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Exact solution of some linear fuzzy fractional differential equation using Laplace transform method

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Abstract

In this paper we have given a procedure to find the exact solution of some linear fractional differential equation(FFDE) with fuzzy initial value using fuzzy Laplace transform method where the fractional differentiation has been considered in terms of Caputo-fractional derivative sense. The Laplace transform method is a powerful tool to find the exact solution of some initial value problems in the field of applied mathematics, physics and engineering. This method can be utilized to solve the different problems in modern field of mathematics like fuzzy fractional differential equation which may be treated as an important type of differential equation. A numerical example has been discussed to compare with our classical ordinary differential equation.

Keywords— Caputo Fractional differentiation, Fuzzy Fractional Differential Equation, Fuzzy Laplace transform.

I. INTRODUCTION

A long mathematical history has been studied so far in [9],[17],[19]. Fractional calculus generalizes the differentiation and integration to an arbitrary order. Fractional differential equations are of great importance in real life problems, since it generalizes our concept more precisely for better description of material properties. Recently fractional calculus has been utilized in development of models area like rehonology, viscoelasticity, electrochemistry, diffusion process etc in terms of fractional differential and fractional integrals[10],[19]. Some theoretical aspects of existence and uniqueness results for fractional differential equation have been considered recently by many authors [11].

A differential and integral calculus for fuzzy valued function was developed in some papers Hukuhara [23], Dubois and Prade [12-14]and Puri and Ralescu [21-22]. The significant results method of fuzzy differential equation and their application has been discussed in the papers [1],[2],[7],[24]. The concept of fuzzy fractional differential equation was introduced by Agarwal, Lakshikantham and Nieto [3]. The aim of this paper is to technique of findings the exact solution of some liner fractional differential equation with initial fuzzy value. Here the fractional differentiations have been assumed in terms of Caputo-fractional-derivative sense. Due to the lack of physical interpretation in initial condition in Riemann-Livullie differentiation, Caputo fractional derivative has been accepted here as it has much more realistic in initial condition of some physical problem [19] pages78-81. Here the obtained solution are expressed in form of Mittag-Leffler fuction, which has been discussed in [20] pages 16-36.

This paper is organized as follows: In section II, we introduce some basic concepts of fuzzy mathematics. The definition and notation of Caputo fractional derivatives and its Laplace transform has been shown in section III. In section IV, the solution of FFDEs are determined under Caputo fractional derivative using fuzzy Laplace transform. In section V, numerical example has been shown. Finally, conclusion and future research are drawn in Section VI.

II. BASIC CONCEPTS

Let E be the set of all upper semi-continuous normal convex fuzzy numbers with bounded r-level intervals. We define the r-level set, if $\tilde{u} \in E$

$\tilde{u}(r)$ ={x$\in$R: u(x)≥r}, 0≤r≤1 which is a closed interval and we denoted by [u(r)]= $[\underline{u}(r),\overline{u}(r)]$ and there exist a $x_0 \in$R such that u(x_0)=1. Two fuzzy numbers $\tilde{u}$ and $\tilde{v}$ are called equal $\tilde{u}=\tilde{v}$ iff u(x)=v(x) for all x$\in$R. it follows that $\tilde{u}=\tilde{v}$ iff [u(r)]=[v(r)] for all r$\in$(0,1).

It is clear that following statements are true

1. $\underline{u}(r)$ is a bounded left continuous non decreasing on [0,1]
2. $\overline{u}(r)$ is a bounded right continuous non decreasing on [0,1]
3. $\underline{u}(r) \le \overline{u}(r)$ for all r$\in$[0,1]

The following arithmetic operation on fuzzy numbers are well known and frequently used bellow.

If u,v $\in$ E, then $[\tilde{u}(r)+\tilde{v}(r)]=[\underline{u}(r)+\underline{v}(r),\bar{u}(r)+\bar{v}(r)]$

$$[\tilde{u}(r)-\tilde{v}(r)]=[\underline{u}(r)-\bar{v}(r),\bar{u}(r)-\underline{v}(r)]$$

And $[\lambda\tilde{u}(r)]=\lambda[\tilde{u}(r)]=\{\begin{matrix}[\lambda\underline{u}(r),\lambda\bar{u}(r)] if \lambda\geq 0\\ [\lambda\bar{u}(r),\lambda\underline{u}(r)] if \lambda<0\end{matrix} \quad \lambda\in R$

For a real interval I=[0,a], a mapping $\tilde{q}$:I $\to$ E is called a fuzzy function. We denote the r-cut representation of fuzzy valued function as $[\tilde{q}(t;r)]$ and defined by $[\tilde{q}(t;r)]=[\underline{q}(t;r),\bar{q}(t;r)]$, for t $\in$ I and r $\in$ (0,1]. The derivative of a fuzzy function $\tilde{q}(t;r)$ is given by $[\tilde{q}'(t;r)]=[\underline{q}'(t;r),\bar{q}'(t;r)]$, r $\in$ (0,1]. Provided that $q'(t)\in$ E. the fuzzy integral $\int_a^b\tilde{q}(t;r)dt$ is defined by $[\int_a^b\tilde{q}(t;r)dt]=[\int_a^b\underline{q}(t;r)dt\int_a^b\bar{q}(t;r)dt]$ provided that the integral on the right side exist.

III. CAPUTO DERIVATIVE AND ITS LAPLACE TRANSFORM

The Caputo–derivative of fractional order ‘α’ of a real valued function q(t) is defined as

$${}^{C}D^{\alpha}q(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{q'(\tau)}{(t-\tau)^{\alpha}}d\tau \text{ where } 0<\alpha\leq 1 \text{ and}$$

$${}^{C}D^{\alpha}q(t)=D^{\alpha}q(t)-\frac{t^{-\alpha}}{\Gamma(1-\alpha)}q(0)$$

Where $D^{\alpha}q(t)=\frac{1}{\Gamma(-\alpha)}\int_0^t\frac{q(\tau)}{(t-\tau)^{\alpha+1}}d\tau$

Let us consider the fuzzy fractional Caputo derivative of fuzzy valued function $\tilde{q}$ (t;r) based on the lower and upper function as following:

$${}^{C}D^{\alpha}\tilde{q}(t;r)=[{}^{C}D^{\alpha}\underline{q}(t;r),{}^{C}D^{\alpha}\bar{q}(t;r)], \text{ for } 0\leq r\leq 1$$

Where ${}^{c}D^{\alpha}\underline{q}(t;r)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\underline{q}'(\tau;r)}{(t-\tau)^{\alpha}}d\tau$ and ${}^{c}D^{\alpha}\bar{q}(t;r)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\bar{q}'(\tau;r)}{(t-\tau)^{\alpha}}d\tau$

The Laplace transform of Caputo fractional derivative of fuzzy valued function $\tilde{q}(t;r)$ is given by $L\{{}^{C}D^{\alpha}\tilde{q}(t;r)\} = s^{\alpha}L\{\tilde{q}(t;r)\} - s^{\alpha-1}\tilde{q}(0;r)$ like as the Laplace transform of the Caputo fractional derivative of a real valued function in[19].

IV. SOLUTION OF LINEAR FFDE

Let us now consider linear fuzzy fractional differential equation

${}^{C}D^{\alpha}\tilde{q}(t;r) = a\tilde{q}(t;r)$ +b , where 0<α≤1 , 0≤r≤1 and a, b are constants. (1.1)

Subject to the fuzzy initial condition; $\tilde{q}(0;r) = (\gamma_1, \gamma_2, \gamma_3)$

${}^{C}D^{\alpha}$ Denotes fractional derivative in Caputo sense.

Hence $\tilde{q}(t;r)$ is the solution to be determined through r-cut approach, we may write the given fuzzy fractional differential equation (1.1) as,

$$[{}^{C}D^{\alpha}\underline{q}(t;r), {}^{C}D^{\alpha}\overline{q}(t;r)] = a[\underline{q}(t;r), \overline{q}(t;r)] + \text{b} \quad (1.2)$$

Case i) let a>0. Then by using the definition of Hukuhara derivative we may write the equation (1.2) as

$$ {}^{C}D^{\alpha}\underline{q}(t;r) = a\underline{q}(t;r) + \text{b} \quad (1.3)$$

$$ {}^{C}D^{\alpha}\overline{q}(t;r) = a\overline{q}(t;r) + \text{b} \quad (1.4)$$

Subject to the initial condition

$$[\underline{q}(0;r), \overline{q}(0;r)] = [\gamma_1 + r(\gamma_2 - \gamma_1), \gamma_3 - r(\gamma_3 - \gamma_2)] \quad (1.5)$$

Adding (1.3) & (1.4) we get,

$$[{}^{C}D^{\alpha}\underline{q}(t;r) + {}^{C}D^{\alpha}\overline{q}(t;r)] = a[\underline{q}(t;r) + \overline{q}(t;r)] + 2\text{b} \quad (1.6)$$

Set u(t;r)= $\underline{q}(t;r) + \overline{q}(t;r)$, (1.6) can be written as

$$ {}^{C}D^{\alpha}u(t;r) = au(t;r) + 2\text{b} \quad (1.7)$$

With u(0;r)= $\gamma_1 + \gamma_3 + r(2\gamma_2 - \gamma_1 - \gamma_3)$ (1.8)

On Subtraction of (1.3) &(1.4) we get,

$$[{}^{C}D^{\alpha}\underline{q}(t;r) - {}^{C}D^{\alpha}\overline{q}(t;r)] = a[\underline{q}(t;r) - \overline{q}(t;r)] \quad (1.9)$$

Set v(t;r)= $\underline{q}(t;r) - \overline{q}(t;r)$, (1.9) can be written as ${}^{C}D^{\alpha}v(t;r) = av(t;r)$ (1.10

With v(0;r) = $\gamma_1 - \gamma_3 + r(\gamma_3 - \gamma_1)$

Taking Laplace transform of (1.7) we get,

$$s^{\alpha}\bar{u}(s) - s^{\alpha-1}u(0;r) = a\bar{u}(s) + \frac{2b}{s}$$

$$\Rightarrow \bar{u}(s) = \frac{s^{\alpha-1}}{s^{\alpha}-a}u(0;r) + \frac{2bs^{-1}}{s^{\alpha}-a}$$

Taking Laplace inverse of above equation by using Mettag-Leffler function $E_{\alpha,\beta}(t)$ we get $u(t;r) = u(0;r)E_{\alpha,1}(at^{\alpha}) + 2bt^{\alpha}E_{\alpha,\alpha+1}(at^{\alpha})$

$$=\{\gamma_1 + \gamma_3 + r(2\gamma_2 - \gamma_1 - \gamma_3)\}E_{\alpha,1}(at^{\alpha}) + 2bt^{\alpha}E_{\alpha,\alpha+1}(at^{\alpha}) \quad (1.11)$$

Again taking Laplace transform (1.10) we get,

$$s^{\alpha}\bar{v}(s) - s^{\alpha-1}v(0;r) = a\bar{v}(s) \quad \Rightarrow \bar{v}(s) = \frac{s^{\alpha-1}}{s^{\alpha}-a}v(0;r)$$

Taking Laplace inverse of above equation and using Mettag-Leffler function $E_{\alpha,\beta}(t)$ we get

$$v(t;r) = v(0;r)E_{\alpha,1}(at^{\alpha}) = \{\gamma_1 - \gamma_3 + r(\gamma_3 - \gamma_1)\}E_{\alpha,1}(at^{\alpha}) \quad (1.12)$$

$$\therefore \underline{q}(t;r) = \frac{1}{2}[u(t;r) + v(t;r)] = \{\gamma_1 + r(\gamma_2 - \gamma_1)\}E_{\alpha,1}(at^{\alpha}) + bt^{\alpha}E_{\alpha,\alpha+1}(at^{\alpha}) \quad (1.13)$$

And $\bar{q}(t;r) = \frac{1}{2}[u(t;r) - v(t;r)] = \{\gamma_3 - r(\gamma_3 - \gamma_2)\}E_{\alpha,1}(at^{\alpha}) + bt^{\alpha}E_{\alpha,\alpha+1}(at^{\alpha})$ (1.14)

Hence the required solution is $\tilde{q}(t;r) = [\underline{q}(t;r), \bar{q}(t;r)]$, $\underline{q}(t;r)$ & $\bar{q}(t;r)$ are given by (1.13) & (1.14).

Case ii) let a<0, then from (1.2) we get ,

$$D^{\alpha}\underline{q}(t;r) = a\bar{q}(t;r) + b \quad (1.15)$$

$$D^{\alpha}\bar{q}(t;r) = a\underline{q}(t;r) + b \quad (1.16)$$

Subject to the initial condition (1.5).

On addition of (1.15) & (1.16) we get, $[D^{\alpha}\underline{q}(t;r) + D^{\alpha}\bar{q}(t;r)] = a[\bar{q}(t;r) + \underline{q}(t;r)]$ +2b (1.17)

Writing u(t;r)= $\underline{q}(t;r) + \bar{q}(t;r)$, we get from (1.17) $D^{\alpha}u(t;r) = au(t;r)$ +2b (1.18)

With the initial condition u(0;r)= $\gamma_1+\gamma_3+r(2\gamma_2-\gamma_1-\gamma_3)$

On subtraction of (1.15) & (1.16) we get, $[D^{\alpha}\underline{q}(t;r)-D^{\alpha}\overline{q}(t;r)]=a[\overline{q}(t;r)-\underline{q}(t;r)]$ (1.19)

Writing v(t;r)= $\underline{q}(t;r)-\overline{q}(t;r)$, we get from (1.19) $D^{\alpha}v(t;r)=-av(t;r)$ (1.20)

With the initial condition v(0;r)= $\gamma_1-\gamma_3+r(\gamma_3-\gamma_1)$

Taking Laplace transform of (1.18) we get,

$$s^{\alpha}\overline{u}(s)-s^{\alpha-1}u(0;r)=a\overline{u}(s)+\frac{2b}{s}$$

$$\Rightarrow \overline{u}(s)=\frac{s^{\alpha-1}}{s^{\alpha}-a}u(0;r)+\frac{2bs^{-1}}{s^{\alpha}-a}$$

Taking Laplace inverse of above equation by using Mettag-Leffler function $E_{\alpha,\beta}(t)$ we get

$$u(t;r)=u(0;r)E_{\alpha,1}(at^{\alpha})+2bt^{\alpha}E_{\alpha,\alpha+1}(at^{\alpha})$$

$$=\{\gamma_1+\gamma_3+r(2\gamma_2-\gamma_1-\gamma_3)\}E_{\alpha,1}(at^{\alpha})+2bt^{\alpha}E_{\alpha,\alpha+1}(at^{\alpha}) \quad (1.21)$$

Again taking Laplace transform (1.20) we get,

$$s^{\alpha}\overline{v}(s)-s^{\alpha-1}v(0;r)=-a\overline{v}(s)$$

$$\Rightarrow \overline{v}(s)=\frac{s^{\alpha-1}}{s^{\alpha}+a}v(0;r)$$

Taking Laplace inverse of above equation by using Mettag-Leffler function $E_{\alpha,\beta}(t)$ we get

$$v(t;r)=v(0;r)E_{\alpha,1}(-at^{\alpha})$$

$$=\{\gamma_1-\gamma_3+r(\gamma_3-\gamma_1)\}E_{\alpha,1}(-at^{\alpha}) \quad (1.22)$$

$$\therefore \underline{q}(t;r)=\frac{1}{2}[u(t;r)+v(t;r)]$$

Or, $\underline{q}(t;r)=\frac{1}{2}[\{\gamma_1+\gamma_3+r(2\gamma_2-\gamma_1-\gamma_3)\}E_{\alpha,1}(at^{\alpha})+\{(\gamma_1-\gamma_3)+r(\gamma_3-\gamma_1)\}$
$E_{\alpha,1}(-at^{\alpha})] \quad +bt^{\alpha}E_{\alpha,\alpha+1}(at^{\alpha})$ (1.23)

And $\overline{q}(t;r) = \frac{1}{2}[u(t;r) - v(t;r)] = \frac{1}{2}[\{\gamma_1 + \gamma_3 + r(2\gamma_2 - \gamma_1 - \gamma_3)\} E_{\alpha,1}(at^{\alpha}) -$

$\{(\gamma_1 - \gamma_3) + r(\gamma_3 - \gamma_1)\}\ E_{\alpha,1}(-at^{\alpha})] + bt^{\alpha} E_{\alpha,\alpha+1}(at^{\alpha})$ (1.24)

Hence the required solution is $\tilde{q}(t;r) = [\underline{q}(t;r), \overline{q}(t;r)]$, $\underline{q}(t;r)$ & $\overline{q}(t;r)$ are given by (1.23) & (1.24).

V. NUMERICAL EXAMPLE

Consider the fuzzy fractional differential equation as,

$D^{\frac{1}{2}}\tilde{q}(t) = \tilde{q}(t) + 1$, with initial condition $\tilde{q}(0)$ =(0.2,0.4,0.6)

Then $\tilde{q}(t) = [\underline{q}(t), \overline{q}(t)]$ will be $\underline{q}(t) = (0.2 + 0.2r)E_{\frac{1}{2}}(\sqrt{t}) + \sqrt{t}E_{\frac{1}{2},\frac{3}{2}}(\sqrt{t})$

And $\overline{q}(t) = (0.6 - 0.2r)E_{\frac{1}{2}}(\sqrt{t}) + \sqrt{t}E_{\frac{1}{2},\frac{3}{2}}(\sqrt{t})$

In the above result (1.23)&(1.24), if α=1, we have, $\underline{q}(t) = \overline{q}(t) = \gamma E_1(t) + tE_{1,2}(t)$ = $(\gamma + 1)e^t - 1$, which is exactly same as the solution of our classical ordinary differential equation $\frac{dq}{dt} = q(t) + 1$ with initial condition q(0)=γ.

VI. CONCLUSION

Procedure of finding exact solution of linear fuzzy fractional differential equation has been discussed here where the fractional derivative has been considered in terms of Caputo sense as it has much more realistic in initial condition in real life problems. This type of FFDE can be utilized to analyses the growth and decay model in more generalized way. For different values of the parameters in the given FFDE, we have some models that applied in class of linear differential dynamical system with fuzzy initial condition.

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