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Small and Classical T-ABSO Fuzzy Submodules

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Abstract

Let M is a unitary R-module over R be a commutative ring with identity and let X be a fuzzy module of an R-module M. Our aim in this paper is studing small T-ABSO fuzzy submodules and classical T-ABSO fuzzy submodules. Many new basic properties and characterizations of these concepts are given and relationship these concepts with T-ABSO fuzzy submodules.

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1. Introduction

In this paper all ring are commutative with $1 \neq 0$ and all modules are unitary. Prime submodule presented by C. P. Lu in [4] where "*A* prime submodule N of an R-module M over a commutative ring R, $N \neq M$ with property $a \in R$, $x \in M$, $ax \in N$ implies that $x \in N$ or $a \in (N : M)$". This concept is generalized to concept of prime fuzzy submodule by Rabi H.J. (see[11]). Layla S. Mahmood in [15] presented the definition of small prime submodule where "a proper submodule N of an R-module M is called small prime submodule if whenever $r \in R$ and $x \in M$ with $< x > \ll M$ and $rx \in N$

implies that either $x \in N$ or $r \in (N :_R M)$. In [13] presented the concept of fuzzy small submodule where “let M be a module over a ring R, and let $A \in F(M)$. Then A is said to be fuzzy small submodule of M if for any $B \in F(M)$ satisfying $B \neq X_M$ implies $A + B \neq X_M$. The notation $A \ll_F M$ indicates that A is fuzzy small submodule of M. Where $F(M)$ is a set of all fuzzy submodule of M. A. Badawi in[2] presented the concept of 2-absorbing ideal where “A proper ideal $I \neq 0$ of R is said to be a 2-absorbing ideal if whenever $r, b, z \in R$ and $rbz \in I$ then $rb \in I$ or $rz \in I$ or $bz \in I$”. Rabah K. in [14] generalized this concept to 2-absorbing fuzzy ideal where “A nonzero proper fuzzy ideal I of R is called 2-absorbing fuzzy ideal if for each f. singletons a_s, b_l, r_k of R, $\forall s, l, k \in L$, and $a_s b_l r_k \subseteq I$, then either $a_s b_l \subseteq I$ or $a_s r_k \subseteq I$ or $b_l r_k \subseteq I$”. while A.Y. Darani, F. Soheilnia in [3]. presented the definition of 2-absorbing submodule where “let $N < M$, N is called 2-absorbing submodule of M if whenever $r, b \in R$, $x \in M$ and $rbx \in N$, then $rx \in N$ or $bx \in N$ or $rb \in (N : M)$”. Abdulrahman A.H. in [1] presented two concepts:small 2-absorbing submodule where “A proper submodule N of an R-module M is called a small 2-absorbing submodule, if whenever $a, b \in R$ and $< m > \ll M$, $abm \in N$ implies that $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$ and small 2-absorbing ideal where “A proper ideal I of a ring R is small 2-absorbing if it is small 2-absorbing submodule of the R-module R. A classical prime submodule presented by M. Behboodi in [12] where “A proper submodule N of an R-module M is called a classical prime submodule, if for each $m \in M$ and elements $a, b \in R$, $abm \in N$ implies that $am \in N$ or $bm \in N$. H. Mostafanasab gave a generalization of classical prime submodule to a classical 2-absorbing submodule where “A proper submodule N of an R-module M is called a classical 2-absorbing submodule, if whenever $a, b, c \in R$ and $m \in M$ with $abcm \in N$, then $abm \in N$ or $acm \in N$ or $bcm \in N$”.

This paper be composed of two sections.
In sec. (1) we present the definitions: T-ABSO fuzzy submodules, small prime fuzzy submodules, small T-ABSO fuzzy ideals and small T-ABSO fuzzy submodules and we give some characterizations of small T-ABSO fuzzy submodules. Also many properties and outcomes of these concepts are given.
In sec. (2) we present the definitions: classical prime submodule and classical T-ABSO fuzzy submodule, many basic properties and outcomes are studied.
Note that we denote to fuzzy: f., module:m., submodule:subm., [0,1]: L , otheroiwse: o.w. and example: ex.

2. Small T-ABSO F. Subm.

In this sec., we introduce the concepts small prime f. subm., T-ABSO f. subm., small T-ABSO f. ideal. and small T-ABSO f. subm. and some of propostions and relationship between small T-ABSO f. subm. with small prime f. subm. and T-ABSO f. subm.

Now, we shall fuzzify the definitions: small prime subm., T-ABSO subm., small T-ABSO ideal and small T-ABSO subm. as follows:

Definition 2.1. A proper f. subm. A of f. m. X of an R-m. M is called small prime f.

subm. if whenever f. singleton a_s of R and $x_v \subseteq X$ with $< x_v >\ll X$ and $a_s x_v \subseteq A$ implies either $x_v \subseteq A$ or $a_s \subseteq (A :_R X)$.

The following proposition specificates small prime f. subm. in terms of its level subm.

Proposition 2.2. Let A is small prime f. subm. of f. m. X of an R-m. M. if and only if the level subm. A_v is small prime subm. of X_v, $\forall v \in L$.

Proof. $(\Rightarrow)$ Let $ax \in A_v$ and $< x >\ll X_v$ for $a \in R$, then $A(ax) \geq v$ and $< x >_v \ll X$ hence $(ax)_v \subseteq A$. So that $a_s x_k \subseteq A$, $\forall s, k \in L$, and $< x_k >\ll X$ where $v = min\{s, k\}$. Since A is small prime f., then either $x_k \subseteq A$ or $a_s \subseteq (A :_R X)$. Hence either $x \in A_v$ or $a \in (A_v :_R X_v)$ since $(A :_R X)_v = (A_v :_R X_v)$ by [7]. Thus A_v is small prime subm. of X_v.
$(\Leftarrow)$ Let $a_s x_k \subseteq A$ and $< x_k >\ll X$ for f. singleton a_s of R, $\forall s, k \in L$. Hence $(ax)_v \subseteq A$ and $< x >_v \ll X$ where $v = min\{s, k\}$, so that $ax \in A_v$ and $< x >\ll X_v$. But A_v is small prime, then either $x \in A_v$ or $a \in (A_v :_R X_v)$, hence either $x_k \subseteq A$ or $a_s \subseteq (A :_R X)$. Thus A is small prime f.subm. of X. ∎

Definition 2.3. Let X be f. m. of an R-m. M. A proper f. subm. A of X is called T-ABSO f. subm. if whenever a_s, b_l be f. singletons of R, and $x_v \subseteq X$, $\forall s, l, v \in L$, such that $a_s b_l x_v \subseteq A$ then either $a_s b_l \subseteq (A :_R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$.

The following proposition specificates T-ABSO f. subm. in terms of its level subm.

Proposition 2.4. Let A is T-ABSO f. subm. of f. m. X of of an R-m. M if and only if the level subm. A_v is T-ABSO subm. of X_v, $\forall v \in L$.

Proof. $(\Rightarrow)$Let $abx \in A_v$ for each $a, b \in R$ and $x \in X_v$, then $A(abx) \geq v$, so $(abx)_v \subseteq A$ implies that $a_s b_l x_k \subseteq A$ where $v = min\{s, l, k\}$. Since A be a T-ABSO f. subm., then either $a_s b_l \subseteq (A :_R X)$ or $a_s x_k \subseteq A$ or $b_l x_k \subseteq A$. hence $(ab)_v \subseteq (A :_R X)$ or $(ax)_v \subseteq A$ or $(bx)_v \subseteq A$. So that $ab \in (A_v :_R X_v)$ since $(A :_R X)_v = (A_v :_R X_v)$ by [7] or $ax \in A_v$ or $bx \in A_v$. Thus A_v is a T-ABSO subm. of X_v.
$(\Leftarrow)$ Let $a_s b_l x_k \subseteq A$ for f. singletons a_s, b_l of R and $x_k \subseteq X$, $\forall s, l, k \in L$, hence $(abx)_v \subseteq A$ where $v = min\{s, l, k\}$ so that $abx \in A_v$. But A_v is T-ABSO subm., then either $ab \in (A_v :_R X_v)$ or $ax \in A_v$ or $bx \in A_v$,since $(A_v :_R X_v) = (A :_R X)_v$ by [7], hence $ab \in (A :_R X)_v$. Then either $(ab)_v \subseteq (A :_R X)$ or $(ax)_v \subseteq A$ or $(bx)_v \subseteq A$, implies either $a_s b_l \subseteq (A :_R X)$ or $a_s x_k \subseteq A$ or $b_l x_k \subseteq A$. Thus A be T-ABSO f. subm. of X. ∎

Definition 2.5. A proper f. subm. A of f. m. X of an R-m. M is called small T-ABSO f. subm. if whenever f. singletons a_s, b_l of R and $x_v \subseteq X$ such that $< x_v >\ll X$, $a_s b_l x_v \subseteq A$ implies either $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$ or $a_s b_l \subseteq (A :_R X)$.

Definition 2.6. A proper f. ideal I of a ring R is a small T-ABSO f. if it is small T-ABSO f. subm. of the R-m. R, equivalent; a proper f. ideal I of a ring R is small T-ABSO f.

if whenever f. singletons a_s, b_l, r_k of R such that $< r_k >\ll R$, $a_s b_l r_k \subseteq I$, then either $a_s r_k \subseteq I$ or $b_l r_k \subseteq I$ or $a_s b_l \subseteq I$.

The following proposition specificates small T-ABSO f. subm. in terms of its level subm.

Proposition 2.7. Let A is small T-ABSO f. subm. of f. m. X of an R-m. M. if and only if the level subm. A_v is small T-ABSO subm. of X_v, for all $v \in L$.

Proof. ($\Rightarrow$) Let $abx \in A_v$ and $< x >\ll X_v$ for $a, b \in R$, then $A(abx) \geq v$ and $< x >_v \ll X$ hence $(abx)_v \subseteq A$. So that $a_s b_l x_k \subseteq A$, $\forall s, l, k \in L$, and $< x_k >\ll X$ where $v = min\{s, l, k\}$. Since A is small T-ABSO f., then either $a_s x_k \subseteq A$ or $b_l x_k \subseteq A$ or $a_s b_l \subseteq (A :_R X)$. Hence either $(ax)_v \subseteq A$ or$(bx)_v \subseteq A$ or $(ab)_v \subseteq (A :_R X)$, so either $(ax) \in A_v$ or $bx \in A_v$ or $ab \in (A_v :_R X_v)$ since $(A :_R X)_v = (A_v :_R X_v)$ by [7]. Thus A_v is small T-ABSO subm. of X_v.
($\Leftarrow$) Let $a_s b_l x_k \subseteq A$ and $< x_k >\ll X$ for f. singletons a_s, b_l of R, $\forall s, l, k \in L$. Hence $(abx)_v \subseteq A$ and $< x >_v \ll X$ where $v = min\{s, l, k\}$, so that $abx \in A_v$ and $< x >\ll X_v$. But A_v is small T-ABSO, then either $ax \in A_v$ or $bx \in A_v$ or $ab \in (A_v :_R X_v)$, hence either $a_s x_k \subseteq A$ or $b_l x_k \subseteq A$ or $a_s b_l \subseteq (A :_R X)$. Thus A is small T-ABSO f.subm. of X. ■

Remark 2.8. There are many remarks and ex. as follows:

1. if A is a small prime f. subm. of f. m. X of an R-m. M, then A is a small T-ABSO f. subm. of X.

Proof. Let $a_s b_l x_v \subseteq A$ for f. singletons a_s, b_l of R and $< x_v >\ll X$, hence $a_s < b_l x_v >\subseteq A$ and $< b_l x_v >\ll X$. Since A is a small prime f. subm., then either $b_l x_v \subseteq A$ or $a_s \subseteq (A :_R X)$. So either $b_l x_v \subseteq A$ or $a_s x_v \subseteq A$. Thus A is small T-ABSO f. subm. ■

The converse incorrect for ex.:
Let $X : Z_{24} \longrightarrow L$ such that $X(y) = \begin{cases} 1 & \text{if } y \in Z_{24} \\ 0 & \text{o.w.} \end{cases}$
It is obvious that X be a f. m. of Z_{24} as Z-m.
Let $A : Z_{24} \longrightarrow L$ such that $A(y) = \begin{cases} v & \text{if } y \in (\bar{12}) \\ 0 & \text{o.w.} \end{cases} \quad \forall v \in L$
Let $B : Z_{24} \longrightarrow L$ such that $B(y) = \begin{cases} v & \text{if } y \in (\bar{6}) \\ 0 & \text{o.w.} \end{cases} \quad \forall v \in L$
Let $C : Z_{24} \longrightarrow L$ such that $C(y) = \begin{cases} v & \text{if } y \in (\bar{0}) \\ 0 & \text{o.w.} \end{cases} \quad \forall v \in L$
It is obvious that A, B and C are f. subm. of X.
Now, $A_v = (\bar{12})$, $B_v = (\bar{6})$, $C_v = (\bar{0})$ and $X_v = Z_{24}$ as Z-m.
where A_v is small T-ABSO subm. since A_v, B_v and C_v are only small subm. in X_v. and $2.1.(\bar{6}) \in A_v$ and $2.(\bar{6}) \in A_v$.

$2.4.(\bar{6}) \in A_v$ and $4.(\bar{6}) \in A_v$.
$3.2.(\bar{2}) \in A_v$ but $(\bar{2})$ not small in X_v
However A_v is not small prime since $2.(\bar{6}) \in A_v$ but $(\bar{6}) \notin A_v$ and $2 \notin (A_v :_R X_v) = 12Z$. Hence A is small T-ABSO f. subm. but A is not small prime f. subm.

1. It is obvious that every T-ABSO f. subm. is small T-ABSO f. subm. However the converse incorrect, for ex.:
   Let $X : Z \longrightarrow L$ such that $X(y) = \begin{cases} 1 & \text{if } y \in Z \\ 0 & \text{o.w.} \end{cases}$
   It is obvious that X be f. m. of Z as Z-m.
   Let $A : Z \longrightarrow L$ such that $A(y) = \begin{cases} v & \text{if } y \in 20Z \\ 0 & \text{o.w.} \end{cases} \quad \forall v \in L$
   It is obvious that A is f. subm. of X.
   $A_v = 20Z$ and $X_v = Z$ as Z-m. where A_v is a proper subm. of X_v, if $abx \in A_v$ for $a, b \in Z$ and $< x > \ll Z$. Since (0) is only small subm. in $X_v = Z$, so that $x = 0$. Then $ax = 0$ and $bx = 0$. Hence A_v is small T-ABSO this satisfies for each proper subm. of Z. But A_v is not T-ABSO subm. since $2.2.5 \in A_v$ but $2.5 \notin A_v$ and $2.2 \notin A_v$. So that A is small T-ABSO f. subm., but it is not T-ABSO f. subm.

The following theorem is a characterization of small T-ABSO f.subm.

Theorem 2.9. Let A be a proper f. subm. of f. m. X of an R-m. M. Then A is a small T-ABSO f. subm. if and only if whenever for singletons a_s, b_l of R, $B \ll X$, $a_s b_l B \subseteq A$, then either $a_s B \subseteq A$ or $b_l B \subseteq A$ or $a_s b_l \subseteq (A :_R X)$.

Proof. ($\Rightarrow$) Suppose that $a_s b_l B \subseteq A$, but $a_s B \nsubseteq A$ and $b_l B \nsubseteq A$, so there exist f. singletons $x_v, y_h \subseteq B$ such that $a_s x_v \nsubseteq A$, $b_l y_h \nsubseteq A$. Then $< x_v > \ll X$ and $< y_h > \ll X$ since $x_v, y_h \subseteq B$ and $B \ll X$ by[13]. Now $a_s b_l x_v \subseteq A$ and $a_s x_v \nsubseteq A$, hence either $b_l x_v \subseteq A$ or $a_s b_l \subseteq (A :_R X)$. If $a_s b_l \subseteq (A :_R X)$ then we are done. If $b_l x_v \subseteq A$. Meditate $a_s b_l (x_v + y_h) \subseteq A$ and $< x_v + y_h > \ll X$ since $< x_v > \ll X$ and $< y_h > \ll X$ by[5]. Since A is a small T-ABSO f. subm., then either $a_s(x_v + y_h) \subseteq A$ or $b_l(x_v + y_h) \subseteq A$ or $a_s b_l \subseteq (A :_R X)$. If $a_s b_l \subseteq (A :_R X)$, then we are done. If $a_s(x_v + y_h) \subseteq A$. Since $a_s x_v \nsubseteq A$, hence $a_s y_h \nsubseteq A$. But $a_s b_l y_h \subseteq A$, $a_s y_h \nsubseteq A$ and $b_l y_h \nsubseteq A$ so that $a_s b_l \subseteq (A :_R X)$. If $b_l(x_v + y_h) \subseteq A$, then similary $a_s b_l \subseteq (A :_R X)$.
($\Leftarrow$) It is obvious. ■

Proposition 2.10. Let A be a proper f. subm. of f. m. X of an R-m. M, then the following expressions are equivalent:

1. A is a small T-ABSO f. subm. of X.

2. $(A :_X I)$ is a small T-ABSO f. subm. for each f. ideal I of R, $IX \nsubseteq A$.

3. $(A :_X a_s)$ is a small T-ABSO f. subm. for each f. singleton a_s of R, $a_s X \nsubseteq A$.

Proof. (1) ⇒ (2) Let $r_k b_l x_v \subseteq (A :_X I)$ and $< x_v > \ll X$ for f. singletons r_k, b_l of R and $x_v \subseteq X$, so $r_k b_l < x_v > \subseteq (A :_X I)$, hence $r_k b_l I < x_v > \subseteq A$. But $< x_v > \ll X$ then $< I x_v > \ll X$ and since A is a small T-ABSO f. subm., then either $r_k < I x_v > \subseteq A$ or $b_l < I x_v > \subseteq A$ or $r_k b_l \subseteq (A :_R X)$ by theorem(5). So that $r_k x_v \subseteq (A :_X I)$ or $b_l x_v \subseteq (A :_X I)$ or $r_k b_l \subseteq (A :_R X)$.
(2) ⇒ (3) It is obvious.
(3) ⇒ (1) It follows directly since $A = (A :_X 1_v)$
where $1_v : R \longrightarrow L$ such that $1_v(y) = \begin{cases} 1 & \text{if } y = 1 \\ 0 & \text{if } y \neq 1 \end{cases} \leq \begin{cases} 1 & \text{if } y \in R \\ 0 & \text{o.w.} \end{cases} = \lambda_R(y) = 1$
by [16]
then A is a small T-ABSO f. subm. of X. ■

Proposition 2.11. Let $f : M_1 \longrightarrow M_2$ be an epimorphism and X_1, X_2 are f. m. of an R-m. M_1, M_2 respectively. Let A be a small T-ABSO f. subm. of X_2, then $f^{-1}(A)$ is a small T-ABSO f. subm. of X_1.

Proof. It is obvious that $f^{-1}(A)$ a proper f. subm. of X_1 since A is a proper f. subm. of X_2. Let $r_k b_l x_v \subseteq f^{-1}(A)$ and $< x_v > \ll X_1$ for f. singletons r_k, b_l of R and $x_v \subseteq X_1$, then $r_k b_l f(x_v) \subseteq A$. But $< f(x_v) > \ll X_2$ by [5]. Since A is a small T-ABSO f. subm., then either $r_k f(x_v) \subseteq A$ or $b_l f(x_v) \subseteq A$ or $r_k b_l \subseteq (A :_R X_2)$. Hence $r_k x_v \subseteq f^{-1}(A)$ or $b_l x_v \subseteq f^{-1}(A)$ or $r_k b_l \subseteq (f^{-1}(A) :_R X_1)$. ■

Remark 2.12. A homorphic image of small T-ABSO f. subm. may be not small T-ABSO f. subm. we can show by the following ex.:
Let $X : Z_{24} \longrightarrow L$ such that $X(y) = \begin{cases} 1 & \text{if } y \in Z_{24} \\ 0 & \text{o.w.} \end{cases}$
It is obvious that X be a f. m. of Z_{24} as Z-m.
Let $A : Z_{24} \longrightarrow L$ such that $A(y) = \begin{cases} v & \text{if } y \in (\bar{2}) \\ 0 & \text{o.w.} \end{cases} \quad \forall v \in L$
It is obvious that A are f. subm. of X.
Now, $A_v = (\bar{2})$, $X_v = Z_{24}$ as Z-m.
where $f : Z_{24} \longrightarrow Z_{24}$ such that $f(y) = 4y \ \forall y \in Z_{24}$, A_V is a small T-ABSO subm. But $f(A_v) =< \bar{8} >$ is not small T-ABSO subm. since 2.2. $< \bar{6} > \in < \bar{8} >$ and $< \bar{6} > \ll X_v$, but 2. $< \bar{6} > \notin < \bar{8} >$ and $2.2 \notin (A_v :_Z X_v) = 8Z$. So that A is small T-ABSO f. subm., but $f(A)$ is not small T-ABSO f. subm.

Proposition 2.13. Let A be a small T-ABSO f. subm. of f. m. of an R-m. M, then $(A :_R X)$ is a small T-ABSO f. ideal of R.

Proof. Let $a_s b_l r_k \subseteq (A :_R X)$ and $< r_k > \ll R$ for f. singletons a_s, b_l, r_k of R. Suppose that $a_s r_k \nsubseteq (A :_R X)$ and $b_l r_k \nsubseteq (A :_R X)$. Now for any f. singleton $x_v \subseteq X$, define $f : R \longrightarrow X$ by $f(d_n) = d_n x_v$. It is obvious that f is well-defined and homomorphism. Since $< r_k > \ll R$, then $< r_k x_v > \ll X$ ...(1). By assumption there exist $y_h, g_m \subseteq X$ such that $a_s r_k y_h \nsubseteq A$ and $b_l r_k g_m \nsubseteq A$. But $a_s b_l (r_k y_h + r_k g_m) \subseteq A$ and by(1) $< r_k y_h > \ll X$, $< r_k g_m > \ll X$, then $< r_k y_h + r_k g_m > \ll X$ by [5]. Then either $a_s(r_k y_h + r_k g_m) \subseteq A$

or $b_l(r_k y_h + r_k g_m) \subseteq A$ or $a_s b_l \subseteq (A :_R X)$. If $a_s b_l \subseteq (A :_R X)$ then we are done. If $a_s(r_k y_h + r_k g_m) \subseteq A$, $a_s r_k y_h \nsubseteq A$, we get $a_s r_k g_m \nsubseteq A$. But $a_s b_l r_k g_m \subseteq A$, $< r_k g_m > \ll X$ and $b_l r_k g_m \nsubseteq A$. Thus $a_s b_l \subseteq (A :_R X)$. By the same method, if $b_l(r_k y_h + r_k g_m) \subseteq A$, then $a_s b_l \subseteq (A :_R X)$. ■

The definitions of faithful f. m. see [11], the finitely generated f. m. see [9] and the multiplication f. m. see [7].

The converse of proposition (2.13) is true under certain conditions as we show by following proposition.

Proposition 2.14. Let A be a proper f. subm. of a faithful finitely generated multiplication f. m. X of an R-m. M. If $(A :_R X)$is a small T-ABSO f. ideal, then A is a small T-ABSO f. subm.

Proof. Let $a_s b_l x_v \subseteq A$ and $< x_v > \ll X$ for f. singletons a_s, b_l of R and $x_v \subseteq X$. But X is a faithful finitely generated multiplication f. m., then $< x_v >= IX$ for some f. ideal I of R and $I \ll R$(since if $I + J = R$ for f. ideal j of R, $IX + JX = RX$, hence $< x_v > +JX = X$. But $< x_v > \ll X$, so that $JX = X$. Thus $J = R$). Hence $a_s b_l IX \subseteq A$, so $a_s b_l I \subseteq (A :_R X)$. Then either $a_s I \subseteq (A :_R X)$ or $b_l I \subseteq (A :_R X)$ or $a_s b_l \subseteq (A :_R X)$ by theorem (3.5). So that $a_s IX \subseteq A$ or $b_l IX \subseteq A$ or $a_s b_l \subseteq (A :_R X)$. Thus $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$ or $a_s b_l \subseteq (A :_R X)$. ■

3. Classical T-ABSO F. Subm.

In this sec. we present the concept of a classical prime f. subm. and a classical T-ABSO f. subm. A classical T-ABSO f. subm. as a generalization of a classical prime, many basic properties, results and relationships between a classical T-ABSO f. subm., T-ABSO f. subm. and a classical prime f. subm. are given.
M. Behboodi presented a classical prime submodule in [12] and H. Mostafanasab gave a generalization of classical prime submodule to a classical 2-absorbing submodule. In this sec. we shall fuzzify these concepts as follows:

Definition 3.1. Let X be f. m. of an R-m. M. A proper f. subm. is called a classical prime f. subm. if for f. singletons a_s, b_l of R and $x_v \subseteq X$ with $a_s b_l x_v \subseteq A$, then either $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$.

Definition 3.2. Let X be f. m. of an R-m. M. A proper f. subm. is called a classical T-ABSO f. subm. if for f. singletons a_s, b_l, r_k of R and $x_v \subseteq X$ with $a_s b_l r_k x_v \subseteq A$, then either $a_s b_l x_v \subseteq A$ or $a_s r_k x_v \subseteq A$ or $b_l r_k x_v \subseteq A$.

The following proposition specificates a classical prime f. subm. in terms of its level subm.

Proposition 3.3. Let A is a classical prime f. subm. of f. m. X of an R-m. M. if and only if the level subm. A_v is a classical prime subm. of X_v, $\forall v \in L$.

Proof. ($\Rightarrow$) Let $abx \in A_v$ for $a \in R$, then $A(abx) \geq v$, hence $(abx)_v \subseteq A$. So that $a_s b_l x_k \subseteq A, \forall s,k \in L$ where $v = min\{s,l,k\}$. Since A is a classical prime f., then either $a_s x_k \subseteq A$ or $b_l x_v \subseteq A$, hence either $ax \in A_v$ or $bx \in A_v$. Thus A_v is classical prime subm. of X_v.
($\Leftarrow$) Let $a_s b_l x_k \subseteq A$ for f. singletons a_s, b_l of R and $x_v \subseteq X$, $\forall s,l,k \in L$. Hence $(abx)_v \subseteq A$ where $v = min\{s,l,k\}$, so that $abx \in A_v$. But A_v is a classical prime, then either $ax \in A_v$ or $bx \in A_v$, hence either $a_s x_k \subseteq A$ or $b_l x_v \subseteq A$. Thus A is classical prime f.sunm. of X. ∎

Now, we give the following proposition specificates classical T-ABSO f. subm. in terms of its level subm.

Proposition 3.4. Let A is a classical T-ABSO f. subm. of f. m. X of of an R-m. M if and only if the level subm. A_v is a classical T-ABSO subm. of X_v, $\forall v \in L$.

Proof. ($\Rightarrow$)Let $abrx \in A_v$ for each $a,b,r \in R$ and $x \in X_v$, then $A(abrx) \geq v$, so $(abrx)_v \subseteq A$ implies that $a_s b_l r_k x_h \subseteq A$ where $v = min\{s,l,k,h\}$. Since A be a classical T-ABSO f. subm., then either $a_s b_l x_h \subseteq A$ or $a_s r_k x_h \subseteq A$ or $b_l r_k x_h \subseteq A$. hence $(abx)_v \subseteq A$ or $(arx)_v \subseteq A$ or $(brx)_v \subseteq A$. So that $abx \in A_v$ or $arx \in A_v$ or $brx \in A_v$. Thus A_v is aclassical T-ABSO subm. of X_v.
($\Leftarrow$) Let $a_s b_l r_k x_h \subseteq A$ for f. singletons a_s, b_l, r_k of R and $x_h \subseteq X$, $\forall s,l,k,h \in L$, hence $(abrx)_v \subseteq A$ where $v = min\{s,l,k,h\}$ so that $abrx \in A_v$. But A_v is classical T-ABSO subm., then either $abx \in A_v$ or $arx \in A_v$ or $brx \in A_v$. Then either $(abx)_v \subseteq A$ or $(arx)_v \subseteq A$ or $(brx)_v \subseteq A$, implies either $a_s b_l x_h \subseteq A$ or $a_s r_k x_h \subseteq A$ or $b_l r_k x_h \subseteq A$. Thus A is a classical T-ABSO f. subm. of X. ∎

Theorem 3.5. Let $f : M_1 \longrightarrow M_2$ be an epimorphism and X_1, X_2 are f. m. of an R-m. M_1, M_2 respectively.

1. If B is a classical T-ABSO f. subm. of X_2, then $f^{-1}(B)$ is a classical T-ABSO f. subm. of X_1.
2. If A is a classical T-ABSO f. subm. of X_1 such that $ker(f) \subseteq A$, then $f(A)$ is a classical T-ABSO f. subm. of X_2.

Proof.

(1) Since f is epimorphism, $f^{-1}(B)$is a proper f. subm. of X_1. Let $a_s b_l r_k x_v \subseteq f^{-1}(B)$ for f. singletons a_s, b_l, r_k of R and $x_v \subseteq X_1$. Hence $a_s b_l r_k f(x_v) \subseteq B$, then either $a_s b_l f(x_v) \subseteq B$ or $a_s r_k f(x_v) \subseteq B$ or $b_l r_k f(x_v) \subseteq B$, hence $a_s b_l x_v \subseteq f^{-1}(B)$ or $a_s r_k x_v \subseteq f^{-1}(B)$ or $b_l r_k x_v \subseteq f^{-1}(B)$. Thus $f^{-1}(B)$ is a classical T-ABSO f. subm. of X_1.

(2) Let $a_s b_l r_k y_h \subseteq f(A)$ for f. singletons a_s, b_l, r_k of R and $y_h \subseteq X_2$, then there exists $x_v \subseteq X_1$ such that $y_h = f(x_v)$ since f is onto, hence $f(a_s b_l r_k x_v) \subseteq f(A)$. Since $ker(f) \subseteq A$, we get $a_s b_l r_k x_v \subseteq A$. So that $a_s b_l x_v \subseteq A$ or $a_s r_k x_v \subseteq A$ or

$b_l r_k x_v \subseteq A$. Hence $a_s b_l y_h \subseteq f(A)$ or $a_s r_k y_h \subseteq f(A)$ or $b_l r_k y_h \subseteq f(A)$.Thus $f(A)$ is a classical T-ABSO f. subm. of X_2. ■

Proposition 3.6. Let X be f. m. of an R-m. M and A, B be classical prime f. subm. of X. Then $A \cap B$ is a classical T-ABSO f. subm. of X.

Proof. Let $a_s b_l r_k x_v \subseteq A \cap B$ for f. singletons a_s, b_l, r_k of R and $x_v \subseteq X$. Since A and B are classical prime f. subm., then we can assume that $a_s x_v \subseteq A$ and $b_l x_v \subseteq B$. Then $a_s b_l x_v \subseteq A \cap B$. So that $A \cap B$ is a classical T-ABSO f. subm. of X. ■

Proposition 3.7. Let A be a proper f. subm. of f. m. X of an R-m. M.

1. If A is T-ABSO f. subm. of X, then A is a classical T-ABSO f. subm. of X.
2. A is a classical prime subm. of X if and only if A is T-ABSO f. subm. of X and $(A :_R X)$ is a prime f. ideal of R.

Proof.

(1) Let $a_s b_l r_k x_v \subseteq A$ for f. singletons a_s, b_l, r_k of R and $x_v \subseteq X$. Since A is T-ABSO f. subm., then either $a_s r_k x_v \subseteq A$ or $b_l r_k x_v \subseteq A$ or $a_s b_l \subseteq (A :_R X)$. If $a_s r_k x_v \subseteq A$ or $b_l r_k x_v \subseteq A$, then we are done. If $a_s b_l \subseteq (A :_R X)$, then $a_s b_l X \subseteq A$, hence $a_s b_l x_v \subseteq A$. Thus is a classical T-ABSO f. subm. of X.

(2) ($\Longrightarrow$) Let A is a classical prime f. subm., then A is T-ABSO f. subm. Now, let $a_s b_l \subseteq (A :_R X)$, then $a_s b_l X \subseteq A$, hence $a_s b_l x_v \subseteq A$ for f. singleton $x_v \subseteq X$. Since A is a classical prime f. subm., $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$, hence $a_s \subseteq (A :_R X)$ or $b_l \subseteq (A :_R X)$. So that $(A :_R X)$ is prime f. ideal.
($\Longleftarrow$) Let $a_s b_l x_v \subseteq A$ for f. singletons a_s, b_l of R and $x_v \subseteq X$. But A is T-ABSO f. subm., then either $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$ or $a_s b_l \subseteq (A :_R X)$. Assume that $a_s x_v \not\subseteq A$ and $b_l x_v \not\subseteq A$. Then $a_s b_l \subseteq (A :_R X)$ and hence $a_s \subseteq (A :_R X)$ or $b_l \subseteq (A :_R X)$. So that $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$ which is a contradication!. Thus A is classical prime f. subm. ■

Remark 3.8. There are some remarks and ex. as follows:

1. The converse of proposition(11)part(1), incorrect for ex.:
Let $X : Z_3 \oplus Z_4 \longrightarrow L$ such that $X(x,y) = \begin{cases} 1 & \text{if } (x,y) \in Z_3 \oplus Z_4 \\ 0 & \text{o.w.} \end{cases}$
It is obvious that X be a f. m. of $Z_3 \oplus Z_4$ as Z-m.
And $A : Z_3 \oplus Z_4 \longrightarrow L$ such that $A(x,y) = \begin{cases} v & \text{if } (x,y) = (\bar{0},\bar{0}) \\ 0 & \text{o.w.} \end{cases} \quad \forall v \in L$
It is obvious that A are f. subm. of X.
Now, $A_v = (\bar{0},\bar{0})$ and $X_v = Z_3 \oplus Z_4$ as Z-m.
where A_v is a classical T-ABSO subm. since $2.3.1(\bar{2},\bar{2}) = (\bar{0},\bar{0})$ then $2.3(\bar{2},\bar{2}) = (\bar{0},\bar{0})$, but A_v is not T-ABSO subm. since $2.3(\bar{2},\bar{2}) = (\bar{0},\bar{0})$, but $2(\bar{2},\bar{2}) = (\bar{1},\bar{0}) \neq$

$(\bar{0},\bar{0})$, $3(\bar{2},\bar{2}) = (\bar{0},\bar{2}) \neq (\bar{0},\bar{0})$ and $2.3 \notin annX_v = 12Z$. So that A is a classical T-ABSO f. subm., but A is not T-ABSO f. subm.

2. However the converse is correct if X is cyclic f. m. this fact they are coincide as follows: Let A be a classical T-ABSO f. subm. and let $a_s b_l x_v \subseteq A$. Since X is cyclic f. m., $X =< y_h >$ for some f. singleton $y_h \subseteq X$, then $x_v = r_k y_h$ for some f. singleton r_k of R. So that $a_s b_l r_k y_h \subseteq A$ and hence either $a_s b_l y_h \subseteq A$ or $a_s r_k y_h \subseteq A$ or $b_l r_k y_h \subseteq A$, implies that $a_s b_l \subseteq (A :_R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. Thus A is T-ABSO f. subm. of X.

3. Every quasi-prime f. subm. see[7], is a classical T-ABSO f., but the converse in general incorrect for ex.:
   Let $X : Z \longrightarrow L$ such that $X(y) = \begin{cases} 1 & \text{if } y \in Z \\ 0 & \text{o.w.} \end{cases}$
   It is obvious that X be a f. m. of Z as Z-m.
   Let $A : Z \longrightarrow L$ such that $A(y) = \begin{cases} v & \text{if } y \in 6Z \\ 0 & \text{o.w.} \end{cases} \quad \forall v \in L$
   It is obvious that A are f. subm. of X.
   Now, $A_v = 6Z$ is T-ABSO subm. of Z , since if $x, y, z, r \in Z$ and $xyzr \in 6Z = A_v$ then at least one of x, y, z and r is even or one of them is 6. Then either $xyr \in A_v$ or $xzr \in A_v$ or $yzr \in A_v$. By[8] then A_v is classical T-ABSO subm. of Z .But A_v is not quasi-prime, since $2.3.1 \in 6Z = A_v$ but $2.1 \notin 6Z = A_v$ and $3.1 \notin 6Z = A_v$. So that A is classical T-ABSO f. subm., but A is not quasi-prime f. subm.

The following theorem gives a characterization of classical T-ABSO f. subm.

Theorem 3.9. Let X be f. m. of an R-m. M and A be a proper f. subm. of X. The following expressions are equivalent:

1. A is a classical T-ABSO f. subm.;
2. For every f. singleton a_s, b_l, r_k of R, $(A :_X a_s b_l r_k) = (A :_X a_s b_l) \cup (A :_X a_s r_k) \cup (A :_X b_l r_k)$;
3. For every f. singleton a_s, b_l of R and $x_v \subseteq X$ with $a_s b_l x_v \nsubseteq A$, $(A :_R a_s b_l x_v) = (A :_R a_s x_v) \cup (A :_R b_l x_v)$;
4. For every f. singleton a_s, b_l of R and $x_v \subseteq X$ with $a_s b_l x_v \nsubseteq A$, $(A :_R a_s b_l x_v) = (A :_R a_s x_v)$ or $(A :_R a_s b_l x_v) = (A :_R b_l x_v)$;
5. For every f. singleton a_s, b_l of R, $x_v \subseteq X$ and every f. ideal I of R with $a_s b_l I x_v \subseteq A$, then either $a_s b_l x_v \subseteq A$ or $a_s I x_v \subseteq A$ or $b_l I x_v \subseteq A$;
6. For every f. singleton a_s of R, $x_v \subseteq X$ and every f. ideal I of R with $a_s I x_v \nsubseteq A$, $(A :_R a_s I x_v) = (A :_R a_s x_v)$ or $(A :_R a_s I x_v) = (A :_R I x_v)$;

7. For every f. singleton a_s of R, $x_v \subseteq X$ and every and every f. ideal I, J of R with $a_s IJx_v \subseteq A$, then either $a_s Ix_v \subseteq A$ or $a_s Jx_v \subseteq A$ or $IJx_v \subseteq A$;

8. For every f. ideal I, J of R and $x_v \subseteq X$ with $IJx_v \nsubseteq A, (A :_R IJx_v) = (A :_R Ix_v)$ or $(A :_R IJx_v)(A :_R Jx_v)$;

9. For every f. ideal I, J, H of R and $x_v \subseteq X$ with $IJH \subseteq A$, then either $IJx_v \subseteq A$ or $IHx_v \subseteq A$ or $JHx_v \subseteq A$;

10. For every f. singleton $x_v \subseteq X \setminus A$, $(A :_R x_v)$ is T-ABSO f. ideal of R.

Proof. (1) $\Longrightarrow$ (2) Assume that A is a classical T-ABSO f. subm. of X. Let $x_v \subseteq (A :_X a_s b_l r_k)$ for f. singleton $x_v \subseteq X$, hence $a_s b_l r_k x_v \subseteq A$, Then either $a_s b_l x_v \subseteq A$ or $a_s r_k x_v \subseteq A$ or $b_l r_k x_v \subseteq A$. Thus $x_v \subseteq (A :_X a_s b_l)$ or $x_v \subseteq (A :_X a_s r_k)$ or $x_v \subseteq (A :_X b_l r_k)$. So that $(A :_X a_s b_l r_k) = (A :_X a_s b_l) \cup (A :_X a_s r_k) \cup (A :_X b_l r_k)$.
(2) $\Longrightarrow$ (3) Let $a_s b_l x_v \nsubseteq A$ for some f. singletons a_s, b_l of R and $x_v \subseteq X$. Suppose that $r_k \subseteq (A :_R a_s b_l x_v)$, hence $a_s b_l r_k x_v \subseteq A$ and so $x_v \subseteq (A :_X a_s b_l r_k)$. Since $a_s b_l x_v \nsubseteq A$, $x_v \nsubseteq (A :_X a_s b_l)$. Then by part(1), $x_v \subseteq (A :_X a_s r_k)$ or $x_v \subseteq (A :_X b_l r_k)$, hence $r_k \subseteq (A :_R a_s x_v)$ or $r_k \subseteq (A :_R b_l x_v)$. Thus $(A :_R a_s b_l x_v) = (A :_R a_s x_v) \cup (A :_R b_l x_v)$;
(3) $\Longrightarrow$ (4) Since $(A :_R a_s b_l x_v)$ is f. ideal of R and $(A :_R a_s b_l x_v) = (A :_R a_s x_v) \cup (A :_R b_l x_v)$. So that either $(A :_R b_l x_v) \subseteq (A :_R a_s x_v)$ or $(A :_R a_s x_v) \subseteq (A :_R b_l x_v)$. Thus $(A :_R a_s b_l x_v) = (A :_R a_s x_v)$ or $(A :_R a_s b_l x_v) = (A :_R b_l x_v)$.
(4) $\Longrightarrow$ (5) Let for some f. singletons a_s, b_l of R, $x_v \subseteq X$ and for f. ideal I of R, $a_s b_l I x_v \subseteq A$. Then $I \subseteq (A :_R a_s b_l x_v)$. If $a_s b_l x_v \subseteq A$, then the proof is complete. Suppose that $a_s b_l x_v \nsubseteq A$. Hence by part(4), we have that $I \subseteq (A :_R a_s x_v)$ or $I \subseteq (A :_R b_l x_v)$; that is $a_s I x_v \subseteq A$ or $b_l I x_v \subseteq A$.
(5) $\Longrightarrow$ (6) $\Longrightarrow$ (7) $\Longrightarrow$ (8) $\Longrightarrow$ (9) The proofs are similar to the previous implications.
(9) $\Longrightarrow$ (10) It is obvious.
(9) $\Longrightarrow$ (1) It is obvious.
(10) $\Longrightarrow$ (1) Is fiddling. ∎

Proposition 3.10. Let X be f. m. of an R-m. M and $\{B_i : i \in I\}$ be a chain of classical T-ABSO f. subm. of X. Then $\cap_{i\in I} B_i$ is a classical T-ABSO f. subm. of X.

Proof. Assume that $a_s b_l r_k x_v \subseteq \cap_{i\in I} B_i$ for some f. singletons a_s, b_l, r_k of R and $x_v \subseteq x$. Suppose that $a_s b_l x_v \nsubseteq \cap_{i\in I} B_i$ and $a_s r_k x_v \nsubseteq \cap_{i\in I} B_i$. Hence there are $n, m \in I$ where $a_s b_l x_v \nsubseteq B_n$ and $a_s r_k x_v \nsubseteq B_m$. Then for every $B_h \subseteq B_n$ and every $B_d \subseteq B_m$, we have that $a_s b_l x_v \nsubseteq B_h$ and $a_s r_k x_v \nsubseteq B_d$. So that for every f. subm. B_u such that $B_u \subseteq B_n$ and $B_u \subseteq B_m$ we have $b_l r_k x_v \subseteq B_u$. Thus $b_l r_k x_v \subseteq \cap_{i\in I} B_i$. ∎

The product $AB = IJX$ where I, J f. ideal of R and A, B f. subm. of a multiplication f. m. of an R-m. M such that $A = IX$ and $B = JX$ see[6].
By using this definition of product of f. subm., we give the following characterization of classical T-ABSO f. subm. under conditions a finitely generated faithful multiplication f. m.

Proposition 3.11. Let X be a multiplication f. m. of an R-m. M and A be a proper f. subm. of X. The following expressions are equivalent:

1. A is a classical T-ABSO f. subm. of X;

2. If $A_1A_2A_3x_v \subseteq A$ for some f. subm. A_1, A_2, A_3 of X and f. singleton $x_v \subseteq X$, then either $A_1A_2x_v \subseteq A$ or $A_1A_3x_v \subseteq A$ or $A_2A_3x_v \subseteq A$.

Proof. (1) $\Longrightarrow$ (2) Let $A_1A_2A_3x_v \subseteq A$ for some f. subm. A_1, A_2, A_3 of X and f. singleton $x_v \subseteq X$. Since X is multiplication f. m., there are f. ideals I, J, K of R such that $A_1 = IX$, $A_2 = JX$ and $A_3 = KX$. Then $IJKx_v \subseteq A$ and so either $IJx_v \subseteq A$ or $IKx_v \subseteq A$ or $JKx_v \subseteq A$. Thus either $A_1A_2x_v \subseteq A$ or $A_1A_3x_v \subseteq A$ or $A_2A_3x_v \subseteq A$.
(2) $\Longrightarrow$ (1) Assume that $IJKx_v \subseteq A$ for f. ideals I, J, K of R and f. singleton $x_v \subseteq X$. It is adequate to set $A_1 = IX$, $A_2 = JX$ and $A_3 = KX$ in part(2), we get A is a classical T-ABSO f. subm. of X. ■

In [10] Quartararo and Butts called that" A commutative ring R is a u-ring provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals; and a um-ring is a ring R with the property that an R-m. which is equal to a finite union of subm. must to equal to one of them."

Now, we shall fuzzify this concept as follows:

Definition 3.12. A commutative ring R is a u-ring provided R has the property that f. ideal contained in a finite union of f. ideals must be contained in one of those f. ideals; and a um-ring is a ring R with the property that an R-m. which is equal to a finite union of f. subm. must to equal to one of them.

Theorem 3.13. Let R be a um-ring, X be f. m. of an R-m. M and A be a proper f. subm. of X. The following expressions are equivalent:

1. A is a classical T-ABSO f. subm.;

2. For every f. singleton a_s, b_l, r_k of R, $(A :_X a_sb_lr_k) = (A :_X a_sb_l)$ or $(A :_X a_sb_lr_k) = (A :_X a_sr_k)$ or $(A :_X a_sb_lr_k) = (A :_X b_lr_k)$;

3. For every f. singleton a_s, b_l, r_k of R and every f. subm. B of X with $a_sb_lr_kB \subseteq A$ implies that $a_sb_lB \subseteq A$ or $a_sr_kB \subseteq A$ or $b_lr_kB \subseteq A$;

4. For every f. singleton a_s, b_l of R and every f. subm. B of X with $a_sb_lB \nsubseteq A$, $(A :_R a_sb_lB) = (A :_R a_sB)$ or $(A :_R a_sb_lB) = (A :_R b_lB)$;

5. For every f. singleton a_s, b_l of R, every f. ideal I of R and every f. subm. B of X with $a_sb_lIB \subseteq A$ implies that $a_sb_lB \subseteq A$ or $a_sIB \subseteq A$ or $b_lIB \subseteq A$;

6. For every f. singleton a_s of R, every f. ideal I of R and every f. subm. B of X with $a_sIB \nsubseteq A$, $(A :_R a_sIB) = (A :_R a_sB)$ or $(A :_R a_sIB) = (A :_R IB)$;

7. For every f. singleton a_s of R, every f. ideal I, J of R and every f.subm. B of X, $a_s IJB \subseteq A$ implies that $a_s IB \subseteq A$ or $a_s JB \subseteq A$ or $IJB \subseteq A$;

8. For every f. ideal I, J of R and every f. subm. B of X with $IJB \nsubseteq A$, $(A :_R IJB) = (A :_R IB)$ or $(A :_R IJB) = (A :_R JB)$;

9. For every f. ideal I, J, H of R and every f. subm. B of X with $IJHB \subseteq A$ implies that $IJB \subseteq A$ or $IHB \subseteq A$ or $JHB \subseteq A$;

10. For every f. subm. B of X not contained in A, $(A :_R B)$ is T-ABSO f. ideal of R.

Proof. By the same method to the proof of theorem (3.9). ■

Proposition 3.14. Let R be a um-ring and A be a proper f. subm. of f. m. X of an R-m. M. Then A is a classical T-ABSO f. subm. of X if and only if A is 3-ABSO f. subm. of X and $(A :_R X)$ is T-ABSO f. idealof R.

Proof. ($\Longrightarrow$) Let A is a classical ABSO f. subm., then A is 3-ABSO f. subm. and $(A :_R X)$ is T-ABSO f. ideal of R by theorem(11).
($\Longleftarrow$) Let $a_s b_l r_k x_v \subseteq A$ for f. singletons a_s, b_l, r_k of R and $x_v \subseteq X$. such that $a_s b_l x_v \nsubseteq A$, $a_s r_k x_v \nsubseteq A$ and $b_l r_k x_v \nsubseteq A$. Thus $a_s b_l r_k \subseteq (A :_R X)$. Hence either $a_s b_l \subseteq (A :_R X)$ or $a_s r_k \subseteq (A :_R X)$ or $b_l r_k \subseteq (A :_R X)$ this is a contradication!. So that A classical T-ABSO f. subm. ■

Proposition 3.15. Let X be f. m. of an R-m. M and A be a classical T-ABSO f. subm. of X. The following expressions hold:

1. For every f. singleton a_s, b_l, r_k of R and $x_v \subseteq X$, $(A :_R a_s b_l r_k x_v) = (A :_R a_s b_l x_v) \cup (A :_R a_s r_k x_v) \cup (A :_R b_l r_k x_v)$;

2. If R is a u-ring, then for every f. singleton a_s, b_l, r_k of R and $x_v \subseteq X$, $(A :_R a_s b_l r_k x_v) = (A :_R a_s b_l x_v)$ or $(A :_R a_s b_l r_k x_v) = (A :_R a_s r_k x_v)$ or $(A :_R a_s b_l r_k x_v) = (A :_R b_l r_k x_v)$.

Proof.

(1) Let f. singletons a_s, b_l, r_k of R and $x_v \subseteq X$. Assume that $c_n \subseteq (A :_R a_s b_l r_k x_v)$, hence $a_s b_l r_k (c_n x_v) \subseteq A$, then either $a_s b_l (c_n x_v) \subseteq A$ or $a_s r_k (c_n x_v) \subseteq A$ or $b_l r_k (c_n x_v) \subseteq A$. Thus $c_n \subseteq (A :_R a_s b_l x_v)$ or $c_n \subseteq (A :_R a_s r_k x_v)$ or $c_n \subseteq (A :_R b_l r_k x_v)$. So that $(A :_R a_s b_l r_k x_v) = (A :_R a_s b_l x_v) \cup (A :_R a_s r_k x_v) \cup (A :_R b_l r_k x_v)$.

(2) By using part(1). ■

Proposition 3.16. Let R be a um-ring, X be a multiplication f. m. of an R-m. M and Abe a proper f. subm. of X. The following expressions are equivalent:

1. A is a classical T-ABSO f. subm. of X;

2. If $A_1A_2A_3A_4 \subseteq A$ for some f. subm.A_1, A_2, A_3, A_4 of X, then either $A_1A_2A_4 \subseteq A$ or $A_1A_3A_4 \subseteq A$ or $A_2A_3A_4 \subseteq A$;

3. If $A_1A_2A_3 \subseteq A$ for some f. subm.A_1, A_2, A_3 of X, then either $A_1A_2 \subseteq A$ or $A_1A_3 \subseteq A$ or $A_2A_3 \subseteq A$;

4. A is T-ABSO f. subm. of X;

5. $(A :_R X)$ is T-ABSO f. ideal of R.

Proof. (1) $\Longrightarrow$ (2) Let $A_1A_2A_3A_4 \subseteq A$ for some f. subm.A_1, A_2, A_3, A_4 of X. Since X is a multiplication f. m., there are f. ideals I, J, K of R such that $A_1 = IX$, $A_2 = JX$, $A_3 = KX$. Hence $IJKA_4 \subseteq A$, then either $IJA_4 \subseteq A$ or $IKA_4 \subseteq A$ or $JKA_4 \subseteq A$. Hence either $A_1A_2A_4 \subseteq A$ or $A_1A_3A_4 \subseteq A$ or $A_2A_3A_4 \subseteq A$ by theorem (3.13).
(2) $\Longrightarrow$ (3) Is simple.
(3) $\Longrightarrow$ (4) Assume that $IJB \subseteq A$ for some f. ideals I, J of R and some f. subm. B of X. It is adequate to set $A_1 = IX$, $A_2 = JX$, $A_3 = B$ in part(3).
(4) $\Longrightarrow$ (1) By part(1) of proposition (3.7).
(4) $\Longrightarrow$ (5) Let $a_sb_lr_k \subseteq (A :_R X)$ for f. singletons a_s, b_l, r_k of R. If $a_sr_k \nsubseteq (A :_R X)$ and $b_lr_k \nsubseteq (A :_R X)$, then there exist f. singletons $x_v, y_h \subseteq X \setminus A$ such that $a_sr_kx_v \nsubseteq A$ and $b_lr_ky_h \nsubseteq A$. Since $a_sb_l(r_k(x_v + y_h)) \subseteq A$ and A is T-ABSO f. subm., then either $a_sb_l \subseteq (A :_R X)$ or $a_sr_k(x_v + y_h) \subseteq A$ or $b_lr_k(x_v + y_h) \subseteq A$. If $a_sr_k(x_v + y_h) \subseteq A$ and $a_sr_kx_v \nsubseteq A$, then we have $a_sr_ky_h \nsubseteq A$. So that $a_sb_l(r_ky_h) \subseteq A$ and $b_lr_ky_h \nsubseteq A$, hence $a_sb_l \subseteq (A :_R X)$. By the same method if $b_lr_k(x_v + y_h) \subseteq A$, we get $a_sb_l \subseteq (A :_R X)$. Thus $(A :_R X)$ is T-ABSO f. ideal of R.
(5) $\Longrightarrow$ (4) Let $IJB \subseteq A$ for some f. subm. B of X. Since X is multiplication f. m., then there is f. ideal K of R such that $B = KX$. So that $IJK \subseteq (A :_R X)$, then either $IJ \subseteq (A :_R X)$ or $IK \subseteq (A :_R X)$ or $JK \subseteq (A :_R X)$. If $IJ \subseteq (A :_R X)$, then the proof is complete. If $IK \subseteq (A :_R X)$, then $IKX = IB \subseteq A$. By the same method if $JK \subseteq (A :_R X)$, then we have $JB \subseteq A$. ∎

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