the limiting values $F^{\pm}(t)=\lim_{\phi\to\pi^{-}}F\left(t\,e^{\pm i\phi}\right),\ F^{+}(t)=\overline{F^{-}(t)}$. Exist for almost all

(i) $F(p)=o(1)$ for $|p|\to\infty$ and $F(p)=o\left(|p|^{-1}\right)$ for $|p|\to 0$ uniformly in any sector $|\arg p|<\pi-\eta,\pi>\eta>0,$

(ii) There exists $\varepsilon>0$ such that for every

$$\frac{F(re^{\pm i\phi})}{1+r}\in L_{1}\left(\mathbb{R}^{+}\right),\left|F(re^{\pm i\phi})\right|\leq a(r),\pi-\varepsilon<\phi\leq\pi,$$

Where $a(r)$ doesn't depend on ϕ and $a(r)e^{-\delta r}\in L_{1}(\mathbb{R}^{+})$ for any $\delta>0$. Then

$$f\ (t) = L^{-1}\left[F\,(s)\right] = \frac{1}{\pi}\int_0^{\infty} \mathrm{Im}\left[F^{-}\,(\eta)\right] e^{-t\eta}\, d\,\eta$$

4. ILLUSTRATIVE EXAMPLES

This section is applied the method presented in the paper and give an exact solution of some linerar fractional differential equations.

Example 4.1. Consider the composite fractional oscillation equation

$$\begin{aligned} &D^{2\alpha} y(x) - aD^{\alpha} y(x) - by(x) = 8, \qquad 0 < \alpha \le 1 \\ &y(0) = y'(0) = 0. \end{aligned} \tag{22}$$

Hence, we have two cases for $0 < \alpha \le \frac{1}{2}$ and $\frac{1}{2} < \alpha \le 1$

For case one using the Laplace transform, $F(s)$ is obtained as follows

$$\begin{aligned} &s^{2\alpha} F(s) - a\, s^{\alpha} F(s) - bF(s) = \frac{8}{s}, \\ &F(s) = \frac{8s^{-1}}{s^{2\alpha} - as^{\alpha} - b}. \end{aligned} \tag{23}$$

Using the **Lemma 3.2.3**, the exact solution of this problem can be obtained as:

$$y(x) = 8x^{2\alpha} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{b^n\ a^k\ \binom{n+k}{k}\ x^{\alpha k + 2\alpha n}}{\Gamma(\alpha k + 2(n+1)\alpha + 1)}. \tag{24}$$

For case two, alpplying the Laplace transform, one has

$$\begin{aligned} &s^{2\alpha} F(s) - a\, s^{\alpha} F(s) - bF(s) = \frac{8}{s}, \\ &F(s) = \frac{8s^{-1}}{s^{2\alpha} - as^{\alpha} - b}. \end{aligned}$$

Which the solution is similar to case $0 < \alpha \le \frac{1}{2}$ when $y(0) = y'(0) = 0$

Example 4.2. Consider the following system of fractional algebraic-differential equations

$$\begin{aligned} &D^{\alpha} x(t) - tD^{\alpha} y(t) + x(t) - (1+t)y(t) = 0, \qquad 0 < \alpha \le 1 \\ &y(t) - \sin t = 0, \end{aligned} \tag{25}$$

subject to the initial conditions $x(0) = 1$, $y(0) = 0.$ (26)

Using the Laplace transform, $F(s) = L[y(t)]$ and $G(s) = L\,[x(t)]$ is obtained as follows

$$\frac{sG(s)-1}{s^{1-\alpha}} + \alpha s^{\alpha-1}F(s) + s^{\alpha}F'(s) + G(s) - F(s) + F'(s) = 0\,,$$
$$F(s) = \frac{1}{s^2+1}, \quad F'(s) = \frac{-2s}{(s^2+1)^2} \tag{27}$$

If we multiplying the above equation by $s^{1-\alpha}$ then we have

$$G(s) = \frac{1}{s^{1-\alpha}+s} + F(s)\frac{s^{1-\alpha}-\alpha}{s+s^{1-\alpha}} - F'(s)\,.$$

Now applying the inverse Laplace transform, one has

$$x(t) = L^{-1}\left\{\frac{1}{s+s^{1-\alpha}}\right\} - L^{-1}\{F'(s)\} + L^{-1}\left\{F(s)\frac{s^{1-\alpha}-\alpha}{s+s^{1-\alpha}}\right\}.$$

According to **Lemma 3.2.1**, we have $L^{-1}\left\{\frac{1}{s+s^{1-\alpha}}\right\} = E_{\alpha}\left(-t^{\alpha}\right).$ Also by defining $H(s) = \frac{s^{1-\alpha}-\alpha}{s+s^{1-\alpha}}$, we can write $L^{-1}\left\{F(s)\frac{s^{1-\alpha}-\alpha}{s+s^{1-\alpha}}\right\} = L^{-1}\{F(s)H(s)\} = \int_0^t y(t-x)h(x)dx$ where $h(t) = L^{-1}\{H(s)\}$.

$$h(t) = L^{-1}\{H(s)\} = L^{-1}\left\{\frac{s^{1-\alpha}-\alpha}{s+s^{1-\alpha}}\right\} = L^{-1}\left\{\frac{s^{1-\alpha}}{s+s^{1-\alpha}} - \frac{\alpha}{s+s^{1-\alpha}}\right\} = L^{-1}\left\{\frac{1}{1+s^{\alpha}}\right\} - \alpha L^{-1}\left\{\frac{1}{s+s^{1-\alpha}}\right\}$$
$$= t^{\alpha-1}E_{\alpha,\alpha}\left(-t^{\alpha}\right) - \alpha E_{\alpha}\left(-t^{\alpha}\right).$$

Therefore $x(t)$ can be obtained as follows

$$x(t) = E_{\alpha}\left(-t^{\alpha}\right) + 2\cos t + \int_0^t \sin(t-x)\left(x^{\alpha-1}E_{\alpha,\alpha}\left(-x^{\alpha}\right) - \alpha E_{\alpha}\left(-x^{\alpha}\right)\right)dx$$

exact solution for $\alpha = 1$ is $x(t) = t\sin t + e^{-t}$. (28)

Example 4.3. Consider the following Volterra sigular integral equation

$$D_x^{\alpha} f(x) = \exp(-ax) + \lambda\int_x^{\infty} J_0\left(2\sqrt{x-t}\right)f(t)dt, f(0) = 0, a < 0, 0 < \alpha \le 1$$

Solution: Taking the Laplace transform to the above integral equation leads to

$$s^{\alpha}F(s)=\frac{1}{s+a}+\lambda\left(\frac{-e^{\frac{1}{s}}}{s}\right)F(s)$$

which implies

$$F(s)=\frac{\frac{1}{s+a}}{\frac{\lambda e^{\frac{1}{s}}}{s}+s^{\alpha}}$$

so we have

$$F(s)=\frac{s+a-a}{s+a}.\frac{1}{\lambda e^{\frac{1}{s}}+s^{\alpha+1}}$$

$$=\frac{1}{\lambda e^{\frac{1}{s}}+s^{\alpha+1}}-\frac{a}{s+a}.\frac{1}{\lambda e^{\frac{1}{s}}+s^{\alpha+1}}=F_1(s)+F_2(s),0<\alpha\leq 1,\lambda\in\Re.$$

$F_1(s)$ has a branch point at $s=0$. Since $F_1(s)$ has no poles on the real negative semi axis, we can use the well-known Tichmarch theorem. $F_2(s)$ has a branch point at $s=0$ and has a simple pole at $s=-a$ too, so that it depends on a sign of a.

Now

$$f_1(x)=L^{-1}\{F_1(s)\}=\frac{1}{\pi}\int_0^{\infty}e^{-xt}\operatorname{Im}F_1^{-}(t)dt,$$

$$F_1^{-}(t)=F_1\left(te^{-i\pi}\right)=\frac{\lambda e^{-\frac{1}{t}}-t^{\alpha+1}\cos\pi\alpha-it^{\alpha+1}\sin\pi\alpha}{\left(\lambda e^{-\frac{1}{t}}-t^{\alpha+1}\cos\pi\alpha\right)^2+\left(t^{\alpha+1}\sin\pi\alpha\right)^2},$$

so

$$\operatorname{Im}F_1^{-}(t)=\frac{-t^{\alpha+1}\sin\pi\alpha}{t^{2\alpha+2}+\lambda e^{-\frac{1}{t}}\left(\lambda e^{-\frac{1}{t}}-2t^{\alpha+1}\cos\pi\alpha\right)}.$$

This implies to

$$f_1(x)=\frac{1}{\pi}\int_0^{\infty}\frac{e^{-xt}\left(-t^{\alpha+1}\sin\pi\alpha\right)}{t^{2\alpha+2}+\lambda e^{-\frac{1}{t}}\left(\lambda e^{-\frac{1}{t}}-2t^{\alpha+1}\cos\pi\alpha\right)}dt,$$

since

$$\frac{\sin \pi\alpha}{\pi} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}.$$

So

$$f_1(x) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^{\infty} \frac{-t^{\alpha+1} e^{-xt}}{t^{2\alpha+2} + \lambda e^{-\frac{1}{t}}\left(\lambda e^{-\frac{1}{t}} - 2t^{\alpha+1} \cos \pi\alpha\right)} dt.$$

$$F_2(s) = \frac{a}{s+a} . \frac{1}{\lambda e^{\frac{1}{s}} + s^{\alpha+1}}$$

$$f_2(x) = L^{-1}\{F_2(s)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{a e^{ts}}{(s+a)\left(\lambda e^{\frac{1}{s}} + s^{\alpha+1}\right)} ds,$$

Since the sign of a is negative,

$$Re\, s\left(F_2(s), s=-a\right) = \frac{a}{\lambda e^{\frac{-1}{a}} + (-a)^{\alpha+1}},$$

Since

$$\int_{AB} F_2(s)ds = \int_{BDE} F_2(s)ds + \int_{EH} F_2(s)ds + \int_{HJK} F_2(s)ds + \int_{KL} F_2(s)ds + \int_{LNA} F_2(s)ds .$$
$$= 2\pi i\, Re\, s\left(F_2(s), s=-a\right)$$

Infact the complete integral are evaluated with the help of Cauchy residue theorem and the Jordan lemma, so that according to the Jordan lemma see Fig. 1, we have

$$\int_{BDE} F_2(s)ds = \int_{AJK} F_2(s)ds = \int_{LNA} F_2(s)ds = 0,$$

and

$$\int_{EH} F_2(s)ds = -\int_R^{\varepsilon} \frac{e^{-xt}}{(a-x)\left(\lambda e^{\frac{1}{x}} + (-x)^{\alpha+1}\right)} dx$$

$$= \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_R^{\varepsilon} \frac{e^{-xt}}{(x-a)\left(\lambda e^{\frac{1}{x}} + (-x)^{\alpha+1}\right)} dx$$

$$= -\int_0^{\infty} \frac{e^{-xt}\left(\lambda e^{-\frac{1}{x}} - x^{\alpha+1}\cos \pi\alpha + i x^{\alpha+1} \sin \pi\alpha\right)}{(x-a)\left(\lambda^2 e^{-\frac{2}{x}} + x^{2\alpha+2} - 2\lambda e^{-\frac{1}{x}} x^{\alpha+1} \cos \pi\alpha\right)} dx$$

$$\int_{KL} F_2(s)ds = -\int_0^\infty \frac{e^{-xt}\left(\lambda e^{-\frac{1}{x}} - x^{\alpha+1}\cos\pi\alpha + ix^{\alpha+1}\sin\pi\alpha\right)}{(x-a)\left(\lambda^2 e^{-\frac{2}{x}} + x^{2\alpha+2} - 2\lambda e^{-\frac{1}{x}} x^{\alpha+1}\cos\pi\alpha\right)} dx.$$

so

$$f_2(x) = L^{-1}\{F_2(s)\} = 2\int_0^\infty \frac{e^{-xt}\left(\lambda e^{-\frac{1}{x}} - x^{\alpha+1}\cos\pi\alpha + ix^{\alpha+1}\sin\pi\alpha\right)}{(x-a)\left(\lambda^2 e^{-\frac{2}{x}} + x^{2\alpha+2} - 2\lambda e^{-\frac{1}{x}} x^{\alpha+1}\cos\pi\alpha\right)} dx + \frac{a}{\lambda e^{\frac{-1}{a}} + (-a)^{\alpha+1}}.$$

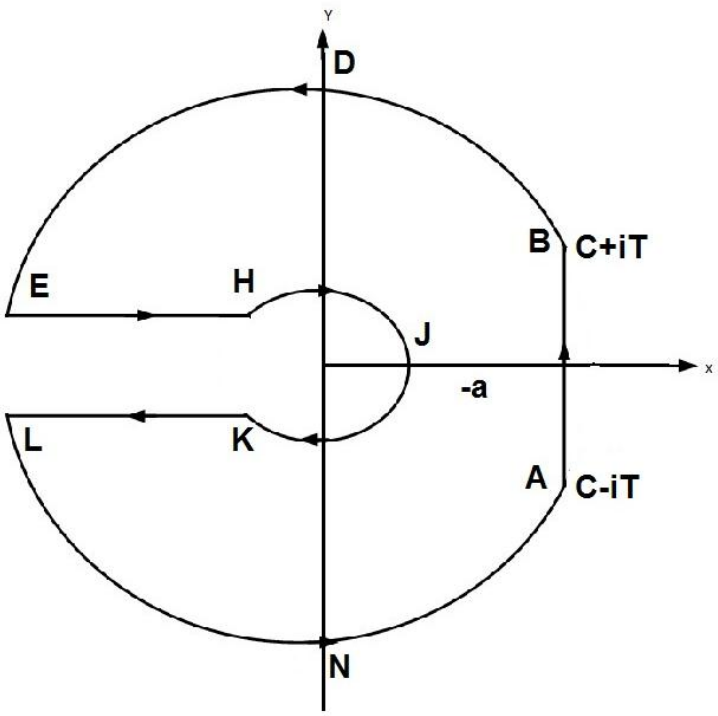

Figure 1.

5. CONCLUSIONS

The Laplace transform is a powerful tool in applied mathematics and engineering and have been applied for solving linear differential equations. In this paper, the application of Laplace transform is investigated to obtain an exact solution of some linerar fractional differential equations. The fractional derivatives are described in the Caputo sense which obtained by Riemann-Liouville fractional integral operator. Solving some problems show that the Laplace transform is a powerful and efficient techniques for obtaining analytic solution of linerar fractional differential equations.

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