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The Dynamics and Optimal Control of a Prey-Predator System

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Abstract

In this paper, we consider a two dimensional continuous prey- predator of first order differential equations. All the possible its equilibria and their local stability are investigated. This autonomous system is then extended to an optimal control model by proposing a control variable, which reduces the risk of extinction of the prey population. Thus the goal of this control is to reduce the number of the predator density. For this aim the Pontryagin's maximum principle has been applied. We also establish the necessary conditions and the characterization for the optimal solutions. Finally, some numerical simulations are presented to support the theoretical conclusions.

Key words: prey-predator, Equilibrium points, Local Stability, optimal control.

1. INTRODUCTION

Modeling the dynamics of a biological ecology systems is the most useful way to understand the complexity of nature. In particular, studying the interaction among biological species or the growth of population. After the famous Lotka-Volterra prey-predator model [14,15] , there are many mathematical models which describe competition and population dynamics. The authors have been studied in details the dynamical behavior of models, see [5,7,9,12,16,17,20,23]. An excellent book reference for prey-predator models is written by Turchin [24], see also Murray [19] and Hasting [10] . In these mathematical models , the ordinary differential equations, the difference equations and the partial differential equations are mostly used to capture the variety of the biological life. Some of authors modeled with stage or age structure for more detail see [1,2,15,21 ,22 ,26] . More realistic model, Beddington-De Angelis, and Crowley-Martin as well as Holling suggest types of functional responses for different species to describe the phenomena of predation [3, 8,11].

The classical continuous two dimension prey-predator system is given by

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(t) f(x_1(t)) - b_1 y_1(t) g(x_1(t)) \\ \frac{dy_1}{dt} &= -r_2 y_1(t) + b_2 y_1(t) g(x_1(t)) \end{aligned} \tag{1.1}$$

Where $x_1(t)$, $y_1(t)$, r_2, b_1, and b_2 are the prey density, the predator density at time t , the predator's death rates, the maximum per capita killing rate, and the conversion rate of predator respectively. The $g(x_1(t))$ is called predator functional response which means the number of the prey which is killed by predator. In this work I consider the functions response based on the traditional function Holling type I [11]. In this model the function $f(x_1(t)) = x_1(t)(1 - \frac{x_1(t)}{k})$ will be taken where the parameter r_1 stands as the growth rates of the prey population , and the constant k is the carrying capacity, the limit population size. So that the system (1.1) becomes

$$\begin{aligned} \frac{dx_1}{dt} &= r_1 x_1^2(t)(1 - \frac{x_1}{k}) - b_1 y_1(t) x_1(t) \\ \frac{dy_1}{dt} &= -r_2 y_1(t) + b_2 y_1(t) x_1(t) \end{aligned} \tag{1.2}$$

In this paper, we give some conditions for the existence and local stability of the feasible equilibira of system. It is then extended to an optimal problem. Here the main goal of the optimality is to protect the prey population from extincting by reducing the density of the predator populations. Pontryagin's maximum principle is used to achieve the optimality. The necessary conditions and the characterization for the optimal control are derived.

This paper is organized as follows: In Section 2, the dynamic behavior of the equilibria is analyzed. In Section 3, optimal control strategy is given, in Section 4 Numerical simulation support the theoretical findings. Finally a general discussion of the model and conclusions are given in section 5.

2. ANALYSIS OF EQUILIBRIA

By using a simple transformation $x(t) = \frac{x_1(t)}{k}$ and $y(t) = y_1(t)$ the system (1.2) becomes

$$\begin{aligned} \frac{dx(t)}{dt} &= a_1 x^2(t)(1 - x(t)) - b_1 y(t)x(t) \\ \frac{dy_1}{dt} &= -r_2 y_1(t) + a_2 y(t)x(t) \end{aligned} \tag{2.1}$$

Where $a_1 = kr_1, a_2 = kb_2$. In order to get the equilibria of the system (2.1) one has to solve the following algebraic equations

$$\begin{aligned} &a_1 x^2(1 - x) - b_1 yx = 0 \\ &-r_2 y_1 + a_2 yx = 0 \end{aligned}$$

After a simple calculating we get this lemma

Lemma 1

1- $E_0 = (0,0)$, the trivial equilibrium is always exists.

2- $E_1 = (1,0)$, the boundary equilibrium is always exists.

3- $E_2 = (x^*, y^*) = (\frac{r_2}{a_2}, \frac{a_1 x^*(1 - x^*)}{b_1})$, the unique positive equilibrium exists if $0 < x^* < 1$

To study the local behavior of the system (2.1) at each equilibrium one needs to compute the Jacobian matrix. This is given by

$$J((x, y)) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \qquad \text{where}$$

$$\begin{aligned} J_{11} &= 2a_1 x - 3a_1 x^2 - b_1 y \\ J_{12} &= -b_1 x \\ J_{21} &= a_2 y \\ J_{22} &= -r_2 + a_2 x \end{aligned}$$

The characteristic polynomial of J can be written as

$f(\lambda)=\lambda^2+P\lambda+Q$ where $P=-trac(J)$ and $Q=\det(J)$. Because of the zero is always root of the characteristic polynomial at E_0. Thus E_0 is always non-hyperbolic points. The local stability of other equilibria is given by the next lemma

Lemma2

1-For the boundary point E_1 we have

i- $E_1=(1,0),$ is sink point if $a_2<r_2$

ii- $E_1=(1,0),$ is saddle point if $a_2>r_2$

iii- $E_1=(1,0),$ is non-hyperbolic point if $a_2=r_2$

2-The unique positive point E_2 is

i- E_2 is sink point if $\frac{1}{2}<\frac{r_2}{a_2}<1$

ii- E_2 is source point if $0<\frac{r_2}{a_2}<\frac{1}{2}$

iii- E_2 is non-hyperbolic point if $a_2=2r_2$

Proof:-

For (1), the roots of the characteristic polynomial are $\lambda_1=-a_1,$ and $\lambda_2=-r_2+a_2$ So that all results can easily be obtained.

For (2) the Jacobian matrix J at E_2

$$J(E_2)=\begin{bmatrix} a_1x^*-2a_1(x^*)^2 & -b_1x^* \\ \frac{a_2a_1(x^*-(x^*)^2)}{b_1} & 0 \end{bmatrix} \qquad (2.2)$$

So that $P(E_2) = -a_1 x^* + 2a_1 (x^*)^2$,and $Q(E_2) = a_2 a_1 (x^*)^2 (1 - x^*)$. Then according to the Routh criterion[18], the result for (i) is completed. Since the roots of the characteristic polynomial at E_2 are

$$\lambda_{1,2} = -\frac{1}{2}[-a_1 x^* + 2a_1 (x^*)^2 \mp \sqrt{(-a_1 x^* + 2a_1 (x^*)^2)^2 - k_1}]$$

Here $k_1 = 4r_2 a_1 x^* (1 - x^*)$ is always positive number. Thus both of roots have positive real part when $0 < \frac{r_2}{a_2} < \frac{1}{2}$. It is clear that E_2 is non-hyperbolic point if $a_2 = 2r_2$

3. THE OPTIMAL CONTROL APPROACH:-

In this section an optimal control problem is investigated for the system (2.1), which is given by

$$\begin{aligned} \frac{dx(t)}{dt} &= a_1 x^2 (t)(1 - x(t)) - b_1 y(t) x(t) + (1 - u(t)) y(t) \\ \frac{dy_1}{dt} &= -r_2 y_1 (t) + a_2 y(t) x(t) - u(t) y(t) \end{aligned} \tag{3.1}$$

the state variables are $x(t)$, and $y(t)$ the prey population density ,and the predator population density respectively. All parameters $a_1, a_2, b_1,$ and r_2 are constants as mention before. The control variable is $u(t)$ such that $0 \le u(t) \le M < 1$, which represents the harvesting amount and M is the maximum harvesting (removing) amount. In this problem our aim is to reduce the number of predators in order to let more prey densities recover. This of course implies that reducing the risk of extincting of prey population. By the analog proof of paper of G. Zaman and S.H. Saker [27], the existences of the solution of system (3.1) is guaranteed. It is clearly that the effect of the harvesting amount on the prey population is positive while it is negative on the predator populations. It is well known that if the objective functional is linear with respect to the control variable then the optimal solution is a bang-bang solution. Here we will use quadratic term of control in the objective functional to penalize the amount of harvesting [13]. Thus the target functional is given by

$$J(u(t)) = \int_0^T (c_1 u(t) + c_2 u^2 (t)) dt \tag{3.2}$$

subject to the state equations which are given by (3.1) with the initial conditions $x(0) = x_0, y(0) = y_0$. Here c_1, c_2 are constants.

Now the optimal control problem is to find the function $u^*(t)$ that minimize the objective functional i.e

$J(u(t)) = \min J(u^*(t))$ for all $u(t) \in U$ where U is the set of controls .

To perform this the Pontryagin's maximum principle will be used [13]. So that we introduce the adjoint variables λ_1, λ_2 in the literature they are commonly called the shadow prices[6], as well as the Hamiltonian H

Which is defined as follows:

$$H(t, x(t), y(t), u(t)) = c_1 y(t) + c_2 u^2(t) + \lambda_1(t)[a_1 x^2(t)(1 - x(t)) - b_1 x(t) y(t) + (1 - u(t)) y(t)] + \lambda_2(t)[-r_2 y(t) + a_2 x(t) y(t) - u(t) y(t)]$$

The adjoint variables satisfy

$$\frac{d\lambda_1(t)}{dt} = \frac{\partial}{\partial x} H(t) = -\lambda_1(t)[2a_1 x(t) - 3a_2 x^2(t) - b_1 y(t)] - \lambda_2(t) a_2 y(t)$$

(3.3)

$$\frac{d\lambda_2(t)}{dy} = -\frac{\partial}{\partial y} H(t) = -c_1 - \lambda_1(t)[-b_1 x(t) + (1 - u(t))] - \lambda_2(t)[-r_2 + a_2 x(t) - u(t)]$$

$\lambda_i(T) = 0, \quad i = 1,2$ (The transversalty conditions)

Then we have the following theorem

Theorem 1

The optimal control $u^*(t)$ with corresponding solutions $x^*(t)$ and $y^*(t)$ are exist to minimize the objective functional $J(u(t))$ for all $u(t) \in U$. Furthermore

$$u^*(t)=\begin{cases} 0 & if \quad \frac{(\lambda_1+\lambda_2)y}{2c_2}\le 0 \\ \frac{(\lambda_1+\lambda_2)y}{2c_2} & if \quad 0<\frac{(\lambda_1+\lambda_2)y}{2c_2}<M \\ M & if \quad M<\frac{(\lambda_1+\lambda_2)y}{2c_2} \end{cases}$$

Proof: By using the Pontryagin's maximum principle. The necessary conditions are given by system (3.3), and the optimality condition is then

$$\frac{\partial H}{\partial u^*}=0\Rightarrow 2c_2u^*-(\lambda_1+\lambda_2)y=0$$

So that the result can directly obtained.

4. NUMERICAL APPROACH:-

In this section, numerical simulations are given to conform the above theoretical results. For the system (2.1) at different sets of parameter the local dynamical behavior of system is studied numerically. For the optimal problem we use numerical method which is described in [13], to find out the optimal solution with corresponding state solutions of the system.

The numerical optimal strategy will be outlined below through the following: First we define all parameters and guess an initial control u_{in} . Second we use the fourth order Range-Kutta method to solve the state system forward with u_{in}, and initial conditions as well as the adjoint system is solved backward with transversality conditions. Third we use a convex combination to update the controls in the previous iteration. Finally this process will be repeated until the values of variables at the last iteration are very close to the ones at current iterations.

We will choose $a_1=20, r_2=0.3, a_2=0.5$, and $b_1=1.3$. The trajectories of the prey population and the predator population are drawn in Figure 1 as a function of time. According to (Lemma 2), the unique positive equilibrium $(x^*,y^*)=(0.6,3.6923)$ will be defined and it is locally stable. Figure 2 illustrates this case. For the optimal control simulation we $a_1=15, r_2=0.2, a_2=0.3, b_1=0.1, c_1=0.5$

and $c_2 = 0.07$. Figures 3 and 4 show the effect of the harvesting on the prey density and the predator density respectively. Finally Figure 5 represents the control variable as a function of time.

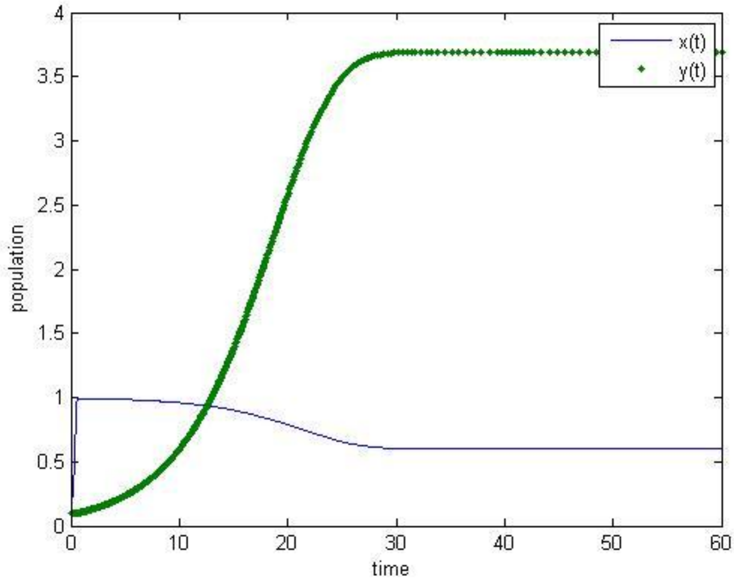

Figure 1: Time series of prey population and predator population of system(2.1). Parameters are $a_1 = 20, r_2 = 0.3, a_2 = 0.5$ and $b_1 = 1.3$.

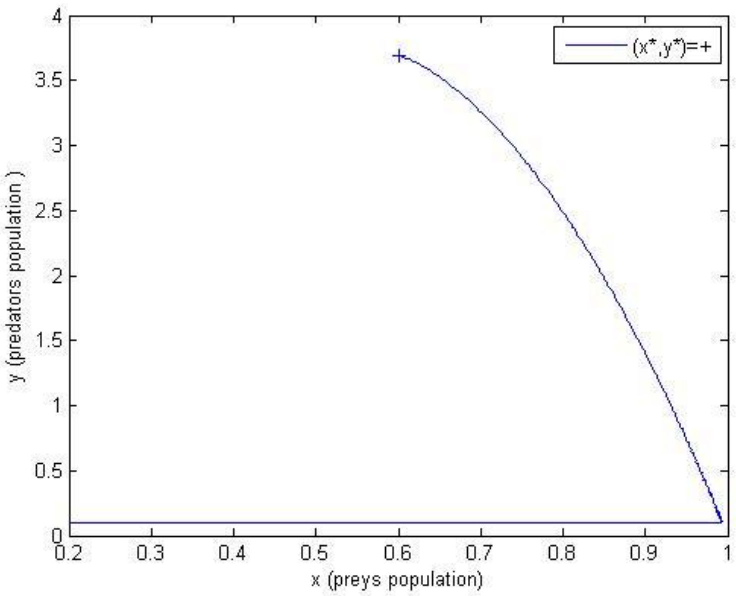

Figure2: local stability of $(x^*, y^*) = (0.6, 3.6923)$. Where (0.2,0.1) is the initial value. $a_1 = 20, r_2 = 0.3, a_2 = 0.5$ and , $b_1 = 1.3$.

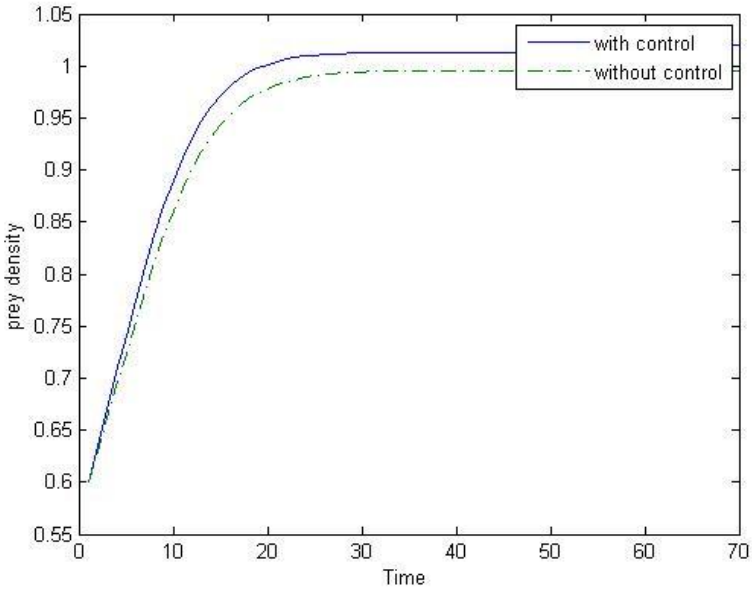

Figure 3: The plot illustrates the population of prey in both with control and without control. The parameters are $a_1 = 15, r_2 = 0.2, a_2 = 0.3, b_1 = 0.1, c_1 = 0.5$ and $c_2 = 0.07$

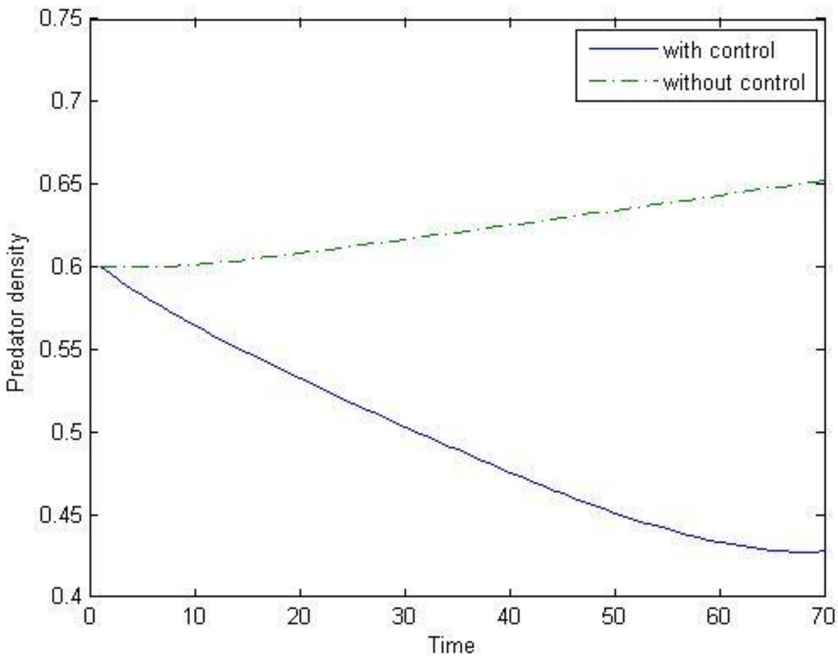

Figure 4: The predator density with control and without control is plotted . The parameters are $a_1 = 15, r_2 = 0.2, a_2 = 0.3, b_1 = 0.1, c_1 = 0.5$ and $c_2 = 0.07$

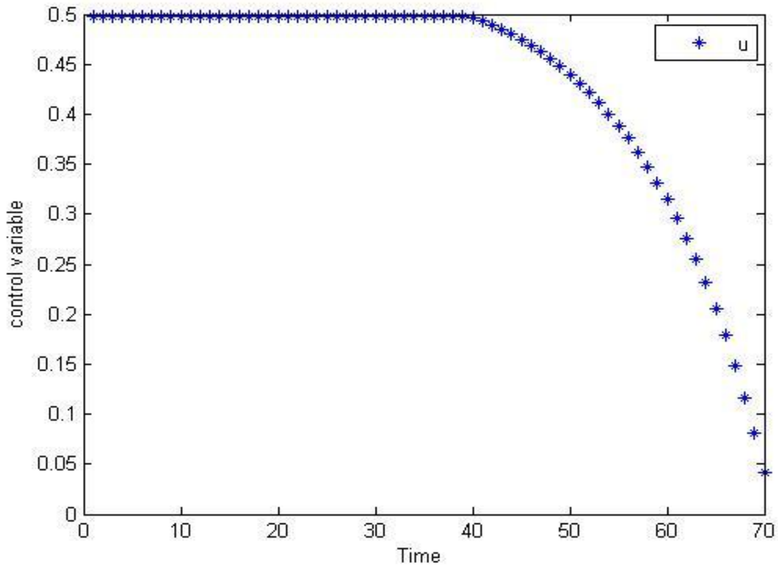

Figure 5:- The control u is plotted . where $c_1 = 0.5$ and $c_2 = 0.07$

5. DISCUSSION AND CONCLUSION

In previous sections a continuous time prey-predator model with Holling type I functional response has been discussed. This model has three equilibrium points, namely E_0, E_1 and E_2. This model exhibits a complex and interesting dynamical behavior. The positive equilibrium is defined when $\frac{r_2}{a_2} < 1$. Our analysis show that the trivial equilibrium point is always non-hyperbolic point while the E_1 can not be source point. The E_2 is never be saddle point. The optimal control and the Pontraygin's maximum principle have been applied to protect the prey population. The solution of the optimal control and the corresponding state solution are given numerically. This analysis put a new impact for studying of the prey-predator model system. One can also investigate this model, but with Holling type II or Holling type III or even any other functional response. The logistic equation for the predator population can also be used.

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