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The Proof of the Method of Variation of Parameter in the General Form

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Abstract

We consider the proof of the method of variation of parameter in the general form. The methodology of this article is done by a choice of another form, other than standard one of Lagrange.

Keywords: Variation of parameters; nonhomogeneous linear ODE; Engineering mathematics

INTRODUCTION

When a nonhomogeneous linear ODE

$$y'' + p(x)y' + q(x)y = r(x)$$

has a general solution y_h, we can get a particular solution y_p by the method of variation of parameters[1-2], providing p, q and r are continuous on some open interval. The method is the

$$y_p(x) = -y_1\int \frac{y_2 r}{W}\,dx + y_2\int \frac{y_1 r}{W}\,dx$$

where y_1, y_2 form a basis of the solution of the corresponding homogeneous ODE, and W is the Wronskian of y_1, y_2, $W = y_1 y_{2'} - y_2 y_{1'}$. Of course, this method naturally can be extended to arbitrary order n as

$$y_p(x) = \sum_{k=1}^{n} y_k(x)\int \frac{W_k(x)}{W(x)} r(x)dx$$

where $W_j (j = 1,\cdots,n)$ is obtained from Wronskian W by replacing the j-th column of W by the column $[0\,0 \cdots 0\,1]^T$.

In the proof of the method, Lagrange insisted that the standard form

$$y'' + p(x)y' + q(x)y = r(x) \quad (1)$$

is essential, not $f(x)y''$. However, having a doubt of the fixed form, we would like to approach a proof of the method in the general form. Consequently, in the above statement, we think that the form would rather change into 'desirable' than 'essential' which is the opinion of Lagrange. Additionally, we have checked the integral form of the method as well. In the previous researches, a nonlinear variation of parameters method for ODEs is obtained by generalizing existing techniques[7], and the method of variation of parameters for dynamic systems was proposed in [6].

To begin with, let us briefly check the existing proof[2]. The method start from determining u and v so that

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x) \quad (2)$$

is a particular solution of the nonhomogeneous linear ODE (1) where y_1, y_2 form a basis of the solution of the corresponding homogeneous ODE. The proof of Lagrange requires the precondition of

$$u'y_1 + v'y_2 = 0. \quad (3)$$

This point is somewhat mysterious. Let us expatiate on the point. Differentiating (2), we have

$$y_p{}' = u'y_1 + uy_{1'}{}' + v'y_2 + vy_{2'}{}'.$$

Here, our aim is to find the values of u and v, and we can get one equation by the substitution of y_p-terms into (1). Since we have to find two values u and v, the other equation is needed. Hence, Lagrange imposed a second condition (3). Next, if we substitute the values of u and v into (2), we find the method of variation of parameters. When we chose the alternative condition $uy_1{}' + vy_2{}' = 0$, we could not found the values of u and v. This is an interest thing, and changing the way, we would like to approach the general form of the method other than the standard one. Finally, we would like to mention that this method can be replaced by integral transforms[3-5].

THE PROOF OF THE METHOD OF VARIATION OF PARAMETER IN THE GENERAL FORM

Let us check the method of variation of parameters in the general form

$$f(x)y'' + g(x)y' + h(x)y = k(x). \quad (4)$$

Here, we assume that the coefficients and $r(x)$ are continuous on an open interval.

Theorem 2.1 *(The proof of the method of variation of parameter in the general form) The equation* $f(x)y'' + g(x)y' + h(x)y = k(x)$, $k(x)/f(x) = r(x)$ *has a particular solution* y_p *of the general form*

$$y_p(x) = -y_1 \int \frac{y_2 r}{W}\, dx + y_2 \int \frac{y_1 r}{W}\, dx$$

where y_1, y_2 form a basis of the solution of the corresponding homogeneous ODE, and W is the Wronskian of y_1 and y_2.

Proof. Let us determine u and v so that

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$$

is a particular solution of the nonhomogeneous linear ODE (4). Differentiating y_p, we have

$$y_p{}' = u'y_1 + uy_{1'}{}' + v'y_2 + vy_{2'}{}',$$

and by the second condition of Lagrange which is $u'y_1 + v'y_2 = 0$, $y_{p'}{}'$ reduces

$$y_{p'}{}' = uy_1{}' + vy_{2'}{}'. \quad (5)$$

Differentiating (5), we have

$$y_p{}'' = u'y_1{}' + uy_1{}'' + v'y_2{}' + vy_2{}''.$$

Substituting these values into (4) and collecting terms with respect to u and v, we have

$$u\{f(x)y_1'' + g(x)y_1' + h(x)y_1\} + v\{f(x)y_2'' + g(x)y_2' + h(x)y_2\} + f(x)\{u'y_1' + v'y_2'\} = k(x).$$

Since y_1 and y_2 are solutions of corresponding homogeneous ODE $f(x)y'' + g(x)y' + h(x)y = 0$, this reduces to

$$f(x)\{u'y_1' + v'y_2'\} = k(x)$$

and so,

$$u'y_1' + v'y_2' = k(x)/f(x) = r(x). \quad (6)$$

This result is the same as Lagrange's, and so, we can obtain the result

$$y_p(x) = -y_1 \int \frac{y_2 r}{W}\, dx + y_2 \int \frac{y_1 r}{W}\, dx$$

We note that this proof is done in general form other than the standard one which is asserted that Lagrange was essential.

Example 2.2 *Solve the nonhomogeneous ODE* $x^2 y'' - 2xy' + 2y = x$.

Solution. Since the left-hand side is Euler-Cauchy equation, the homogeneous ODE of the above equation has a basis x and x^2. This gives the Wronskian $W(x, x^2) = x^2$. From theorem 2.1, we have the particular solution

$$y_p = -x\int 1/x\, dx + x^2 \int 1/x^2\, dx$$

where, $r(x) = k(x)/f(x) = 1/x$. Calculating this integral, we have $y_p = -x\ln x - x$, and so, we obtain the answer

$$y = c_1 x + c_2 x^2 - x\ln x - x = (c_1 - 1)x + c_2 x^2 - x\ln x$$

(c_1 and c_2 are arbitrary constants).

Corollary 2.3 *The equation* $y'' + p(x)y' + q(x)y = r(x)$ *has a particular solution* y_p *of the integral form*

$$y_p(x) = y_1 \int e^{-h}\left(\int e^h r / y_1\, dx + C\right) dx + y_2 \int y_2^{-2} / e^{\int p\, dx}\, dx$$

under the condition of $u''y_1 + 2u'y_{1'}{}' + pu'y_1 = r$. Where, y_1, y_2 form a basis of the solutions of the corresponding homogeneous ODE on some open interval on which the coefficients and $r(x)$ are continuous.

Proof. Let us assume that a general solution of the given equation is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x).$$

Next, we would like to determine u and v so that

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$$

is a particular solution of the nonhomogeneous linear ODE (1). Substituting $y_p{}' = u'y_1 + uy_1{}' + v'y_2 + vy_2{}'$ and $y_p{}'' = u''y_1 + 2u'y_1{}' + uy_1{}'' + v''y_2 + 2v'y_2{}' + vy_2{}''$ into the given equation and organizing the equality, we have

$$(u''y_1 + v''y_2) + 2(u'y_1{}' + v'y_2{}') + p(u'y_1 + v'y_2) = r.$$

Once again, organizing the equation, we obtain

$$(u''y_1 + 2u'y_1{}' + pu'y_1) + (v''y_2 + 2v'y_2{}' + pv'y_2) = r. \quad (7)$$

Here, we need a second condition of

$$u''y_1 + 2u'y_1{}' + pu'y_1 = r, (8)$$

and of course it is similar to Lagrange's method because the numbers of unknown parameters is two.

From (7) and (8), we have

$$v''y_2 + 2v'y_2{}' + pv'y_2 = 0. (9)$$

By the method of reduction of order proposed by Lagrange, the equations (8) and (9) can be rewritten as

$$U'y_1 + (2y_1{}' + y_1p)U = r$$
$$V'y_2 + (2y_2{}' + py_2)V = 0.$$

provided $u' = U$ and $v' = V$. Separation of variables gives

$$V'/V = -2y_2{}'/y_2 - p.$$

By integration, we have

$$\ln V = -2\ln|y_2| - \int p\,dx.$$

By taking exponents we obtain

$$V = y_2{}^{-2}/e^{\int p\,dx}.$$

On the other hand, $U'y_1 + (2y_1{}' + y_1p)U = r$ can be represented by

$$U' + (2y_1{}'/y_1 + p)U = r/y_1.$$

Since this equation is linear in both the unknown function U and its derivative U', its solution is

$$U = e^{-h}\left(\int e^h r/y_1\,dx + C\right)$$

where,

$$h = \int(2y_1{}'/y_1 + p)\,dx = 2\ln|y_1| + \int p\,dx$$

and C is a constant. Since $U = u'$ and $V = v'$,

$$u = \int e^{-h}\left(\int e^h r/y_1\,dx + C\right)dx$$

and

$$v = \int y_2{}^{-2}/e^{\int p\,dx}\,dx$$

and so,

$$y_p(x) = y_1\int e^{-h}\left(\int e^h r/y_1\,dx + C\right)dx + y_2\int y_2{}^{-2}/e^{\int p\,dx}\,dx.$$

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