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Stability of Finite Difference Method for Frost Heaving Calculation in 2D Space

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Abstract

Designing of gas-pipe and condensate lines and lines in conditions of water-inundated soil, high water-level, boggy ground with no runoff, seasonal freezing ground or negative temperature of the pumped product has a high risk of frost heaving of soils. Frosty heaving is the process of increasing of soil volume during the frigidness. Increasing of the volume is partially connected with an increasing of the volume of pore water during the frigidness. But mostly the volume increase due to migration of liquid water from the underlying layers of soil into the freezing zone. Water freezes and increases in volume after crossing the freezing front. The pressure capable of balancing the movement of liquid water is more than 1.2 MPa/ºC, that significantly exceeds the base load-bearing capacity. Therefore, the migration of water and frost heaving does not stop at zone with negative temperature and source of liquid water. Pipelines is deforming by expanding soils in short sections, that leads to an emergency depressurization of the internal space. Authors developed a mathematical model of frost heaving to design engineering protection. This model consist of system of nonlinear parabolic differential equations. Authors encountered the problem of instability of finite differences method on an implicit difference scheme, even with a convergent iterative process. Using the spectral condition of stability, authors derived expressions for correlation between the time step and other parameters of the problem (step in space, capacitive and conducting properties of the medium) for the 2D case. Expressions show that the implicit scheme can be unstable under certain combinations of problem parameters. The greatest influence on the stability has the term responsible to the volume source of heat or matter.

Keywords: frost heaving, system of nonlinear parabolic differential equations, stability of finite difference method

INTRODUCTION

The practice of running the gas-pipe and condensate lines in permafrost soils or water-inundated soil shows the need for engineering protection against frost heaving even if the temperature of the pumped product is positive [1-2]. If in the autumn-winter period the transported product leaves the station with a low positive or negative temperature and intersecting the permafrost, then the temperature of product decreases. At the end of the permafrost, the product, cooled to negative temperatures, enters the melted soil zone. The ground begins to freeze around the pipeline. At the same time, the capillary-sorption potential of underground moisture decreases, which leads to the migration of liquid water from the thawed zone to the frozen zone [3-6]. After the freezing front, liquid water freezes and forming clusters of ice schliere. The pore pressure capable of balancing this process is more than 1.2 MPa/ºC that significantly exceeds the base load-bearing capacity at depths of 1-3 m. Thus the process of migration of liquid water does not stop during the cold period of the year, that leads to the accumulation of huge amounts of ice. Increased in the volume soil deforms the pipeline in short sections about 1-30 meters that leads to an emergency depressurization of the internal space [7-8].

OBJECT AND PURPOSE

Prediction of frost heaving and choosing the best methods of engineering protection requires solving the system of differential equations of heat and mass transfer in soil. For a five-component soil system (solid particles, water, dissolved salt, undissolved salt, ice), it looks as follows [9]:

$$\left(\wp_s c_p^s + \wp_w c_p^w + \wp_i c_p^i + \wp_s c_p^s + \wp_{ns} c_p^{ns}\right)\frac{\partial T}{\partial t}$$
$$= \nabla\cdot\left(\lambda_T \vec{\nabla} T\right) - \left\{\vec{j}_w^{ws}\left(c_p^w - c_p^s\right) + \vec{j}_{ws}\left(\omega_w c_p^w + \omega_s c_p^s\right)\right\}\vec{\nabla} T$$
$$+\left(L_w + \left(c_p^w - c_p^{л}\right)(T - T_{wл})\right)\frac{\partial \wp_{л}}{\partial t} + L_s\frac{\partial \wp_{ns}}{\partial t}, \qquad (1)$$

$$\frac{\partial \wp_w}{\partial \psi_w}\frac{\partial \psi_w}{\partial t} = -\frac{\partial \wp_i}{\partial t}$$
$$+\nabla\cdot\left(\omega_w\frac{\lambda_P \rho_{ws}}{g}\left(\omega_w \vec{\nabla}\psi_w - hav(\varepsilon_V)\vec{g}\right) - D_{ws}\wp_{ws}\vec{\nabla}\omega_s\right), \qquad (2)$$

$$\left(\frac{\wp_{ws}}{\omega_w}\right)\frac{\partial\omega_s}{\partial t} = -\frac{\omega_s}{\omega_w}\left(\frac{\partial\wp_w}{\partial\psi_w}\right)\frac{\partial\psi_w}{\partial t} - \frac{\partial\wp_{ns}}{\partial t}$$
$$+\nabla\cdot\left(\omega_s\frac{\lambda_P\rho_{ws}}{g}\left(\omega_w\vec{\nabla}\psi_w - hav(\varepsilon_V)\vec{g}\right) + D_{ws}\wp_{ws}\vec{\nabla}\omega_s\right), \quad (3)$$

$$\frac{\partial\wp_i}{\partial t} = \frac{\Omega_w - \Omega_\mathrm{i}}{\tau_\Omega}, \quad (4)$$

$$\frac{\partial\wp_{ns}}{\partial t} = \frac{\wp_{ws}(\omega_s - \omega_s^{max})}{\tau_s}, \quad (5)$$

$$\vec{J}_{ws} = -\frac{\lambda_P\rho_{ws}}{g}\left(\omega_w\vec{\nabla}\psi_w - hav(\varepsilon_V)\vec{g}\right), \quad (6)$$

$$\vec{J}_w^{ws} = -D_{ws}\wp_{ws}\vec{\nabla}\omega_w, \quad (7)$$

where T – temperature, K; c_p^s, c_p^w, c_p^i, c_p^s, c_p^{ns} – heat capacity at constant pressure of solid particles, water, ice, dissolved salt and undissolved salt, J/(kg·K); $\wp_s$ – dry unit weight of soil, kg/m^3; $\wp_w$, $\wp_i$, $\wp_s$, $\wp_{ns}$ – mass content of water, ice, dissolved salt and undissolved salt in soil, kg/m^3; $\wp_{\mathrm{ws}}$ - mass content of solution in soil, kg/m^3; λ_T – heat-conduction coefficient, W/(m·K); ρ_{ws} – density of solution, kg/m^3; ω_s – concentration of salt in solution, u.f.; ω_w – concentration of water in solution, u.f.; L_w – heat of fusion of ice at normal condition, J/kg; T_{wl} – fusion temperature of at normal condition, K; L_s – heat of dilution, J/kg; ψ_w – soil-water potential, J/kg; λ_P – moisture conduction coefficient, m/s; ρ_{ws} – density of soil-water solution, kg/m^3; D_{ws} – diffusivity coefficient in soil, m^2/s; $(\Omega_w - \Omega_\mathrm{i})$ – difference of total thermodynamic potential of water and ice, J/kg; τ_Ω – parameter of relaxation of ice crystallization, J·s/kg^2; ω_s^{max} – concentration of saturated solution, u.f.; τ_s – time of relaxation of salt crystallization, s; $hav(x)$ – Heaviside function of argument x; $\vec{g}$ – vector of acceleration of gravity, m/s^2.; ε_V – relative deformation of frost heaving, u.f.

The boundary value problem with using the finite differences method reduces to solving a system of nonlinear equations. The method of successive approximations allow to get the convergence with a sufficient number of iterations and the norm of the Jacobian matrix less than 1 throughout the iteration process ($\sum_{j=1}^{N}|\alpha_{n,j}|<1$). The choice of the optimal step is not difficult because the value of the norm of the Jacobian matrix depends linearly of the time step. However, we are faced with the instability when using implicit difference scheme, even with a convergent iterative process. It was required to do additional investigation of the stability of finite difference schemes for equations (1-7) by spectral condition and to determine the correlations between time step, space step, and other parameters included in equations (1-7).

RESEARCH METHOD

The system of equations (1-7) contains three parabolic differential equations. In the general case, each of these equations can be written in the linearized form as follows:

$$\frac{\partial\mathcal{F}}{\partial t} = A\nabla\cdot\left(B\vec{\nabla}\mathcal{F}\right) + A\vec{C}\cdot\vec{\nabla}\mathcal{F} + A\nabla\cdot\left(\vec{D}\mathcal{F}\right) + Aq\mathcal{F} + Aq_0, \quad (8)$$

where A – an arbitrary positive constant characterizing the capacity; B – an arbitrary positive constant characterizing the conductivity of the medium; $\vec{C}$ –an arbitrary vector characterizing the rate of matter transfer; $\vec{D}$ – an arbitrary vector characterizing the velocity of matter moving under the influence of gravity; $E \approx q\mathcal{F} + q_0$ – the linearized form of a volume source of heat or matter; $\mathcal{F}$ – an arbitrary function (T, ψ_w, ω_s).

Since in our problem there is no constant source of heat or matter, the term Aq_0 can be neglected.

Next, we transform (8) into an implicit finite-difference divergence form taking into account the nonlinear dependence of the coefficients A, B, $\vec{C}$, $\vec{D}$, q on $\mathcal{F}$ and the constant grid density over the 2D space:

$$\frac{\mathcal{F}_{j,k}^{\tau+1} - \mathcal{F}_{j,k}^{\tau}}{\theta} = A_{j,k}^{\tau+1}q_{j,k}^{\tau+1}\mathcal{F}_{j,k}^{\tau+1}$$
$$+\frac{A_{j,k}^{\tau+1}}{h_z}\left(B_{j+\frac{1}{2},k}^{\tau+1}\frac{\mathcal{F}_{j+1,k}^{\tau+1} - \mathcal{F}_{j,k}^{\tau+1}}{h_z} - B_{j-\frac{1}{2},k}^{\tau+1}\frac{\mathcal{F}_{j,k}^{\tau+1} - \mathcal{F}_{j-1,k}^{\tau+1}}{h_z}\right)$$
$$+\frac{A_{j,k}^{\tau+1}}{h_y}\left(B_{j,k+\frac{1}{2}}^{\tau+1}\frac{\mathcal{F}_{j,k+1}^{\tau+1} - \mathcal{F}_{j,k}^{\tau+1}}{h_y} - B_{j,k-\frac{1}{2}}^{\tau+1}\frac{\mathcal{F}_{j,k}^{\tau+1} - \mathcal{F}_{j,k-1}^{\tau+1}}{h_y}\right)$$
$$+A_{j,k}^{\tau+1}C_{j,k}^{\tau+1}\left(\frac{\mathcal{F}_{j+1,k}^{\tau+1} - \mathcal{F}_{j-1,k}^{\tau+1}}{2h_z}\right) + A_{j,k}^{\tau+1}C_{j,k}^{\tau+1}\left(\frac{\mathcal{F}_{j,k+1}^{\tau+1} - \mathcal{F}_{j,k-1}^{\tau+1}}{2h_y}\right)$$
$$+A_{j,k}^{\tau+1}\left(\frac{D_{j+1,k}^{\tau+1}\mathcal{F}_{j+1,k}^{\tau+1} - D_{j-1,k}^{\tau+1}\mathcal{F}_{j-1,k}^{\tau+1}}{2h_z}\right)$$
$$+A_{j,k}^{\tau+1}\left(\frac{D_{j,k+1}^{\tau+1}\mathcal{F}_{j,k+1}^{\tau+1} - D_{j,k-1}^{\tau+1}\mathcal{F}_{j,k-1}^{\tau+1}}{2h_y}\right), \quad (9)$$

where h_z, h_y – grid steps over the axes z, y.

We transform equation (9) into a coefficient form:

$$\left(1 + \theta a_{j,k}\right)\mathcal{F}_{j,k}^{\tau+1} + \theta b_{j,k}\mathcal{F}_{j+1,k}^{\tau+1} + \theta c_{j,k}\mathcal{F}_{j-1,k}^{\tau+1} +$$
$$+\theta d_{j,k}\mathcal{F}_{j,k+1}^{\tau+1} + \theta f_{j,k}\mathcal{F}_{j,k-1}^{\tau+1} = \mathcal{F}_{j,k}^{\tau} \quad (10)$$

where coefficients $a_{j,k}$, $b_{j,k}$, $c_{j,k}$, $d_{j,k}$, $f_{j,k}$, $g_{j,k}$ is determine by next expressions:

$$a_{j,k} = A_{j,k}^{\tau+1}\left\{\left(\frac{B_{j+\frac{1}{2},k}^{\tau+1} + B_{j-\frac{1}{2},k}^{\tau+1}}{h_z^2}\right) + \left(\frac{B_{j,k+\frac{1}{2}}^{\tau+1} + B_{j,k-\frac{1}{2}}^{\tau+1}}{h_y^2}\right) - q_{j,k}^{\tau+1}\right\}, \quad (11)$$

$$b_{j,k} = A_{j,k}^{\tau+1}\left(-\frac{B_{j+\frac{1}{2},k}^{\tau+1}}{h_z^2} - \frac{C_{j,k}^{\tau+1} + D_{j+1,k}^{\tau+1}}{2h_z}\right), \quad (12)$$

$$c_{j,k} = A_{j,k}^{\tau+1}\left(-\frac{B_{j-\frac{1}{2},k}^{\tau+1}}{h_z^2} + \frac{C_{j,k}^{\tau+1} + D_{j-1,k}^{\tau+1}}{2h_z}\right), \quad (13)$$

$$d_{j,k} = A_{j,k}^{\tau+1}\left(-\frac{B_{j,k+\frac{1}{2}}^{\tau+1}}{h_y^2} - \frac{C_{j,k}^{\tau+1} + D_{j,k+1}^{\tau+1}}{2h_y}\right), \quad (14)$$

$$f_{j,k} = A_{j,k}^{\tau+1}\left(-\frac{B_{j,k-\frac{1}{2}}^{\tau+1}}{h_y^2} + \frac{C_{j,k}^{\tau+1} + D_{j,k-1}^{\tau+1}}{2h_y}\right). \quad (15)$$

To investigate the stability of the difference scheme (10), we introduce the Fourier modes with respect to two spatial coordinates:

$$\mathcal{F}_{j,k}^{\tau} = V(\tau\theta)e^{i\omega_z j h_z}e^{i\omega_y k h_y}, \quad (16)$$

$$\mathcal{F}_{j+1}^{\tau+1} = V(\tau\theta + \theta)e^{i\omega_z j h_z}e^{i\omega k h_y}, \quad (17)$$

where $V(\tau\theta)$ – the harmonic amplitude, ω_z, ω_y – the wave numbers.

We substitute the Fourier modes (16) - (17) into equation (10), group the terms and divide by $e^{i\omega_z j h_z}e^{i\omega_y k h_y}$:

$$V(\tau\theta + \theta) = \epsilon V(\tau\theta), \quad (18)$$

$$\epsilon = \frac{1}{1+\theta\left(a_{j,k} + b_{j,k}e^{i\omega_z h_z} + c_{j,k}e^{-i\omega_z h_z} + d_{j,k}e^{i\omega_y h_y} + f_{j,k}e^{-i\omega_y h_y}\right)}. \quad (19)$$

For the stability of the difference scheme (10), it is necessary to check the conditions wherein the modulus of the transition coefficient between next and current step ε is less than unity, i.e. $|\epsilon| \le 1$. We transform the exponents into trigonometric functions and get:

$$\Big|1 + \theta\big(a_{j,k} + (c_{j,k} + b_{j,k})cos(\omega_z h_z) + (f_{j,k} + d_{j,k})cos(\omega_y h_y) + (b_{j,k} - c_{j,k})i \cdot sin(\omega_z h_z) + i(d_{j,k} - f_{j,k})sin(\omega_y h_y)\big)\Big| \ge 1 \quad (20)$$

We seek the solution of inequality (20) in the sense of choosing a vector on the complex plane with length greater than unity. The length of vector $R = x + iy$ on the complex plane is $R = \sqrt{x^2 + y^2}$. Since the sinus can take any value, the worst case of stability corresponds to the zero of the imaginary part (20). Since cosines in left part (20) has any values in range [-1; 1], inequality (20) is true always, if inequality (21) is true:

$$\Big||1 + \theta a_{j,k}| - \theta|c_{j,k} + b_{j,k}| - \theta|f_{j,k} + d_{j,k}|\Big| \ge 1 \quad (21)$$

It is quite obvious that the solution of (21) always exists, because it is a linear function under the sing of the modulus.

Expand the external modulus and the left inner modulus and get the following four inequalities:

1) over $1 + \theta a_{j,k} > 0$:

$$a_{j,k} - |c_{j,k} + b_{j,k}| - |f_{j,k} + d_{j,k}| \ge 0 \quad (22)$$

$$\theta a_{j,k} - \theta|c_{j,k} + b_{j,k}| - \theta|f_{j,k} + d_{j,k}| \le -2 \quad (23)$$

2) over $1 + \theta a_{j,k} < 0$:

$$-\theta a_{j,k} - \theta|c_{j,k} + b_{j,k}| - \theta|f_{j,k} + d_{j,k}| \ge 2 \quad (24)$$

$$-a_{j,k} - |c_{j,k} + b_{j,k}| - |f_{j,k} + d_{j,k}| \le 0 \quad (25)$$

The stability of the difference scheme (10) requires that at least one of the inequalities (22) - (25) be true. The algorithm of using inequalities for choosing the time step is as follows. If inequalities (22) or (25) is true, then the time step can be any. Otherwise, the time step is selected by the following expressions:

$$\theta = \frac{-2}{a_{j,k} - |c_{j,k} + b_{j,k}| - |f_{j,k} + d_{j,k}|}, \text{ over } 1 + \theta a_{j,k} > 0 \quad (26)$$

$$\theta = \frac{2}{-a_{j,k} - |c_{j,k} + b_{j,k}| - |f_{j,k} + d_{j,k}|}, \text{ over } 1 + \theta a_{j,k} < 0 \quad (27)$$

Thus, for finite-difference schemes of the authors obtained the expressions (22-27) for the time step that ensure the stability of the solution of parabolic differential equations in system (1) - (7).

The expressions (22) - (27) show that the implicit difference scheme (9) is not absolutely stable and depends on the capacitive and conductive properties of the medium, as well as the intensity of the phase translation.

CONCLUSION

The convergence and stability of the finite difference method for a non-linear differential equation of parabolic type (9) is studied. To study of the stability, a spectral feature was chosen.

The authors obtained the expressions (22) - (27), which allow to choose the time step, which ensures the stability of the solution. It is shown that implicit difference scheme is not always absolutely stable. The time step depends on all of the parameters entering into equation (9).

Simulation performed by the authors showed that the greatest influence on the stability is made by the term describing the volume source Aq, because it has very large values. This is associated with large heat capacity of phase translation.

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