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On decomposition of a singularly perturbed dynamical system model with perturbation parameters of different orders of magnitude

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Abstract

A problem of decomposition of a singularly perturbed nonlinear dynamical system model with several small perturbation parameters of different orders of magnitude is considered. The difficulties in solving this problem arise due to the need to investigate the stability properties of solutions of nonlinear subsystems of equations of the original model – associated (or boundary-layer) systems – whose number depends on the number of different orders of magnitude of the perturbation parameters. The approach which allows solving this problem by investigating the stability of a set of linearized associated systems is proposed in this paper. A procedure for consecutive determination of reduced models of different orders and the corresponding associated systems is given. The linearized associated systems obtained by linearization of the nonlinear ones along the trajectory of the corresponding reduced systems are introduced. The assertion that the stability property of the solution of the nonlinear associated system can be deduced from the stability property of its linearized model is proved. The requirements to the system model under which this assertion gives necessary and sufficient conditions are given. The obtained results permit simplifying the solution of the problem of the original model decomposition.

Keywords: singular perturbation; singularly perturbed model of dynamical system; model order reduction; perturbation parameters of different orders of magnitude; linearized model

INTRODUCTION

The description of nonlinear systems with processes of different speed often leads to construction of singularly perturbed models. For the wide-spread case of the system with two proper motions with very different speeds the model is a singularly perturbed system of differential equations containing a small parameter multiplying a part of derivatives [1-4]. Zeroing the small parameter allows to get a simplified model of a reduced order. This simplified model is called reduced (or degenerate) and under certain conditions can adequately describe the processes in the dynamical system on the interval of their observation except for the initial part or the boundary layer. The value of the boundary layer is determined by the value of the small parameter which was thrown away in the passing to the reduced model. The interval of the boundary layer is determined mainly by the fast proper motions which are not taken into account by the reduced model. The satisfaction of the conditions under which the reduced model is adequate to the initial model guarantees that the system dynamics in the boundary layer interval will lead the system to the state described by the initial model. These conditions were stated in the well-known theorem of A.N. Tikhonov [5-8]. The description of the dynamic of the systems characterized by several kinds of proper motions with very different speed is very interesting. Their models can be represented by a singularly perturbed system of differential equations with small parameters of different orders of magnitude. The use of them gives us a possibility to get several models of different orders which are correct in different intervals of the processes observation for the system. They are necessary, for example, when together with the investigation of the normal mode of the system on the basis of the reduced model with zeroing all small parameters it is necessary to control some rapidly changing process. This process that can cause an abnormal situation demands to take into consideration in the model some parameter of the certain order. For singularly perturbed equations system with the perturbance parameters of different orders of magnitude Tikhonov also received the conditions under which the reduced system is adequate to the initial system [5, 7]. The main difficulties while condition testing in practice are the necessity of stability investigation for the forced motions of the nonlinear equations systems called associated (or boundary layer) systems. The approach considered in this paper allows us to remove this difficulty.

INITIAL CONDITIONS AND PROBLEM STATEMENT

Suppose that the processes in a nonlinear nonautonomous dynamical system are modeled by a singularly perturbed model of s order containing perturbances of different orders of magnitude:

$$\begin{cases} \dfrac{d\vec{y}}{dt} = f\left(\vec{y}, \vec{z}^{(1)}, \ldots, \vec{z}^{(m)}, t\right), \\ \mu^{(j)} \dfrac{d\vec{z}^{(j)}}{dt} = F^{(j)}\left(\vec{y}, \vec{z}^{(1)}, \ldots, \vec{z}^{(m)}, t\right), \end{cases} \quad 0 \le t \le T, \qquad (1)$$

when

$$\vec{y}(0) = \vec{y}_0, \ \vec{z}^{(j)}(0) = \vec{z}_0^{(j)}, \ j = \overline{1, m}, \qquad (2)$$

where $\vec{y} = (y_1, \ldots, y_n)^T$, $f = (f_1, \ldots, f_n)^T$ - n-dimensional vectors, $\vec{z}^{(j)} = (\vec{z}_1^{(j)}, \ldots, \vec{z}_{n_j}^{(j)})^T$, $F^{(j)} = (F_1^{(j)}, \ldots, F_{n_j}^{(j)})^T$ - n_j - dimensional vectors, $j = \overline{1, m}$;

parameter $\mu^{(j+1)}$ is of the higher magnitude order than $\mu^{(j)}$ so that when they are approaching zero $\dfrac{\mu^{(j+1)}}{\mu^{(j)}} \to 0$, $j = \overline{1, m}$;

$s = n + \sum_{j=1}^{m} n_j$.

Let us suppose that all functions in the right parts of the equations are continuous and all the solutions of all differential equations exist and are unique. Besides that it is supposed that the variation of the factors defining the nonautonomy of the system is not faster than the change of its processes.

If we set $\mu^{(1)} = \ldots = \mu^{(k)} = 0$ in (1) we obtain *k-fold reduced model*:

$$\begin{cases} \dfrac{d\vec{y}}{dt} = f\left(\vec{y}, \vec{z}^{(1)}, \ldots, \vec{z}^{(m)}, t\right), \\ \mu^{(j)} \dfrac{d\vec{z}^{(j)}}{dt} = F^{(j)}\left(\vec{y}, \vec{z}^{(1)}, \ldots, \vec{z}^{(m)}, t\right), \quad (1 \le j \le m - k) \\ \quad 0 = F^{(j)}\left(\vec{y}, \vec{z}^{(1)}, \ldots, \vec{z}^{(m)}, t\right), \quad (m - k + 1 \le j \le m) \end{cases} \qquad (3)$$

the initial conditions for $\vec{y}$, $\vec{z}^{(1)}$, …, $\vec{z}^{(m-k)}$ coincide with (2).

The order of the received differential equations system (3) is lower than the order of the initial system s (1) by the value $h_k = \sum_{j=m-k+1}^{m} n_j$, because the constraints on the variables in k vector equations $F^{(j)} = 0$ are holonomic and do not contain derivatives.

The fact that the model (3) is adequate to the initial model when $t > t_k$ (where t_k is the value of the boundary layer) depends on the behavior of the system in the boundary layer which is determined by the character of the associated equations systems solutions.

The associated system of k-order (i.e. the associated system corresponding the kth order of the initial model reduction) is obtained from equations with the small parameters of the initial system (1) when $m - k + 1 \le j \le m$. It is the system of equations:

$$\frac{d\tilde{\vec{z}}^{(j)}}{d\tau_k} = F^{(j)}\left(\vec{y}, \vec{z}^{(1)}, \ldots, \vec{z}^{(j-1)}, \tilde{\vec{z}}^{(j)}, t\right), \text{ (here } j = m - k + 1) \qquad (4)$$

where the independent variable is $\tau_k = t / \mu^{(j)}$ and $\vec{z}^{(1)}$, …, $\vec{z}^{(j-1)}$, t are considered as parameters.

The equilibrium point of the associated system is the root $\vec{z}^{(j)} = \varphi^{(j)}(\vec{y}, \vec{z}^{(1)}, \ldots, \vec{z}^{(j-1)}, t)$, when $j = m - k + 1$, the root is obtained as the equations system solution

$$\begin{cases} 0 = F^{(j)}\left(\vec{y}, \vec{z}^{(1)}, \ldots, \vec{z}^{(j)}, t\right) \\ \quad \left(m - k + 1 \le j \le m\right) \end{cases},$$

from where the variables $\vec{z}^{(\nu)}$ with $\nu > j$ are excluded.

Taking into consideration the expression for the root $\vec{z}^{(j)}$ the description of the k-fold reduced system can be represented as

$$\begin{cases} \dfrac{d\vec{y}}{dt} = f\left(\vec{y}, \vec{z}^{(1)}, \ldots, \vec{z}^{(j)}, t\right), \\ \mu^{(j)} \dfrac{d\vec{z}^{(j)}}{dt} = F^{(j)}\left(\vec{y}, \vec{z}^{(1)}, \ldots, \vec{z}^{(j)}, t\right). \quad \left(1 \le j \le m - k\right) \end{cases} \qquad (5)$$

According to [5], the feature of the reduction of the singularly perturbed model with the values of the perturbance parameters of different orders of magnitude to the completely reduced model with k = m is the demand that the theorem condition testing is carried out consecutively respecting the reduced models of different reduction order k from 1 to m included. To obtain (k + 1)-fold reduced model the k-fold reduced model obtained earlier is considered as initial in which it is necessary to zero the small parameter of the highest order of magnitude $\mu^{(m-k)}$ and so on.

Let us state that the root $\vec{z}^{(j)} = \varphi^{(j)}(\vec{y}, \vec{z}^{(1)}, \ldots, \vec{z}^{(j-1)}, t)$ is an isolated root of k-order if in its ε-surrounding (where ε > 0) there are no other roots.

The isolated root of k-order $\vec{z}^{(j)} = \varphi^{(j)}(\vec{y}, \vec{z}^{(1)}, \ldots, \vec{z}^{(j-1)}, t)$ is considered to be a stable root in some limited closed domain D_k of the parameter space ($\vec{y}$, $\vec{z}^{(1)}$, ... , $\vec{z}^{(j-1)}$, t) if for any values $(\vec{y}, \vec{z}^{(1)}, \ldots, \vec{z}^{(j-1)}, t) \in D_k$ the equilibrium points $\vec{z}^{(j)} = \varphi^{(j)}(\vec{y}, \vec{z}^{(1)}, \ldots, \vec{z}^{(j-1)}, t)$ are asymptotically stable in the sense of Lyapunov.

Checking the fulfillment of conditions of the theorem [5] assumes among other things the need to investigate the stability properties of the associated systems roots for all orders of reduction. For nonlinear systems this investigating has practical difficulties. The approach which allows to overcome these difficulties using the first approximation equations was proposed by authors in [9, 10] for models with perturbation parameters of the same order of magnitude. An opportunity of use of different types of linearization was considered at the same time. In this paper the models of the associated systems linearized along the trajectories of the corresponding reduced systems are introduced. It is shown that the stability property of the roots of the nonlinear associated system can be deduced from the stability property of its linearized model.

LINEARIZATION OF ASSOCIATED SYSTEMS

Let us consider linearized associated systems. Let us suppose that functions $\varphi^{(j)}$, $j=\overline{1,m}$, have limited partial derivatives with respect to all variables. First let us consider the associated system of a singly reduced system. For this purpose let us zero the smallest from the small parameters $\mu^{(m)}$ in (1). We obtain a singly reduced system (5) for k = 1 and let us find its solution when the initial conditions $\vec{y}(0)=\vec{y}_0$, $\vec{z}^{(j)}(0)=\vec{z}_0^{(j)}$, $j=\overline{1,m-1}$, coincide with the initial conditions from (2):

$$\vec{\bar{y}}(t),\ \vec{\bar{z}}^{(1)}(t),\ \dots,\ \vec{\bar{z}}^{(m-1)}(t),\ \vec{\bar{z}}^{(m)}(t)=\varphi^{(m)}(\vec{\bar{y}},\vec{\bar{z}}^{(1)},\dots,\vec{\bar{z}}^{(m-1)},t).$$

Let us linearize the associated system (4) when $k = 1$

$$\frac{d\vec{\tilde{z}}^{(m)}}{d\tau_1}=F^{(m)}\left(\vec{y},\vec{z}^{(1)},\dots,\vec{z}^{(m-1)},\vec{\tilde{z}}^{(m)},t\right)$$

in the surrounding of the obtained solution of the singly reduced system.

This way we obtain the linearized associated system of the first order:

$$\frac{d\Delta\vec{\tilde{z}}^{(m)}}{d\tau_1}=\bar{F}^{(m)}_{z^{(m)}}\cdot\Delta\vec{\tilde{z}}^{(m)}, \tag{6}$$

where

$$\bar{F}^{(m)}_{z^{(m)}}=\left.\frac{\partial F^{(m)}(\vec{y},\vec{z}^{(1)},\dots,\vec{z}^{(m)},t)}{\partial\vec{z}^{(m)}}\right|_{L_1}=$$

$$=\left[\left.\frac{\partial F_i^{(m)}(\vec{y},\vec{z}^{(1)},\dots,\vec{z}^{(m)},t)}{\partial z_k^{(m)}}\right|_{L_1}\right]_{k=\overline{1,n_m}}^{i=\overline{1,n_m}},$$

$$\tau_1=t/\mu^{(m)},\ \Delta\vec{\tilde{z}}^{(m)}=\vec{\tilde{z}}^{(m)}-\varphi^{(m)},$$

$$L_1=L_1(t)=\left\{\left(\vec{y},\vec{z}^{(1)},\dots,\vec{z}^{(m)},t\right):\ \vec{y}=\vec{\bar{y}}(t);\ \vec{z}^{(1)}=\vec{\bar{z}}^{(1)}(t);\right.$$

$$\dots;\ \vec{z}^{(m-1)}=\vec{\bar{z}}^{(m-1)}(t);\ \vec{z}^{(m)}=\vec{\bar{z}}^{(m)}(t)=$$

$$\left.=\varphi^{(m)}(\vec{\bar{y}},\vec{\bar{z}}^{(1)},\dots,\vec{\bar{z}}^{(m-1)},t);\ 0\le t\le T\right\}$$

- the curve corresponding the solution of the singly reduced system. (Coefficients of matrix $\bar{F}^{(m)}_{z^{(m)}}$ depend on time, parameters $\mu^{(1)}$,…, $\mu^{(m-1)}$, which are not small in comparison to $\mu^{(m)}$, and on the selection of the root $\varphi^{(m)}$ of the equation $F^{(m)}\left(\vec{y},\vec{z}^{(1)},\dots,\vec{z}^{(m)},t\right)=0$, if it has several solutions).

In order to find a linearized associated system of the second order the singly reduced model (5) is considered as initial when k = 1. By zeroing the next small parameter $\mu^{(m-1)}$ a doubly reduced system can be obtained in the points of the solution of which

$$L_2=L_2(t)=\left\{\left(\vec{y},\vec{z}^{(1)},\dots,\vec{z}^{(m-1)},t\right):\ \vec{y}=\vec{\bar{\bar{y}}}(t);\ \vec{z}^{(1)}=\vec{\bar{\bar{z}}}^{(1)}(t);\right.$$

$$\dots;\ \vec{z}^{(m-2)}=\vec{\bar{\bar{z}}}^{(m-2)}(t);\vec{z}^{(m-1)}=\vec{\bar{\bar{z}}}^{(m-1)}(t)=$$

$$\left.=\varphi^{(m-1)}(\vec{\bar{\bar{y}}},\vec{\bar{\bar{z}}}^{(1)},\dots,\vec{\bar{\bar{z}}}^{(m-2)},t);\ 0\le t\le T\right\}$$

when the initial conditions are $\vec{\bar{\bar{y}}}(0)=\vec{y}_0$, $\vec{\bar{\bar{z}}}^{(l)}(0)=\vec{z}_0^{(l)}$, $l=\overline{1,m-2}$, the linearization of the associated system of the second order (4) (when k = 2) with the independent variable $\tau_2=t/\mu^{(m-1)}$ with the pass to the new variable $\Delta\vec{\tilde{z}}^{(m-1)}=\vec{\tilde{z}}^{(m-1)}-\varphi^{(m)}$ is carried out.

In the same way we obtain linearized associated systems by mth order included.

The general form of the linearized associated system of kth order:

$$\frac{d\Delta\vec{\tilde{z}}^{(j)}}{d\tau_k}=\bar{F}^{(j)}_{z^{(j)}}\cdot\Delta\vec{\tilde{z}}^{(j)},\ (j=m-k+1) \tag{7}$$

where

$$\bar{F}^{(j)}_{z^{(j)}}=\left.\frac{\partial F^{(j)}(\cdot)}{\partial\vec{z}^{(j)}}\right|_{L_k}+\left.\frac{\partial F^{(j)}(\cdot)}{\partial\varphi^{(j+1)}}\right|_{L_k}\cdot\left.\frac{\partial\varphi^{(j+1)}(\cdot)}{\partial\vec{z}^{(j)}}\right|_{L_k}+\dots+$$

$$+\left.\frac{\partial F^{(j)}(\cdot)}{\partial\varphi^{(m)}}\right|_{L_k}\cdot\left(\left.\frac{\partial\varphi^{(m)}(\cdot)}{\partial\vec{z}^{(j)}}\right|_{L_k}+\left.\frac{\partial\varphi^{(m)}(\cdot)}{\partial\varphi^{(j+1)}}\right|_{L_k}\cdot\left.\frac{\partial\varphi^{(j+1)}(\cdot)}{\partial\vec{z}^{(j)}}\right|_{L_k}+\dots+\right.$$

$$\left.+\left.\frac{\partial\varphi^{(m)}(\cdot)}{\partial\varphi^{(m-1)}}\right|_{L_k}\cdot\left.\frac{\partial\varphi^{(m-1)}(\cdot)}{\partial\vec{z}^{(j)}}\right|_{L_k}\right),$$

$$F^{(j)}(\cdot)=F^{(j)}\left(\vec{y},\vec{z}^{(1)},\dots,\vec{z}^{(j)},\varphi^{(j+1)}(\vec{y},\vec{z}^{(1)},\dots,\vec{z}^{(j)},t),\right.$$

$$\varphi^{(j+2)}(\vec{y},\vec{z}^{(1)},\dots,\vec{z}^{(j)},\varphi^{(j+1)}(\cdot),t),\dots,$$

$$\varphi^{(m)}(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(j)},\varphi^{(j+1)}(\cdot),\ldots,\varphi^{(m-1)}(\cdot),t),t\big);$$

$$L_k = L_k(t) = \Big\{\big(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(j)},t\big):\ \vec{y} = \vec{\bar{y}}(t);\vec{z}^{(1)} = \vec{\bar{z}}^{(1)}(t);\ldots;$$

$$\vec{z}^{(j)} = \vec{\bar{z}}^{(j)}(t) = \varphi^{(j)}(\vec{\bar{y}},\vec{\bar{z}}^{(1)},\ldots,\vec{\bar{z}}^{(j-1)},t);\ 0 \le t \le T\Big\}$$

- the curve corresponding to the solution of the *k*-fold reduced system; $j = m-k+1$, $\Delta\vec{\bar{z}}^{(j)} = \vec{\bar{z}}^{(j)} - \varphi^{(j)}$. The independent variable in this system is $\tau_k = t/\mu^{(j)}$ and *t* is considered as a parameter.

Let us get down to the consideration of the main assertion of the paper allowing us to determine the stability property of the roots $\vec{z}^{(j)} = \varphi^{(j)}$, $j = \overline{1,m}$.

ASSERTION

Let us assume that the matrix determinants $\bar{F}^{(j)}_{z^{(j)}}$ depending from t, $\mu^{(1)},\ldots,\mu^{(j-1)}$, differ from zero when $\forall\ t \in [0,T]$ for each $j = \overline{1,m}$.

Assertion: For stability of the roots $\vec{z}^{(j)} = \varphi^{(j)}$ of the nonlinear associated systems of kth (k = m – j + 1) order (4), for every $k = \overline{1,m}$, it is sufficient that the corresponding linearized associated systems of kth order (7), $k = \overline{1,m}$, are asymptotically stable according to Lyapunov.

If the eigenvalues of the matrixes $\bar{F}^{(j)}_{z^{(j)}}$, $j = \overline{1,m}$, have a non-zero real part when $\forall\ t \in [0,T]$ then this condition is not only sufficient but also necessary.

Proof: First of all let us prove the sufficiency of the assertion.

It is necessary to prove that if linearized associated systems (7), $k = \overline{1,m}$, are asymptotically stable according to Lyapunov, then the roots $\vec{z}^{(j)} = \varphi^{(j)}$ of the nonlinear associated system (4), $j = \overline{1,m}$, are stable in the sense of the earlier introduced determination of the root stability.

Let us assume that the linearized associated systems (7), $k = \overline{1,m}$, are asymptotically stable according to Lyapunov. Let us consider the nonlinear associated system of kth order (4) and the corresponding linearized associated system (7). Let us assume for simplicity that k = 2 (the case of k = 1 is trivial):

$$\frac{d\vec{z}^{(m-1)}}{d\tau_2} = F^{(m-1)}(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(m-2)},\vec{\bar{z}}^{(m-1)},$$

$$\varphi^{(m)}(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(m-2)},\vec{\bar{z}}^{(m-1)},t),t), \qquad (8)$$

$$\frac{d\Delta\vec{\bar{z}}^{(m-1)}}{d\tau_2} = \bar{F}^{(m-1)}_{z^{(m-1)}} \cdot \Delta\vec{\bar{z}}^{(m-1)}, \qquad (9)$$

where

$$\bar{F}^{(m-1)}_{z^{(m-1)}} = \left.\frac{\partial F^{(m-1)}(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(m-1)},\varphi^{(m)}(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(m-1)},t),t)}{\partial\vec{z}^{(m-1)}}\right|_{L_2} +$$

$$+\left.\frac{\partial F^{(m-1)}(\cdot)}{\partial\varphi^{(m)}}\right|_{L_2} \cdot \left.\frac{\partial\varphi^{(m)}(\cdot)}{\partial\vec{z}^{(m-1)}}\right|_{L_2}.$$

Elements of the matrix $\bar{F}^{(m-1)}_{z^{(m-1)}}$ depend on t and parameters $\mu^{(1)},\ldots,\mu^{(m-2)}$, which are not small while investigating this associated system. In the both associates systems (8) and (9) all variables in the right part except for $\vec{\bar{z}}^{(m-1)}$ и $\Delta\vec{\bar{z}}^{(m-1)}$ respectively are considered as parameters.

Because in all points of the curve $L_2(t)$ the Jacobian $\det\bar{F}^{(m-1)}_{z^{(m-1)}}(t)$ differs from zero, according to the theorem of the implicit function [11] in some surrounding of the each point of the curve $L_2(t)$ the vector equation

$$F^{(m-1)}\big(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(m-1)},\varphi^{(m)}(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(m-1)},t),t\big) = 0$$

has a unique solution $\vec{z}^{(m-1)} = \varphi^{(m-1)}(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(m-2)},t)$. So the root of the second order $\vec{z}^{(m-1)} = \varphi^{(m-1)}$ is isolated which is a necessary condition of the Tikhonov theorem. Now we have to show that this root is an asymptotically stable equilibrium point of the nonlinear associates system of the second order (8).

The necessary and sufficient condition of the asymptotical stability according to Lyapunov for the linear associated system (9) (in which t is considered as a parameter) is the fulfilment of the condition

$$\mathrm{Re}\,\bar{\lambda}_i(t) < 0,\ i = \overline{1,n_{m-1}},\ \text{for}\ 0 \le t \le T, \qquad (10)$$

where $\bar{\lambda}_i(t)$, $i = \overline{1,n_{m-1}}$, are the eigenvalues of the matrix $\bar{F}^{(m-1)}_{z^{(m-1)}}$ from (9).

Let us denote by $\lambda_i\big(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(m-2)},t\big)$, $i = \overline{1,n_{m-1}}$, the eigenvalues of the matrix

$$F^{(m-1)}_{z^{(m-1)}} = \left[\left.\frac{\partial F_i^{(m-1)}(\cdot)}{\partial z_j^{(m-1)}}\right|_{\vec{z}^{(m-1)}=\varphi^{(m-1)}(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(m-2)},t)}\right]_{j=\overline{1,n_{m-1}}}^{i=\overline{1,n_{m-1}}}$$

elements of which differ from elements of $\bar{F}^{(m-1)}_{z^{(m-1)}}$ because they are calculated not in the points of the curve $L_2(t)$ but in the points of the root $\varphi^{(m-1)}$. It is evident that $\bar{\lambda}_i(t)=\lambda_i\left(\bar{\vec{y}}(t),\bar{\vec{z}}^{(1)}(t),\ldots,\bar{\vec{z}}^{(m-2)}(t),t\right)$ (where $\bar{\vec{y}}(t)$, $\bar{\vec{z}}^{(1)}(t)$, …, $\bar{\vec{z}}^{(m-2)}(t)$, $\bar{\vec{z}}^{(m-1)}(t)=\varphi^{(m-1)}(\bar{\vec{y}},\bar{\vec{z}}^{(1)},\ldots,\bar{\vec{z}}^{(m-2)},t)$) is the solution for the doubly reduced system.

As it is shown in [12] if the condition (10) is fulfilled because of the continuity of $\lambda_i\left(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(m-2)},t\right)$, that is provided by the continuity of $F^{(m-1)}_{z^{(m-1)}}$, there exists a parameter domain $\vec{y}$, $\vec{z}^{(1)}$,..., $\vec{z}^{(m-2)}$, t:

$$\hat{D}_2=\left\{\left(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(m-2)},t\right):\ \left\|\vec{y}-\bar{\vec{y}}(t)\right\|\le\eta,\left\|\vec{z}^{(1)}-\bar{\vec{z}}^{(1)}(t)\right\|\le\eta,\ \ldots;\right.$$

$$\left.\left\|\vec{z}^{(m-2)}-\bar{\vec{z}}^{(m-2)}(t)\right\|\le\eta;\ 0\le t\le T\right\}\subseteq D_2$$

($\eta>0$ - some constant),

such that $\operatorname{Re}\lambda_i\left(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(m-2)},t\right)<0$ when $\left(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(m-2)},t\right)\in\hat{D}_2$, $i=\overline{1,n_{m-1}}$.

It follows herefrom that $\bar{\vec{z}}^{(m-1)}=\varphi^{(m-1)}$ is an asymptotically stable equilibrium point of the nonlinear associated system (8) uniformly in respect to the parameter domain $\hat{D}_2\subseteq D_2$.

In the same way one can consider associated systems of other orders k ($k=\overline{1,m}$) and show that from the asymptotical stability of linearized associated systems (7) the asymptotical stability of the nonlinear associated systems (4) roots $\bar{\vec{z}}^{(j)}=\varphi^{(j)}$ ($j=m-k+1$) in small follows, because the character of the induction is completely determined in the given proof for the case when k = 2.

Let us prove the necessity of the assertion.

It is necessary to show that if the eigenvalues of the matrixes $\bar{F}^{(j)}_{z^{(j)}}$, $j=\overline{1,m}$, has a nonzero real part when $\forall\ t\in[0,T]$ (t is a parameter), then from the stability of the roots $\vec{z}^{(j)}=\varphi^{(j)}$ of the nonlinear associated systems (4), $j=\overline{1,m}$, it follows that the linearized associated systems (7), $k=\overline{1,m}$, - linear systems with constant coefficients – are stable according to Lyapunov.

Let the roots $\vec{z}^{(j)}=\varphi^{(j)}$ of nonlinear associated systems (7), $k=\overline{1,m}$, be stable. Let us consider a nonlinear and linearized associated systems of *k*th order. Let us take, as it was done earlier, k = 2. Because the equilibrium point $\bar{\vec{z}}^{(m-1)}=\varphi^{(m-1)}(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(m-2)},t)$ of the nonlinear associated system (8) is asymptotically stable according to Lyapunov, then according to the first Lyapunov method the eigenvalues $\lambda_i\left(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(m-2)},t\right)$, $i=\overline{1,n_{m-1}}$, of the matrix $F^{(m-1)}_{z^{(m-1)}}$ have nonpositive real part for all $\left(\vec{y},\vec{z}^{(1)},\ldots,\vec{z}^{(m-2)},t\right)\in D_2$. Because of that, $\bar{\lambda}_i(t)=\lambda_i\left(\bar{\vec{y}}(t),\bar{\vec{z}}^{(1)}(t),\ldots,\bar{\vec{z}}^{(m-2)}(t),t\right)$, $i=\overline{1,n_{m-1}}$, also have a nonpositive real part. But because according to the condition of the statement $\operatorname{Re}\bar{\lambda}_i(t)\neq 0$ for $\forall\ t\in[0,T]$, $i=\overline{1,n_{m-1}}$, then $\operatorname{Re}\bar{\lambda}_i(t)<0$ for $\forall\ t\in[0,T]$. That means that the linearized associated system (9) (which is a linear system with constant coefficients) is asymptotically stable according to Lyapunov.

In the same way it can be shown that for the other $k=\overline{1,m}$ linearized associated systems (7) are asymptotically stable according to Lyapunov. So, the second part of the assertion is proved.

CONCLUSION

The models of the associated systems linearized along the trajectories of the reduced systems for different orders of reduction have been introduced in the paper. The assertion proved allows us solving the problem of investigating the stability properties of nonlinear associated systems roots by investigating the stability of linearized models which are linear systems with constant parameters. The investigation must be carried out for different sets of the parameters. This assertion gives sufficient conditions. For a wide class of models the conditions of the assertion are also necessary.

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