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# On Semi Generalized $\omega\alpha$-Closed Sets in Topological Spaces

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**Abstract**

The aim of this paper is to introduce the new class of closed sets called semi generalized $\omega\alpha$-closed (briefly sg$\omega\alpha$-closed) sets in topological spaces which is properly lies between the class of semi-closed sets and the class of gs-closed sets. Further we define sg$\omega\alpha$-closure and sg$\omega\alpha$-interior in topologicsl spaces and obtained some of their properties.

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## 1. Introduction

In the year 1963, Levine [11] introduced and investigated the weaker forms of open sets called semi-open sets. Later Biswas [6] introduced the notion of semi-closed set. The concept of generalized closed (briefly g-closed) sets as a generalization of closed set is defined by Levine [12] in 1970. Sundaram and Sheik John [19] introduced and studied the notion of $\omega$-closed sets in topological spaces. Recently Benchalli et al. [4] defined and studied the concept of $\omega\alpha$-closed sets in topological spaces.

The aim of this paper is to introduce the new weaker forms of closed sets called sg$\omega\alpha$-closed sets and studied the some of their charecterizations and also we define and study the sg$\omega\alpha$-closure and sg$\omega\alpha$-interior and some of their basic properties are investigated.

## 2. Preliminaries

Throughout this paper, the space $(X, \tau)$ (or simply X) always means a topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of a space (X, $\tau$), then cl(A), int(A) and $A^c$ denote the closure of A, the interior of A and the compliment of A in X respectively.

**Definition 2.1.** A subset A of a topological space X is called

(i) regular open [18] if A =int(cl(A)) and regular closed if A = cl(int(A)).

(ii) semi-open set [11] if A $\subseteq$ cl(int(A)) and semi-closed set if int(cl(A)) $\subseteq$ A.

(iii) pre-open set [16] if A $\subseteq$ int(cl(A)) and pre-closed set if cl(int(A))$\subseteq$ A.

(iv) $\alpha$-open set [17] if A $\subseteq$ int(cl(int(A))) and $\alpha$-closed set if cl(int(cl(A)))$\subseteq$ A.

(v) semi-pre open set [2] (= $\beta$-open [1]) if A $\subseteq$ cl(int(cl(A))) and semi-pre closed set [2] (= $\beta$-closed [1]) if int(cl(int(A))) $\subseteq$ A.

The intersection of all semi-closed (resp. semi-open) subsets of $(X, \tau)$ containing $A$ is called the semi-closure (resp. semi-kernel) of $A$ and by scl($A$) (resp.sker($A$)). Also the intersection of all preclosed (resp. semi-preclosed and $\alpha$-closed) subsets of $(X, \tau)$ containing $A$ is callled the pre-closure (resp. semi-preclosure and $\alpha$-closure) of $A$ and is denoted by pcl($A$) (resp. spcl($A$) and $\alpha$-cl($A$)).

**Definition 2.2.** A subset A of a topological space X is called a

(i) generalized closed (briefly g-closed) set [12] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X.

(ii) generalized semi-closed (briefly gs-closed) set [3] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X.

(iii) $\alpha$-generalized closed (briefly $\alpha$g-closed) set [14] if $\alpha$cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X.

(iv) generalized $\alpha$-closed (briefly g$\alpha$-closed) set [13] if $\alpha$cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $\alpha$-open in X.

(v) generalized pre-closed (briefly gp-closed) set [15] if pcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X.

(vi) generalized semi-preclosed (briefly gsp-closed) set [7] if spcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X.

(vii) generalized pre-regular-closed (briefly gpr-closed) set [8] if pcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is regular-open in X.

(viii) $\omega$-closed [19] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open in X.

(ix) g$^*$-closed set [20] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is g-open set in X.

(x) $\alpha$-generalized regular closed (briefly $\alpha$gr-closed) set [22] if $\alpha$cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is regular-open in X.

(xi) pre g$^*$-closed (briefly pg$^*$-closed)set [10] if pcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $\omega\alpha$-open in X.

(xii) g$^*$-preclosed (briefly g$^*$p-closed) set [21] if pcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is g-open set in X.

(xiii) $\omega\alpha$-closed [4] if $\alpha$cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $\omega$-open in X.

(xiv) generalized $\omega\alpha$-closed (briefly g$\omega\alpha$-closed) set [5] if $\alpha$cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $\omega\alpha$-open set in X.

**Definition 2.3.** A topological space $(X, \tau)$ is said to be semi-normal [3] if for each pair of disjoint semiclosed sets A and B of X, there exist disjoint open sets U and V in X such that A $\subseteq$ U and B $\subseteq$ V.

## 3. sg$\omega\alpha$-Closed Sets in Topological Spaces

In this section, we indroduce semi generalized $\omega\alpha$-closed (briefly sg$\omega\alpha$-closed) sets in topological spaces and obtained some of their properties.

**Definition 3.1.** A subset A of a topological space $(X, \tau)$ is called semi generalized $\omega\alpha$-closed (briefly sg$\omega\alpha$-closed) set if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $\omega\alpha$-open in X. We denote the set of all sg$\omega\alpha$-closed sets in $(X, \tau)$ by SG$\omega\alpha(X, \tau)$.

**Theorem 3.2.** Every closed set is sg$\omega\alpha$-closed.

*Proof.* Let A be a closed and G be an $\omega\alpha$-open set containing A in X. Since A is closed, we have $cl(A) = A$. But scl(A) $\subseteq cl(A)$ is always true. So that scl(A) $\subseteq$ cl(A) $\subseteq$ G. Therefore scl(A) $\subseteq$ G. Hence A is sg$\omega\alpha$-closed set. The converse of the above theorem need not be true as seen from the following example. ■

**Example 3.3.** Let $X$ = {a, b, c } and $\tau$ = {X, $\phi$, {a},{b, c}}. Then the set {a,b} is sg$\omega\alpha$-closed but not a closed set in X.

**Theorem 3.4.** Every $\alpha$-closed set is sg$\omega\alpha$-closed.

*Proof.* Let A be an $\alpha$-closed set and G be an $\omega\alpha$-open set in X such that A $\subseteq$ G. Since A is $\alpha$-closed, we have $\alpha$cl(A) = A. But scl(A) $\subseteq$ $\alpha$cl(A) is always true. So that scl(A) $\subseteq$ $\alpha$cl(A) $\subseteq$ G. Therefore scl(A) $\subseteq$ G. Hence A is sg$\omega\alpha$-closed set. The converse of the above theorem need not be true as seen from the following example. ■

**Example 3.5.** In Example 3.3, the set {a,b} is sg$\omega\alpha$-closed but not an $\alpha$-closed set in X.

**Theorem 3.6.** Every semi-closed set is sg$\omega\alpha$-closed but not conversely.

*Proof.* Let A be semi-closed set and G be an $\omega\alpha$-open set in X such that A $\subseteq$ G. Since A is semi-closed, we have scl(A) = A $\subseteq$ G. Therefore scl(A) $\subseteq$ G. Hence A is sg$\omega\alpha$-closed set.

**Example 3.7.** In Example 3.3, the set {a,b} is sg$\omega\alpha$-closed but not semi-closed in X.

**Theorem 3.8.** Every g$\omega\alpha$-closed set is sg$\omega\alpha$-closed.

*Proof.* Let A be g$\omega\alpha$-closed set and G be an $\omega\alpha$-open set in X such that A $\subseteq$ G. Since A is g$\omega\alpha$-closed, we have $\alpha$cl(A) $\subseteq$ G. But scl(A) $\subseteq$ $\alpha$cl(A) is always true. So that scl(A) $\subseteq$ G. Therefore scl(A) $\subseteq$ G. Hence A is sg$\omega\alpha$-closed set. The converse of the above theorem need not be true as seen from the following example. ■

**Example 3.9.** Let $X$ = {a, b, c } and $\tau$ = {X, $\phi$, {a}, {b}, {a,b}}. Then the set {b} is sg$\omega\alpha$-closed but not g$\omega\alpha$-closed in X.

**Theorem 3.10.** Every sg$\omega\alpha$-closed set is gsp-closed.

*Proof.* Let A be sg$\omega\alpha$-closed set and G be an open set in X such that A $\subseteq$ G. Since every open set is $\omega\alpha$-open set and A is sg$\omega\alpha$-closed, we have scl(A) $\subseteq$ G. But spcl(A) $\subseteq$ scl(A) is always true. So that spcl(A) $\subseteq$ G. Hence A is gsp-closed. The converse of the above theorem need not be true as seen from the following example. ■

**Example 3.11.** Let $X$ = {a, b, c } and $\tau$ = {X, $\phi$, {a}}. Then the set {a, b} is gsp-closed but not sg$\omega\alpha$-closed in X.

**Theorem 3.12.** Every sg$\omega\alpha$-closed set is gs-closed (resp. wg-closed).

*Proof.* The proof follows from the definitions. The converse of the above theorem need not be true as seen from the following example. ■

**Example 3.13.** In Example 3.11, the set {a, b} is gs-closed (resp. wg-closed) but not sg$\omega\alpha$-closed in X.

**Remark 3.14.** The concept of sg$\omega\alpha$-closed set is independent of the concept of sets namely pre-closed, semi-preclosed, g-closed, gp-closed, $\alpha$g-closed, gpr-closed, $\alpha$gr-closed, $g^*$-closed, $g^*$p-closed, $\omega\alpha$-closed sets as seen from the following exmaples.

**Example 3.15.** In Example 3.11, the set A = {a, c} is $\omega\alpha$-closed, gp-closed, g-closed, $\alpha$gr-closed but not sg$\omega\alpha$-closed in X.

**Example 3.16.** Let X= {a, b, c, d} and $\tau$ = {X, $\phi$, {b, c}, {b, c, d}, {a, b, c}}. Then the set A = {b} is pre-closed, semi-preclosed but not sg$\omega\alpha$-closed in X.

**Example 3.17.** Let X= {a, b, c } and $\tau$ = {X, $\phi$, {a}, {a, c}}. Then the set A = {a, b} is g-closed, $g^*$-closed, $g^*$p-closed but not sg$\omega\alpha$-closed in X.

**Example 3.18.** In Example 3.9, the set A = {b} is sg$\omega\alpha$-closed but not pre-closed, semi-preclosed, g-closed, gpr-closed, $\alpha$gr-closed, $g^*$-closed, $g^*$p-closed, gp-closed, $\omega\alpha$-closed in X.

**Remark 3.19.** Union of two sg$\omega\alpha$-closed sets need not be a sg$\omega\alpha$-closed set as seen from the following example.

**Example 3.20.** In Example 3.9, the sets {a} and {b} are sg$\omega\alpha$-closed sets but their union {a} ∪ {b} = {a, b} is not a sg$\omega\alpha$-closed set in X.

**Theorem 3.21.** If a subset A of X is sg$\omega\alpha$-closed, then scl(A)-A does not contain any non empty $\omega\alpha$-closed set in $(X, \tau)$.

*Proof.* Suppose that A is sg$\omega\alpha$-closed set and F be a non empty $\omega\alpha$-closed subset of scl(A)-A. Then F ⊆ scl(A) ∩ (X-F). Since (X - F) is $\omega\alpha$-open and A is sg$\omega\alpha$-closed, scl(A) ⊆ (X - F). Therefore F ⊆ (X - scl(A)) Then F ⊆scl(A) ∩ (X - scl(A)) = $\phi$. That is F = $\phi$. Thus scl(A) - A does not contain any non-empty $\omega\alpha$-closed set in $(X, \tau)$.

The converse of the above theorem need not be true as seen from the following example. ■

**Example 3.22.** In Example 3.16, the set A = {a, b}, then scl(A) - A = {c, d} does not contain non empty $\omega\alpha$-closed set. But A is not sg$\omega\alpha$-closed set in $(X, \tau)$.

**Theorem 3.23.** If a subset A of a topological space X is sg$\omega\alpha$-closed such that A ⊆ B ⊆ scl(A), then B is also sg$\omega\alpha$-closed.

*Proof.* Let U be an $\omega\alpha$-open set in X such that B ⊆ U, then A ⊆ U. Since A is sg$\omega\alpha$-closed, scl(A) ⊆ U. By hypothesis scl(B) ⊆ scl(scl(A)) = scl(A) ⊆ U. Consequently, scl(B) ⊆ U. Therefore B is also sg$\omega\alpha$-closed set in $(X, \tau)$. The converse of the above theorem need not be true as seen from the following example. ■

**Example 3.24.** In Example 3.3, the set A = {a} and B = {a, b} such that A and B are sg$\omega\alpha$-closed sets but A ⊆ B ⊈scl(A).

**Theorem 3.25.** If A is open and gs-closed set, then A is sg$\omega\alpha$-closed set in X.

*Proof.* Let A be an open and gs-closed set in X. Let A ⊆ U and let U be $\omega\alpha$-open in X. Now A ⊆ A. By hypothesis scl(A) ⊆ A. Thst is scl(A) ⊆ U. Thus A is sg$\omega\alpha$-closed in X. ■

**Theorem 3.26.** If A is $\omega\alpha$-open and sg$\omega\alpha$-closed set, then A is semi-closed set in X.

*Proof.* Let A ⊆ A, where A is $\omega\alpha$-open. Then scl(A) ⊆ A as A is sg$\omega\alpha$-closed in X. But A ⊆ scl(A) is always true. Therefore A = scl(A). Hence A is semi-closed in X. ■

**Theorem 3.27.** For each $x \in X$, either x is $\omega\alpha$-closed or $x^c$ is sg$\omega\alpha$-closed in X.

*Proof.* Suppose {x} is not $\omega\alpha$-closed in X, then $x^c$ is not $\omega\alpha$-open and the only $\omega\alpha$-open set containing $x^c$ is the space X itself. Therefore scl($x^c$) ⊆ X, and thus $x^c$ is sg$\omega\alpha$-closed in X. ■

**Theorem 3.28.** If A is a sg$\omega\alpha$-closed set in X and A ⊆ Y ⊆ X, then A is a sg$\omega\alpha$-closed set relative to Y.

*Proof.* Let A ⊆ Y ∩ G where G is an $\omega\alpha$-open set in X. Then A ⊆ Y and A ⊆ G. Since A is sg$\omega\alpha$-closed set in X, so scl(A) ⊆ G, which implies that Y ∩ scl(A) ⊆ Y ∩ G. Hence A is sg$\omega\alpha$-closed set relative to Y. ■

**Theorem 3.29.** Let $(X, \tau)$ be a $s$-normal space and if Y is sg$\omega\alpha$-closed subset of X, then the subspace Y is $s$-normal spcae.

*Proof.* If $G_1$ and $G_2$ are disjoint semi-closed sets in X such that $(Y \cap G_1) \cap (Y \cap G_2) = \phi$. Then $Y \subseteq (G_1 \cap G_2)^c$ and $(G_1 \cap G_2)^c$ is $\omega\alpha$-open and Y is sg$\omega\alpha$-closed in X. Therefore $scl(Y) \subseteq (G_1 \cap G_2)^c$ and hence $(scl(Y) \cap G_1) \cap (scl(Y) \cap G_2) = \phi$. Since X is s-normal space, there exists disjoint open sets A and B such that $scl(Y) \cap G_1 \subseteq A$ and $scl(Y) \cap G_2 \subseteq B$ such that $Y \cap G_1 \subseteq Y \cap A$ and $Y \cap G_2 \subseteq Y \cap B$. Hence Y is s-normal space. ■

**Theorem 3.30.** A regular open, sg$\omega\alpha$-closed is semi-closed and hence clopen.

*Proof.* Let A be a regular open sg$\omega\alpha$-closed. Since regular open set is $\omega\alpha$-open [4], scl(A) ⊆ A. This implies A is semi-closed(regular) open setis (regular) closed. Hence A is clopen. ■

**Theorem 3.31.** Let $(X, \tau)$ be a topological space. Then if X is hyperconnected if and only if every subset of X is sg$\omega\alpha$-closed and X is connected.

*Proof.* Necessity: Let X be hyperconncted. Then by Jankovic [9], the only regular open subsets of X are trivial ones. Hence every subset of X is sg$\omega\alpha$-closed. Also every hyperconnected space is trivially connected.

Sufficiently: Let A be a non-void proper regular open subset X. Then A is sg$\omega\alpha$-closed, by hypothesis. From Theorem 3.30, it follows that A is clopen, which contradicts the hypothesis, since A is connected. Hence X is hyperconnected. ■

**Definition 3.32. [4]** The intersection of all $\omega\alpha$-open subsets of $(X, \tau)$ containing A is callled $\omega\alpha$-kernel of A and is denoted by $\omega\alpha$-ker(A).

**Theorem 3.33.** A subset A of $(X, \tau)$ is sg$\omega\alpha$-closed if and only if scl(A) $\subseteq$ $\omega\alpha$-ker(A).

*Proof.* Suppose that A is sg$\omega\alpha$-closed. Then scl(A) $\subseteq$ U, whenever A $\subseteq$ U and U is $\omega\alpha$-open. Let x $\in$ scl(A). If x $\notin$ $\omega\alpha$-ker(A), then there is a $\omega\alpha$-open set U containing A such that x $\notin$ U. Since U is $\omega\alpha$-open set containing A. We have x $\notin$ scl(A). This is a contradiction. Hence scl(A) $\subseteq$ $\omega\alpha$-ker(A). Conversely, let scl(A) $\subseteq$ $\omega\alpha$-ker(A). If U is only $\omega\alpha$-open set containing A, then scl(A) $\subseteq$ $\omega\alpha$-ker(A) $\subseteq$ U. Hence A is sg$\omega\alpha$-closed. ■

**Theorem 3.34.** Let A be sg$\omega\alpha$-closed in $(X, \tau)$, then A is semi-closed if and only if scl(A) - A is $\omega\alpha$-closed.

*Proof.* Necessity: Suppose A is semi-closed, then scl(A) = A and so scl(A) - A = $\phi$. Which is $\omega\alpha$-closed.

Sufficiently: Suppose scl(A) -A is $\omega\alpha$-closed. Then scl(A) - A = $\phi$, Since A is sg$\omega\alpha$-closed, that is scl(A) = A or A is semi-closed. ■

Now we introduce the following.

**Definition 3.35.** A subset A of a topological space $(X, \tau)$ is called semi generalized $\omega\alpha$-open (briefly sg$\omega\alpha$-open) set if its complement $A^c$ is sg$\omega\alpha$-closed set in X.

**Theorem 3.36.** A subset A of a topological space X is sg$\omega\alpha$-open, then F $\subseteq$ sint(A) whenever F is $\omega\alpha$-closed in $(X, \tau)$.

*Proof.* Assume that A is sg$\omega\alpha$-open. Then $A^c$ is sg$\omega\alpha$-closed. Let F be a $\omega\alpha$-closed set in X contained in A. Then $F^c$ is $\omega\alpha$-open set containing $A^c$ in (X, $\tau$). Since $A^c$ is sg$\omega\alpha$-closed, this implies that scl(A) $\subseteq$ $F^c$. Taking complements on both sides, we have F $\subseteq$ sint(A). ■

**Theorem 3.37.** A subset A is sg$\omega\alpha$-open in $(X, \tau)$ then G = X, whenever G is $\omega\alpha$-open and sint(A) $\cup$ (X - A) $\subseteq$ G.

*Proof.* Let A be sg$\omega\alpha$-open, G be $\omega\alpha$-open and sint(A) $\cup$ (X-G) $\subseteq$ G. This gives X-

G ⊆ (X-sint(A)) ∩ (X-(X-A)) = (X-sint(A))-(X-A) = scl(X-A)-(X-A). Since X-A is sg$\omega\alpha$-closed and X-G is $\omega\alpha$-closed. Then by Theorem 3.21, it follows that X-G = $\phi$. Therefore X = G. ■

**Theorem 3.38.** Let A and B be any two subsets of X. If sint(A) ⊆ B ⊆ A and A is sg$\omega\alpha$-open in $(X, \tau)$ then B is a sg$\omega\alpha$-open set in $(X, \tau)$.

*Proof.* By hypothesis, sint(A) ⊆ B ⊆ A, then (X-A) ⊆ (X-B) ⊆ X-sint(A) = scl(X-A). Since A is sg$\omega\alpha$-open set, then (X-A) is sg$\omega\alpha$-closed, By Theorem 3.23, (X-B)is sg$\omega\alpha$-closed set in X. Therefore sg$\omega\alpha$-open in X. ■

## 4. sg$\omega\alpha$-Closure and sg$\omega\alpha$-Interior

In this section, the notion of sg$\omega\alpha$-closure and sg$\omega\alpha$-interior in topological spaces are defined and some of its properties are studied.

**Definition 4.1.** For a subset A of $(X, \tau)$, sg$\omega\alpha$-closure of A is denoted by sg$\omega\alpha$cl(A) and is defined as sg$\omega\alpha$cl(A) = ∩ {G: A ⊆ G, G is sg$\omega\alpha$-closed in $(X, \tau)$}.

**Definition 4.2.** For a subset A of $(X, \tau)$, sg$\omega\alpha$-interior of A is denoted by sg$\omega\alpha$int(A) and is defined as sg$\omega\alpha$int(A) = ∪ {G: G ⊆ A, G is sg$\omega\alpha$-open in $(X, \tau)$}. That is, sg$\omega\alpha$int(A) is the union of all sg$\omega\alpha$-open sets contained in A.

**Theorem 4.3.** For any x ∈ X, x ∈ sg$\omega\alpha$cl(A) if and only if A ∩ V $\neq \phi$ for every sg$\omega\alpha$-open set V containing x.

*Proof.* Let x ∈ sg$\omega\alpha$cl(A). Suppose there exists a sg$\omega\alpha$-open set V containing x such that V ∩ A = $\phi$. Then A ⊆ X-V, sg$\omega\alpha$cl(A) ⊆ X-V. This implies x ∉ sg$\omega\alpha$cl(A) which is a contradiction. Hence V ∩ A $\neq \phi$. Conversely Suppose x ∉ sg$\omega\alpha$cl(A), then there exists sg$\omega\alpha$-closed set G containing A such that x ∉ G. Then x ∈ X-G is sg$\omega\alpha$-open. Also (X-G) ∩ A = $\phi$. Which is contradiction to the hypothesis. Hence x ∈ sg$\omega\alpha$cl(A).■

**Theorem 4.4.** If A ⊆ X then, A ⊆ sg$\omega\alpha cl(A)$ ⊆ cl(A).

*Proof.* Since every closed set is sg$\omega\alpha$-closed, the proof follows. ■

**Remark 4.5.** Both containment relations in the Theorem 4.4 may be proper as seen from the following example.

**Example 4.6.** Let X= {a, b, c } and $\tau$ = {X, $\phi$, {a, b}}, the set A = {a} then sg$\omega\alpha$cl(A) = {a, c} and cl(A) = X and so A ⊆ sg$\omega\alpha$cl(A) ⊆ cl(A).

**Remark 4.7.** For any subset A of X, int(A) ⊆ sg$\omega\alpha int(A)$ ⊆ A.

**Theorem 4.8.** Let A be any subset of a space X, then

(i) sg$\omega\alpha$cl($\phi$) = $\phi$ and sg$\omega\alpha cl(X)$ = X.

(ii) sg$\omega\alpha cl(A)$ is a sg$\omega\alpha$-closed set in X.

*Proof.* The proof follows from the definitin 4.1. ■

**Theorem 4.9.** Let A and B be any two subsets of a space X, then the following properties are true

(i) A is sg$\omega\alpha$-closed set if and only if sg$\omega\alpha$cl(A) = A.

(ii) A is sg$\omega\alpha$-closed set in X, then sg$\omega\alpha cl(A)$ is the smallest sg$\omega\alpha$-closed subset of X containing A.

(iii) If A $\subseteq$ B, then sg$\omega\alpha cl(A) \subseteq$ sg$\omega\alpha cl(B)$.

(iv) sg$\omega\alpha$cl(A $\cup$ B) = sg$\omega\alpha cl(A) \cup$ sg$\omega\alpha cl(B)$.

(v) sg$\omega\alpha$cl(A $\cap$ B) $\subseteq$ sg$\omega\alpha cl(A) \cap$ sg$\omega\alpha cl(B)$.

(vi) sg$\omega\alpha$cl(sg$\omega\alpha cl(A)$) = sg$\omega\alpha cl(A)$.

*Proof.*

(i) Let A be a sg$\omega\alpha$-closed set in X. Since A $\subseteq$ A and A $\in$ { F $\subseteq$ X: A $\subseteq$ F and F is sg$\omega\alpha$-closed set } which implies that A = $\cap$ { F $\subseteq$ X: A $\subseteq$ F and F is sg$\omega\alpha$-closed set } $\subseteq$ A.Then sg$\omega\alpha cl(A) \subseteq$ A. But A $\subseteq$ sg$\omega\alpha cl(A)$ is always true. Hence A = sg$\omega\alpha cl(A)$
Conversely, Suppose A = sg$\omega\alpha cl(A)$, then sg$\omega\alpha cl(A)$ is sg$\omega\alpha$-closed set in X which implies that A is sg$\omega\alpha$-closed set in X.

(ii) Suppose A is sg$\omega\alpha$-closed set in X , then sg$\omega\alpha cl(A)$ = $\cap$ { F $\subseteq$ X: A $\subseteq$ F and F is sg$\omega\alpha$-closed set }. Since A $\subseteq$ A and A is sg$\omega\alpha$-closed set in X, From (i), A = sg$\omega\alpha$cl(A) is the smallest sg$\omega\alpha$-closed set in space X containing A.

(iii) We know that B $\subseteq$ sg$\omega\alpha cl(B)$,then A $\subseteq$ B $\subseteq$ sg$\omega\alpha cl(B)$. So sg$\omega\alpha cl(B)$ is the sg$\omega\alpha$-closed set containing A. $\dashrightarrow$ (a).
But sg$\omega\alpha cl(A)$ is the smallest sg$\omega\alpha$-closed set containing A.$\dashrightarrow$(b).
From (a) and (b) we get, sg$\omega\alpha$cl(A) is smaller than sg$\omega\alpha$ cl(B).Therfore sg$\omega\alpha$cl(A) $\subseteq$ sg$\omega\alpha cl(B)$

(iv) We know that A $\subseteq$ A $\cup$ B and B $\subseteq$ A $\cup$ B,from (iii), sg$\omega\alpha$cl(A) $\subseteq$ sg$\omega\alpha$ cl(A $\cup$ B) and sg$\omega\alpha$cl(B) $\subseteq$ sg$\omega\alpha$ cl(A $\cup$ B).Then sg$\omega\alpha$cl(A) $\cup$ sg$\omega\alpha cl(B) \subseteq$ sg$\omega\alpha$ cl(A $\cup$ B) $\dashrightarrow$ (a).
Let x $\in$ X be any point such that x $\notin$ sg$\omega\alpha cl(A) \cup$ sg$\omega\alpha cl(B)$, then there exist sg$\omega\alpha$-closed sets G and H such that A $\subseteq$ G and B $\subseteq$ H, x $\notin$ G and x $\notin$ H, it implies that x $\notin$ G $\cup$ H and A $\cup$ B $\subseteq$ G $\cup$ H. By the Theorem 3.28, G $\cup$ H is sg$\omega\alpha$-closed, then x $\notin$ sg$\omega\alpha$cl(A $\cup$ B). Therefore sg$\omega\alpha$cl(A $\cup$ B) $\subseteq$ sg$\omega\alpha$cl(A) $\cup$ sg$\omega\alpha$cl(B) $\dashrightarrow$ (b).
From (a) and (b) we have , sg$\omega\alpha$cl(A $\cup$ B) = sg$\omega\alpha$cl(A) $\cup$ sg$\omega\alpha$cl(B).

(v) We know that A ∩ B ⊆ A and A ∩ B ⊆ B. From (iii) we get, sg$\omega\alpha$cl(A ∩ B) ⊆ sg$\omega\alpha$cl(A) and sg$\omega\alpha$ cl(A ∩ B) ⊆ sg$\omega\alpha$cl(B). Then sg$\omega\alpha$ cl(A ∩ B) ⊆ sg$\omega\alpha$cl(A) ∩ sg$\omega\alpha$cl(B).

(vi) From definition 4.1 we have, sg$\omega\alpha$cl(A) is a sg$\omega\alpha$-closed set in X. Let sg$\omega\alpha$cl(A) = G, then G is a sg$\omega\alpha$-closed set in X. From (i) we have, sg$\omega\alpha$cl(G) = G. Which implies that sg$\omega\alpha$cl(sg$\omega\alpha$cl(A)) = sg$\omega\alpha$cl(A). The converse of property (ii) and property (v) are need not be true as seen in the following examples. ■

**Example 4.10.** In the Example 3.11, the subset A = { a, c }, then sg$\omega\alpha cl(A)$ = X which is the smallest sg$\omega\alpha$-closed set in X containing A but A is not a sg$\omega\alpha$-closed set in X.

**Example 4.11.** Let X= {a, b, c } and $\tau$ = {X, $\phi$, {a,b}}, the subset A = {a} and B = {b}, then sg$\omega\alpha cl(A)$ = {a, c} and sg$\omega\alpha cl(B)$ = {b, c} and sg$\omega\alpha$cl(A ∩ B) = $\phi$. Hence { c } = sg$\omega\alpha$cl(A) ∩ sg$\omega\alpha$cl(B) ⊈ sg$\omega\alpha$cl(A ∩ B) = $\phi$.

**Remark 4.12.** The following example shows that for any two subsets A and B of X, A ⊆ B implies sg$\omega\alpha$cl(A) $\neq$ sg$\omega\alpha$cl(B)

**Example 4.13.** In the Example 3.17, the subset A = {a} and B = { a, c }, then A ⊆ B. Now sg$\omega\alpha cl(A)$ = {c} and sg$\omega\alpha cl(B)$ = X. Hence sg$\omega\alpha$cl(A) $\neq$ sg$\omega\alpha$cl(B).

**Remark 4.14.** For a subset A of (X, $\tau$), sg$\omega\alpha$cl(A) $\neq$ cl(A) as seen from the following example.

**Example 4.15.** In the Example 3.17, the subset A = {c} ⊆ X,sg$\omega\alpha$cl(A) = {c} and cl(A) = {b, c}. Therefore sg$\omega\alpha$cl(A) $\neq$ cl(A).

**Remark 4.16.** For any two subsets A and B of (X, $\tau$), sg$\omega\alpha$cl(A) = sg$\omega\alpha$cl(B) does not imply that A = B. This is shown by the following example.

**Example 4.17.** In the Example 3.11, the subset A = {a} and B = { a, c }, then sg$\omega\alpha cl(A)$ = sg$\omega\alpha cl(B)$ = X, but A $\neq$ B.

**Definition 4.18.** Let (X, $\tau$) be a topological space and let x ∈ X. A subset N of X is said to be sg$\omega\alpha$-neighborhood of point x ∈ X if there exist a sg$\omega\alpha$-open set G such that x ∈ G ⊆ N.

**Definition 4.19.** Let (X, $\tau$) be a topological space and A be a subset of X, A subset N of X is said to be sg$\omega\alpha$-neighborhood (briefly sg$\omega\alpha$-nbd) of A is if there exists a sg$\omega\alpha$-open set G such that A ∈ G ⊆ N.

The collection of all sg$\omega\alpha$-neighborhood of x ∈ X is called sg$\omega\alpha$-neighborhood system of x and shall be denoted by sg$\omega\alpha$N(x).

**Theorem 4.20.** A subset A of topological space X is sg$\omega\alpha$-open then it is a sg$\omega\alpha$-neighborhood of each of its points.

*Proof.* Let a subset G of a topological space X be sg$\omega\alpha$-open. Then for every x ∈ X, x ∈ G ⊆ G and therefore G is a sg$\omega\alpha$-neighborhood of each of its points. The converse of the above theorem need not be true as seen from the following example. ■

**Example 4.21.** In the Example 3.11,the subset A = { b, c} is sg$\omega\alpha$-neighborhood of each of its points b and c, but A is not sg$\omega\alpha$-open.

**Theorem 4.22.** A subset A of topological space X is sg$\omega\alpha$-closed and x ∈ sg$\omega\alpha$cl(A) if and only if N ∩ A $\neq \phi$ for any sg$\omega\alpha$-nbd N of x in X.

*Proof.* If x ∉ sg$\omega\alpha$cl(A). Then there exist sg$\omega\alpha$-closed set F of X such that A ⊆ F and x ∉ F. Thus x ∈ (X - F) and (X - F) is sg$\omega\alpha$-open in X. But A ∩ (X-F) = $\phi$, which is a contradicton. Hence x ∈ sg$\omega\alpha$cl(A). Conversely, suppose that there exist a sg$\omega\alpha$-nbd N of a point x ∈ X such that N ∩ A = $\phi$. Then there exist an sg$\omega\alpha$-open set F of X such that x ∈ F ⊆ N. Therefore we have F ∩ A = $\phi$ and A ∈ (X-F). Then sg$\omega\alpha$cl(A) ⊆ (X-F) and x ∉ sg$\omega\alpha$cl(A), which is a contradiction to hypothesis that x ∈ sg$\omega\alpha$cl(A). Therefore N ∩ A $\neq \phi$. ■

**Definition 4.23.** Let A be a subset of topological space X. Then a point x ∈ X is said to be a sg$\omega\alpha$-limit point of A if every sg$\omega\alpha$-open set of x contains a point of A other than x, that is [G - { x }] ∩ A $\neq \phi$ for every sg$\omega\alpha$-open set G of x.
In topological space X, the set of all sg$\omega\alpha$-limit points of a given subset A of X is called sg$\omega\alpha$-derived set of A and is denoted by sg$\omega\alpha$d(A).

**Theorem 4.24.** Let A and B be any two subsets of a space X, then the following properties are true:

(i) sg$\omega\alpha$d($\phi$) = $\phi$.

(ii) If A ⊆ B, then sg$\omega\alpha d(A)$ ⊆ sg$\omega\alpha d(B)$.

(iii) If x ∈ sg$\omega\alpha$d(A), then x ∈ sg$\omega\alpha$d[$A - \{x\}$].

(iv) sg$\omega\alpha$d(A ∪ B) = sg$\omega\alpha$d(A) ∪ sg$\omega\alpha$d(B).

(v) sg$\omega\alpha$d(A ∩ B) ⊆ sg$\omega\alpha$d(A) ∩ sg$\omega\alpha$d(B).

*Proof.*

(i) Let x ∈ X and x ∈ sg$\omega\alpha$d($\phi$). Then for every sg$\omega\alpha$-open set G containing x, we should have [G - { x }] ∩ A $\neq \phi$,which is impossible.Therfore sg$\omega\alpha$d($\phi$) = $\phi$.

(ii) Let x ∈ sg$\omega\alpha$d(A) then x is a limit point of A. From definition 4.23, [G - { x }] ∩ A $\neq \phi$ for every sg$\omega\alpha$-nbd G containing x. Since A ⊆ B, then [G - { x }] ∩ A ⊆ [G - { x }] ∩ B.Therefore x ∈ sg$\omega\alpha$d(B). Hence sg$\omega\alpha d(A)$ ⊆ sg$\omega\alpha d(B)$.

(iii) If x ∈ sg$\omega\alpha$d(A), by definition 4.23, every sg$\omega\alpha$-open set G containing x contains at least one point other than x of $A - \{x\}$. Hence x is sg$\omega\alpha$-limit point of $A - \{x\}$ and it belongs to sg$\omega\alpha$d$[A - \{x\}]$. Hence x ∈ sg$\omega\alpha$d$[A - \{x\}]$.

(iv) Since A ⊆ A ∪ B and B ⊆ A ∪ B, then from property (ii), sg$\omega\alpha$d(A) ∪ sg$\omega\alpha$d(B) ⊆ sg$\omega\alpha$d(A ∪ B).−−→(a).
On the other hand if x ∉ sg$\omega\alpha$d(A) ∪ sg$\omega\alpha$d(B), then x ∉ sg$\omega\alpha$d(A) and x ∉ sg$\omega\alpha$d(B). Therefore there exist sg$\omega\alpha$-nbds $G_1$ and $G_2$ of x such that $G_1$ ∩ (A- { x }) = $\phi$ and $G_2$ ∩ (B- { x }) = $\phi$. Since $G_1 \cap G_2$ is sg$\omega\alpha$-nbd of x, then we get ($G_1 \cap G_2$ ) ∩ [A- { x }] = $\phi$. Therefore x ∉ sg$\omega\alpha$d(A ∪ B). Therfore sg$\omega\alpha$d(A ∪ B) ⊆ sg$\omega\alpha$d(A) ∪ sg$\omega\alpha$d(B).−−→(b).
Therefore from (a) and (b) we get, sg$\omega\alpha$d(A ∪ B) = sg$\omega\alpha$d(A) ∪ sg$\omega\alpha$d(B).

(v) Since A ∩ B ⊆ A and A ∩ B ⊆ B, then from property (ii) we have sg$\omega\alpha$d(A ∪ B) ⊆ sg$\omega\alpha$d(A) and sg$\omega\alpha$d(A ∩ B) ⊆ sg$\omega\alpha$d(B).Consequently, sg$\omega\alpha$d(A ∩ B) ⊆ sg$\omega\alpha$d(A) ∩ sg$\omega\alpha$d(B). ■

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