Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 9 (2017), pp. 5213-5224

http://www.ripublication.com

Submersion of Semi-invariant Submanifolds of Contact Manifolds

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Abstract

In this paper, we discuss submersion of semi-invariant submanifolds of contact manifolds and derive some results on its geometry. We also derive some curvature relations.

Keywords: Semi-invariant submanifold, almost contact manifold, Submersion.

AMS Subject Classification: 53C40, 53C32.

1. INTRODUCTION

The field of submersion has been enriched by works of many authors. Papaghuice [1] studied the submersion of semi-invariant submanifolds of a Sasakian manifold whereas Kobayashi [2] studied submersion of $CR-$submanifolds and obtained interesting results. Motsumoto et al. [3] studied the submersion of semi-invariant submanifolds of trans-Sasakian manifold. Submersion of $CR-$submanifolds of nearly trans-Sasakian manifold were studied by Mohammed Jamali and Mohammad Hasan Shahid [4]. If the characteristic vector field ξ, belongs to the $(k,\mu)-$nullity distribution, then a contact metric manifold is called $(k,\mu)-$contact metric manifold. In 1995, Blair, Koufogiorgas and Papantonion [5] introduced the notion of $(k,\mu)-$contact metric manifold and a full classification of such a manifold is given by E. Boeckx (2000) [6]. A $(k,\mu)-$contact metric manifold with $\mu=0$ is called $N(k)$-

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contact metric manifold [12]. Our aim in this paper is to study the submersion of semi-invariant submanifolds of contact manifolds.

2. PRELIMINARIES

Let $\bar{M}$ be an almost contact metric manifold [7] with an almost contact metric structure (ϕ,ξ,η,g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a $1-$form and g is a compatible Riemannian metric such that

(2.1) $\phi^2 = -I + \eta \otimes \xi,$

(2.2) $\eta(\xi) = 1, \quad \phi(\xi) = 1, \quad \eta o \phi = 0,$

(2.3) $g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y),$

(2.4) $g(X, \phi Y) = -g(\phi X, Y), \quad g(X,\xi) = \eta(X),$

for all $X,Y \in T\bar{M}$. It is known that on a contact metric manifold, N^3 vanishes if and only if ξ is a killing vector field. A contact metric structure for which ξ is a killing vector field is called a $K-$contact metric manifold. As the tensor N^3 satisfies many important properties in a contact metric manifold, we define a tensor field h by $h = \frac{1}{2} Ł_{\xi}\phi = \frac{1}{2} N^3$, where $Ł_{\xi}$ denotes the operator of Lie-differentiation. The tensor field h is symmetric and satisfies the properties

(2.5) $h\xi = 0, \quad h\phi + \phi h = 0, \quad \nabla\xi = -\phi - \phi h, \quad Trace h = Trace \phi h = 0,$

where ∇ is Levi-Civita connection.

The $(k,\mu)-$nullity distribution on a contact metric manifold $\bar{M}(\phi,\xi,\eta,g)$ is defined as [5]

$$N(k,\mu): p \to N_P(k,\mu) = \{Z \in T_pM : R(X,Y)Z$$
$$= k[g(Y,Z)X - g(X,Z)Y] + \mu[g(Y,Z)hX - g(X,Z)hY]\},$$

for any $X,Y \in T_p\bar{M}$. Hence if the characteristic vector field belongs to the $(k,\mu)-$ nullity distribution, then we have

(2.6) $$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],$$

where k,μ are constants. A contact metric manifold satisfying the equation (2.6) is called a $(k,\mu)-$contact metric manifold and the examples (for both Sasakian and non-Sasakian cases) of such manifolds are given in [5]. A contact metric manifold with ξ, belonging to the $(k,\mu)-$nullity distribution has been studied in [6],[8],[9],[10],[11]. In a $(k,\mu)-$contact metric manifold the following relations hold:

(2.7) $$h^2=(k-1)\phi^2,$$

(2.8) $$(\nabla_X\phi)Y=g(X+hX,Y)\xi-\eta(Y)(X+hX),$$

(2.9) $$(\nabla_X\eta)Y=g(X+hX,\phi Y).$$

In particular, if $\mu=0$ then, we obtain the condition of $k-$nullity distribution introduced by Tanno [12].

$$N(k):p\to N_P(k)=\{Z\in T_pM:R(X,Y)Z=k[g(Y,Z)X-g(X,Z)Y]\},$$

for any $X,Y\in T_p\bar{M}$. Also if the characteristic vector field ξ belongs to the $k-$nullity distribution, then we have

(2.10) $$R(X,Y)\xi=k[\eta(Y)X-\eta(X)Y].$$

A contact metric manifold with $\xi\in N(k)$, is called a $N(k)$-contact metric manifold. Thus a $N(k)$-contact metric manifold is a contact metric manifold satisfying the relation (2.10). In a $N(k)$-contact metric manifold equations (2.7), (2.8) and (2.9) also holds.

Let M be a $n-$dimensional isometrically immersed submanifold of $\bar{M}$ and tangent to ξ. Let g be the metric tensor field on M as well as the induced metric on $\bar{M}$.

Definition 2.1: A $m-$dimensional Riemannian submanifold M of a $(k,\mu)-$contact metric manifold $\bar{M}$ is called a semi-invariant submanifold if ξ is tangent to M and it is endowed with a pair of orthogonal differentiable distributions $(D,D^\perp)$, which satisfies

(1) $TM=D\oplus D^\perp\oplus\xi$, where $\oplus$ denotes the orthogonal direct sum,(2) the distribution $D_x:x\to D\subset T_xM$ is invariant under ϕ i.e $\phi D_x\subset D_x$ for each $x\in M$ (3) the orthogonal

complementary distribution $D^{\perp}: x \to D^{\perp} \subset T_x M$ of the distribution D on M is totally real i.e $\phi D^{\perp} \subset T_x^{\perp} M,$ where $T_x M$ and $T_x^{\perp} M$ are the tangent space and the normal space of M at x respectively.

Let the dimension of D (resp. $D^{\perp}$) be $2p$ (resp. q) where $2p+q=m-1.$ If $p=0$ (resp. $q=0$) the submanifold M becomes anti-invariant (resp. invariant) submanifold. A generic submanifold M satisfies $D^{\perp} = \dim T_x^{\perp} M.$ A submanifold is called proper if it is neither invariant nor anti-invariant. It is easy to see that any hypersurface to which the vector field ξ is tangent is a typical example of semi-invariant submanifold.

Where D and $D^{\perp}$ are the horizontal and vertical distribution respectively. Let $\bar{\nabla}$ (resp. ∇) be the covariant differentiation with respect to Levi-Civita connection on $\bar{M}$ (resp. M). The Gauss and Weingarten formulas for M are respectively given by

(2.11) $$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)$$

(2.12) and $$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}{}_X N,$$

for $X, Y \in TM,$ $N \in T^{\perp} M,$ where σ (resp. A) is the second fundamental form(resp. tensor) of M in $\bar{M}$ and $\bar{\nabla}$ denote the operator of the normal connection.

(2.13) $$g(\sigma(X,Y),N) = g(A_N X, Y).$$

The projection of TM to D and $D^{\perp}$ are denoted by h and v respectively i.e, for any $X \in TM,$ we have

(2.14) $$X = \sigma X + vX + \eta(X)\xi.$$

The normal bundle to M has the decomposition

(2.15) $$T^{\perp} M = \phi D^{\perp} \oplus n_1,$$

where $g(\phi D^{\perp}, n_1) = 0.$ For any $U \in T^{\perp} M,$ we put

(2.17) $$U = nU + mU,$$

where $nU \in \phi D^{\perp}, mU \in n_1.$ From the above equation we have

$$\phi U = \phi nU + \phi mU, U \in T^{\perp} M, \phi nU \in D^{\perp}, \phi mU \in n_1.$$

Definition 2.2: Let M be a semi-invariant submanifold of a $(k,\mu)-$contact metric manifold $\bar{M}$ and M' be an almost contact metric manifold with the almost contact metric structure (ϕ',ξ',η',g'). Assume that there is a submersion $\pi: M \to M'$ such that

(i) $D^{\perp} = \ker \pi_* : TM \to TM'$ is the tangent mapping to π

(ii) $\pi_* : D_p \oplus \{\xi\} \to T_{\pi(p)}M'$ is an isometry for each $p \in M$ which satisfies $\pi_* o\phi = \phi' o\pi_*; \eta = \eta' o\pi_*; \pi_*(\xi_p) = \xi'_{\pi(p)}$, where $T_{\pi(P)}M'$ denotes the tangent space of M' at $\pi(P)$.

A vector X on M is said to be basic if, $X \in D_p \oplus \xi$ and X is π-related to a vector field on M' i.e there exists a vector field $X_* \in TM'$ such that $\pi_*(X_p) = X_{*\pi(p)}$ for each $p \in M$. Note that, by condition (ii) of the above definition 2.2, we have that the structural vector field ξ is a basic vector field.

Lemma 2.3: Let X, Y be basic vector fields on M. Then

(i) $g(X,Y) = g'(X_*, Y_*) o\pi,$

(ii) the component $\sigma([X,Y]) + \eta([X,Y]\xi) = [X_*, Y_*],$

(iii) $[U, X] \in D^{\perp}$ for any $U \in D^{\perp},$

(iv) $\sigma(\nabla_X Y) + \eta(\nabla_X Y)\xi$ is a basic vector field corresponding to $\nabla^*{}_{X_*} Y_*$, where ∇^* denotes the Levi-Civita connection on M'.

For basic vector fields on M, we define the operator $\tilde{\nabla}^*$ corresponding to ∇^* by setting $\tilde{\nabla}^*_X Y = \sigma([X,Y]) + \eta([X,Y]\xi)$ for $X, Y \in (D_p \oplus \{\xi\})$. By(iv) of Lemma 2.3, $\nabla^*{}_X Y$ is a basic vector field and we have

(2.17) $$\pi_*\left(\tilde{\nabla}^*_X Y\right) = \nabla^*{}_{X_*} Y_*.$$

Define the tensor field C by

(2.18) $$\nabla_X Y = \tilde{\nabla}^*_X Y + C(X,Y),$$

$X, Y \in (D_p \oplus \{\xi\})$, where $C(X,Y)$ is the verticle part of $\nabla_X Y$. It is known that C is skew-symmetric and satisfies

(2.19) $$C(X,Y)=\frac{1}{2}v[X,Y],$$

where $X,Y\in(D_p\oplus\{\xi\})$.

The curvature tensor R, R^* of the connection ∇, ∇^* on M and M' respectively are related by

(2.20) $$\begin{aligned}R(X,Y,Z,W)&=R^*(X_*,Y_*,Z_*,W_*)-g(C(Y,Z),C(X,W))\\&+g(C(X,Z),C(Y,W))+2g(C(X,Y),C(Z,W)),\end{aligned}$$

for $X,Y,Z,W\in(D_p\oplus\{\xi\})$, where $\pi_*X=X_*,\pi_*Y=Y_*,\pi_*Z=Z_*$ and $\pi_*W=W_*\in\chi M'$. For $(k,\mu)-$contact metric manifold we prove:

Proposition 2.4: Let $\pi:M\to M'$ be a submersion of semi-invariant submanifold of a $(k,\mu)-$contact metric manifold $\bar{M}$ on to almost contact metric manifold M'. Then we have

(2.21) $$(\tilde{\nabla}^*_X\phi)Y=g(X+hX,Y)\xi-\eta(Y)(X+hX),$$

(2.22) $$C(X,\phi Y)=\phi n\sigma(X,Y),$$

(2.23) $$\phi C(X,Y)=n\sigma(X,\phi Y),$$

(2.24) $$\phi mh(X,\phi Y)=m\sigma(X,\phi Y),$$

Proof: For any $X,Y\in(D_p\oplus\{\xi\})$ and by using Gauss formula (2.11), decomposition equation (2.16) and (2.18), we obtain

(2.25) $$\begin{aligned}\bar{\nabla}_XY&=\nabla_XY+\sigma(X,Y)=\nabla_XY+n\sigma(X,Y)+m\sigma(X,Y)\\&=\tilde{\nabla}^*_XY+C(X,Y)+n\sigma(X,Y)+m\sigma(X,Y).\end{aligned}$$

And

(2.26) $$\phi\bar{\nabla}_XY=\phi\tilde{\nabla}^*_XY+\phi C(X,Y)+\phi n\sigma(X,Y)+\phi m\sigma(X,Y).$$

Putting $Y=\phi Y$ in equation (2.25) ,we get

(2.27) $$\tilde{\nabla}_X\phi Y=\tilde{\nabla}^*_X\phi Y+C(X,\phi Y)+n\sigma(X,\phi Y)+m\sigma(X,\phi Y).$$

Using the definition of (k,μ)-contact metric manifold, we find

(2.28) $$(\tilde{\nabla}^*_X\phi)Y=\tilde{\nabla}_X\phi Y-\phi\tilde{\nabla}_XY=g(X+hX,Y)\xi-\eta(Y)(X+hX).$$

Substituting (2.26) and (2.27) in (2.28) ,we get

(2.29)
$$\begin{aligned}&\tilde{\nabla}^*_X\phi Y + C(X,\phi Y) + n\sigma(X,\phi Y) + m\sigma(X,\phi Y) - \phi\tilde{\nabla}^*_X Y - \phi C(X,Y)\\ &-\phi n\sigma(X,Y) - \phi m\sigma(X,Y) = g(X+hX,Y)\xi - \eta(Y)(X+hX).\end{aligned}$$

Comparing the components of $(D_p \oplus \{\xi\})$, $D^\perp$, $\phi D^\perp$ and n_1 respectively on both sides in the above equation, we get the required results.

Corollary (2.5) : Let $\pi : M \to M'$ be a submersion of semi-invariant submanifold of a $N(k)$-contact metric manifold $\bar{M}$ on to almost contact metric manifold M'. Then we have

(i) $$\left(\tilde{\nabla}^*_X\phi\right)Y = g(X+hX,Y)\xi - \eta(Y)(X+hX),$$

(ii) $$C(X,\phi Y) = \phi n\sigma(X,Y),$$

(iii) $$\phi C(X,Y) = n\sigma(X,\phi Y),$$

(iv) $$\phi mh(X,\phi Y) = m\sigma(X,\phi Y),$$

Proposition 2.6: Let $\pi : M \to M'$ be a submersion of semi-invariant submanifold of a $(k,\mu)-$contact metric manifold $\bar{M}$ on to almost contact metric manifold M'. Then M' is also a $(k,\mu)-$contact metric manifold.

Proof: From equation (2.21) of proposition (2.4) ,we have

(2.30) $$\left(\tilde{\nabla}^*_X\phi\right)Y = g(X+hX,Y)\xi - \eta(Y)(X+hX).$$

Applying π_* to the above equation and using Lemma 2.3, equation (2.17) and definition of submersion, we derive

(2.31) $$\left(\tilde{\nabla}^*_X\phi'\right)Y_* = g(X_* + h'X_*, Y_*)\xi - \eta'(Y_*)(X_* + h'X_*).$$

Hence M' is $(k,\mu)-$contact metric manifold.

Corollary 2.7: Let $\pi : M \to M'$ be a submersion of semi-invariant submanifold of a $N(k)$ -contact metric manifold $\bar{M}$ on to almost contact metric manifold M'. Then M' is also a $N(k)$-contact metric manifold.

Proposition 2.8: Let $\pi : M \to M'$ be a submersion of semi-invariant submanifold of a $(k,\mu)-$contact metric manifold on to almost contact metric manifold M'. Then

(i) $n\sigma(\phi X,\phi Y)+n\sigma(\phi X,Y)=0,$

(ii) $n\sigma(\phi X,\phi Y)=n\sigma(\phi X,Y),$

(iii) $m\sigma(\phi X,\phi Y)-m\sigma(X,Y),$

(iv) $C(\phi X,\phi Y)=C(X,Y),$ for any $X,Y\in(D_p\oplus\{\xi\})$.

Proof: (i) Interchaging X and Y in equation (2.23) gives

(2.32) $$\phi C(Y,X)=n\sigma(Y,\phi X)=n\sigma(X,\phi Y),$$

Then

$$n\sigma(X,\phi Y)+n\sigma(\phi X,Y)=\phi C(Y,X)+\phi C(X,Y)$$
$$=\phi C(Y,X)-\phi C(Y,X)=0.$$

(ii) Putting $X=\phi X$ in (2.23) , we get

(2.33) $$n\sigma(\phi X,\phi Y)=\phi C(\phi X,Y)=-\phi C\sigma(Y,\phi X)..$$

Using (2.22) in (2.33), we deduce

$$n\sigma(\phi X,\phi Y)=-\phi C(Y,\phi X)=-\phi(\phi n\sigma(Y,X))=\phi^2 n\sigma(Y,X).$$

$$=n\sigma(Y,X)-\eta(\sigma(X,Y))\xi=n\sigma(Y,X).$$

(iii) Putting $X=\phi X$ in(2.23) and using again the same equation, we find

$$m\sigma(\phi X,\phi Y)=\phi m\sigma(\phi X,Y)=\phi m\sigma(Y,\phi X)$$
$$=\phi^2 m\sigma(Y,X)=-m\sigma(X,Y).$$

(iv) Putting $X=\phi X$ in (2.22) and then using(2.23) yields

$$C(\phi X,\phi Y)=\phi n\sigma(\phi X,Y)=\phi n\sigma(Y,\phi X)$$
$$=\phi^2 C(Y,X)=-C(Y,X)+\eta(C(Y,X))\xi=C(X,Y).$$

Corollary 2.7: Let $\pi : M \to M'$ be a submersion of semi-invariant submanifold of a $N(k)$-contact metric manifold $\bar{M}$ on to almost contact metric manifold M'.Then we have

(i) $n\sigma(\phi X,\phi Y)+n\sigma(\phi X,Y)=0,$

(ii) $n\sigma(\phi X,\phi Y)=n\sigma(\phi X,Y)$,

(iii) $m\sigma(\phi X,\phi Y)-m\sigma(X,Y)$,

(iv) $C(\phi X,\phi Y)=C(X,Y)$, for any $X,Y\in(D_p\oplus\{\xi\})$.

3. CURVATURE RELATION

Proposition 3.1. Let $\pi:M\to M'$ be a submersion of semi-invariant submanifold of a $(k,\mu)-$contact metric manifold $\bar{M}$ on to almost contact metric manifold M'. Then the $\phi-$sectional curvature of $\bar{M}$ of M′ are related by

$$\bar{B}(X,Y)=B'(X_*,Y_*)-2\|n\sigma(X,Y)\|^2-2\|n\sigma(X,\phi Y)\|^2$$
$$-2(n\sigma(X,X),n\sigma(Y,Y))+2\|m\sigma(X,Y)\|^2,$$

where $X,Y\in(D_p\oplus\{\xi\})$.

Proof: We know

$$\bar{B}(X,Y)=\bar{R}(X,\phi X,\phi Y,Y).$$

Putting $Y=\phi X,Z=\phi Y,W=Y$ in Gauss equation

$$\bar{R}(X,Y,Z,W)=R(X,Y,Z,W)-g(\sigma(X,W),\sigma(Y,Z))$$
$$+g(\sigma(X,Z),\sigma(Y,W)),$$

we get

$$\bar{R}(X,\phi X,\phi Y,Y)=R(X,\phi X,\phi Y,Y)-g(\sigma(X,Y),\sigma(\phi X,\phi Y))$$
$$+g(\sigma(X,\phi Y),\sigma(\phi X,Y)).$$

Subsitituting $\sigma=n\sigma+m\sigma,$ in above equation, we get

$$\bar{R}(X,\phi X,\phi Y,Y)=R(X,\phi X,\phi Y,Y)-g(n\sigma(X,Y)+m\sigma(X,Y),n\sigma(\phi X,\phi Y)+m\sigma(\phi X,\phi Y))$$
$$+g(n\sigma(X,\phi Y)+m\sigma(X,\phi Y),n\sigma(\phi X,Y)+m\sigma(\phi X,Y)),$$
$$=R(X,\phi X,\phi Y,Y)-g(n\sigma(X,Y),n\sigma(\phi X,\phi Y))-g(n\sigma(X,Y),m\sigma(\phi X,\phi Y))$$
$$-g(m\sigma(X,Y),n\sigma(\phi X,\phi Y))-g(m\sigma(X,Y),m\sigma(\phi X,\phi Y))+g(n\sigma(X,\phi Y),n\sigma(\phi X,Y))$$
$$+g(n\sigma(X,\phi Y),m\sigma(\phi X,Y))+g(m\sigma(X,\phi Y),n\sigma(\phi X,\phi Y))+g(m\sigma(X,\phi Y),m\sigma(\phi X,Y)),$$

$$\bar{R}(X,\phi X,\phi Y,Y) = R(X,\phi X,\phi Y,Y) - g(n\sigma(X,Y),n\sigma(\phi X,\phi Y))$$
$$-g(m\sigma(X,Y),m\sigma(\phi X,\phi Y)) + g(n\sigma(X,\phi Y),n\sigma(X,\phi Y))$$
$$+g(\phi m\sigma(X,Y),\phi m\sigma(X,Y)),$$

(3.1) $$\bar{R}(X,\phi X,\phi Y,Y) = R(X,\phi X,\phi Y,Y) - \|n\sigma(X,Y)\|^2 + 2\|m\sigma(X,Y)\|^2 - \|n\sigma(X,\phi Y)\|^2,$$

Putting $Y = \phi X, Z = \phi Y, W = Y$ in equation (2.20) it follows

$$R(X,\phi X,\phi Y,Y) = R^*(X_*,\phi' X_*,\phi' Y_*,Y_*) - g(C(\phi X,\phi Y),C(X,Y))$$
(3.2) $$+g(C(X,\phi Y),C(\phi X,Y)) + 2g(C(X,\phi Y),C(\phi Y,Y)).$$

Applying ϕ on both side of equation (2.23), we obtain

(3.3) $$C(X,Y) = -\phi n\sigma(X,\phi Y).$$

Using equation (3.3) in equation (3.2) give

$$R(X,\phi X,\phi Y,Y) = R^*(X_*,\phi' X_*,\phi' Y_*,Y_*) - \|n\sigma(X,Y)\|^2$$
$$-\|n\sigma(X,\phi Y)\|^2 - 2g(n\sigma(X,X),n\sigma(Y,Y)).$$

Putting the value of $R(X,\phi X,\phi Y,Y)$ in (3.1) we obtain

$$\bar{R}(X,\phi X,\phi Y,Y) = R^*(X_*,\phi' X_*,\phi' Y_*,Y_*) - \|n\sigma(X,Y)\|^2$$
$$-\|n\sigma(X,\phi Y)\|^2 - 2g(n\sigma(X,X),n\sigma(Y,Y)) - \|n\sigma(X,Y)\|^2$$
$$+2\|m\sigma(X,Y)\|^2 - \|n\sigma(X,\phi Y)\|^2,$$

or

$$\bar{B}(X,Y) = B'(X_*,Y_*) - 2\|n\sigma(X,Y)\|^2 - 2\|n\sigma(X,\phi Y)\|^2$$
$$-2g(n\sigma(X,X),n\sigma(Y,Y)) + 2\|m\sigma(X,Y)\|^2.$$

Corollary 3.2. Let $\pi: M \to M'$ be a submersion of semi-invariant submanifold of a $N(k)-$contact metric manifold $\bar{M}$ on to almost contact metric manifold M'. Then the $\phi-$sectional curvature of $\bar{M}$ of M' are related by

$$\bar{B}(X,Y) = B'(X_*,Y_*) - 2\|n\sigma(X,Y)\|^2 - 2\|n\sigma(X,\phi Y)\|^2$$
$$-2g(n\sigma(X,X), n\sigma(Y,Y)) + 2\|m\sigma(X,Y)\|^2.$$

where $X,Y \in (D_p \oplus \{\xi\})$.

Proposition 3.3. Let $\pi: M \to M'$ be a submersion of semi-invariant submanifold of a $(k,\mu)-$contact metric manifold $\bar{M}$ on to almost contact metric manifold M'. Then the $\phi-$sectional curvature of $\bar{M}$ of M′ are related by

$$\bar{H}(X) = H'(X_*) - 4\|n\sigma(X,X)\|^2 + 2\|m\sigma(X,X)\|^2,$$

where $X,Y \in (D_p \oplus \{\xi\})$.

Proof: Putting $X = Y$ in equation (3.4) we obtain

$$\bar{B}(X,X) = \bar{H}(X) = H'(X_*) - 2\|n\sigma(X,X)\|^2 - 2\|n\sigma(X,\phi X)\|^2$$
$$-2g(n\sigma(X,X), n\sigma(X,X)) + 2\|m\sigma(X,X)\|^2$$
$$= H'(X_*) - 4\|n\sigma(X,X)\| - 2\|n\sigma(X,\phi X)\|^2$$
$$+2\|m\sigma(X,X)\|^2$$

Putting $Y = X$ in (2.23), we get

$$n\sigma(X,\phi X) = \phi C(X,X) = 0.$$

Thus we get

$$\bar{H}(X) = H'(X_*) - 4\|n\sigma(X,X)\|^2 + 2\|m\sigma(X,X)\|^2.$$

Corollary 3.4. Let $\pi: M \to M'$ be a submersion of semi-invariant submanifold of a $N(k)$-contact metric manifold $\bar{M}$ on to almost contact metric manifold M'. Then the

$\phi-$-sectional curvature of $\bar{M}$ of M' are related by

$$\bar{H}(X)=H'(X_*)-4\|n\sigma(X,X)\|^2+2\|m\sigma(X,X)\|^2,$$

where $X,Y\in(D_p\oplus\{\xi\})$.

ACKNOWLEDGEMENT

The first author is thankful to University Grants Commission, New Delhi, India for financial support in the form UCG- Dr D. S. Kothari Post Doctoral Fellowship.

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