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Fixed point results in J-cone metric space over Banach algebra

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Abstract: In this article, we establish some coupled fixed point results for generalized Lipschitz mappings verifying in J-cone metric space over Banach algebra. Our results generalize, extend, and unify several well-known comparable results in the literature. An example is also presented.

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INTRODUCTION

Huang and Zhang [14] generalized the notion of metric spaces by replacing the set of real numbers by an ordered Banach space, named as cone metric space and obtained some fixed point theorems for mapping satisfying different contractive conditions in normal cone metric space. Afterwards, the results were generalized by Rezapour and Hamlbarani in [21] by omitting the assumption of normality in cone metric space, which became a breakthrough in the development of fixed point theory in cone metric space. For further details on this theme, one can be referred to [2], [5], [6], [7].

In recent past, several authors reported that some published results dealing with a cone metric space with normal cones can be considered as a simple consequence of the existence theorems in the setting of the usual metric spaces. Due to which they made a conclusion that cone metric spaces are equivalent to metric spaces where the real-valued metric function d^* is defined by a nonlinear scalarization function ξ_e [1] or by Minkowski functional q_e; see [18].

But the present failure transformed not long ago when Liu and Xu [19] introduced the notion of cone metric spaces over Banach algebra. They showed that their fixed point results with normal cones cannot be obtained by the corresponding results on usual metric spaces and gave examples to support their main results. This remarkable specialty attracted the attention of Xu and Radenović [24]. They generalized the results of [19] without the normality of cones. Motivated by these ideas, many authors further studied various cone metric spaces over Banach algebra (see [3], [4], [8]–[12]).

Very recently, Fernandez et al. [13] introduced J-cone metric space over Banach algebras which generalizes the concepts of G_{p_b}-metric space [16] and cone metric space over Banach algebra. The authors defined generalized Lipschitz and expansive maps and prove some fixed point theorems for such maps.

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The aim of this paper is to investigate some coupled fixed point results in the context of J-cone metric space over Banach algebra. Our results extend and generalize several results of [20]. An example is given to verify the strength of our main results.

PRELIMINARIES

Now we give the following definitions which are useful in our study.

Huang and Zhang [14] introduced the notion of a cone metric space and Liu and Radenović [19] introduced the concept of cone metric space over Banach algebra as follows:

Let A always be a real Banach algebra. That is, A is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in A$, $\alpha \in R$)

1. $(xy)z = x(yz)$,
2. $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$,
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$,
4. $\|xy\| \leq \|x\| \, \|y\|$.

Throughout this paper, we shall assume that a Banach algebra has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in A$. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details, we refer the reader to [22].

The following proposition is given in [22].

Proposition 1. Let A be Banach algebra with a unit e, and $x \in A$. If the spectral radius $\rho(x)$ of x is less than 1, i.e.

$$\rho(x) = \lim_{n\to\infty} \left\|x^n\right\|^{\frac{1}{n}} = \inf \left\|x^n\right\|^{\frac{1}{n}} < 1.$$

then $e - x$ is invertible. Actually,

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Remark 2. From [22] we see that the spectral radius $\rho(x)$ of x satisfies $\rho(x) \leq \|x\|$ for all $x \in A$, where A is a Banach algebra with a unit e.

Remark 3. (See [24]) In Proposition 1, if the condition $'\rho(x) < 1'$ is replaced by $\|x\| \leq 1$, then the conclusion remains true.

Remark 4. (See [24]) If $\rho(x) < 1$ then $\|x^n\| \to 0 (n \to \infty)$.

A subset P of A is called a cone if

1. P is non-empty closed and $\{\theta, e\} \subset P$;
2. $\alpha P + \beta P \subset P$ for all non-negative real numbers α, β;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{\theta\}$,

where θ denotes the null of the Banach algebra A. For a given cone $P \subset A$, we can define a partial ordering $\preccurlyeq$ with respect to P by $x \preccurlyeq y$ if and only if $y - x \in P$. $x \prec y$ will stand for $x \preccurlyeq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int P$, where $int P$ denotes the interior of P. If $int P \neq \emptyset$ then P is called a solid cone.

The cone P is called normal if there is a number $M > 0$ such that, for all $x, y \in A$,

$$\theta \preccurlyeq x \preccurlyeq y \Rightarrow \|x\| \leq M \|y\|.$$

The least positive number satisfying the above is called the normal constant of P [14].

In the following we always assume that A is a Banach algebra with a unit e, P is a solid cone in A and $\preccurlyeq$ is the partial ordering with respect to P.

Definition 5. ([14, 19]) Let X be a nonempty set. Suppose the mapping $d : X \times X \to A$ satisfies

1. $\theta \prec d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \preccurlyeq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over Banach algebra A.

For other definitions and related results on cone metric spaces over Banach algebra we refer to [19].

J-CONE METRIC SPACE OVER BANACH ALGEBRA

Definition 6. ([13]) Let X be a nonempty set. Suppose that the mapping $J : X \times X \times X \to A$ satisfies the following conditions:

(J_1) $x = y = z$ if $J(x,y,z) = J(z,z,z) = J(y,y,y) = J(x,x,x)$;
(J_2) $J(x,x,x) \preccurlyeq J(x,x,y) \preccurlyeq J(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$;
(J_3) $J(x,y,z) = J(p\{x,y,z\})$, where p is any permutation of x, y or z (symmetry in all three variables);
(J_4) $J(x,y,z) \preccurlyeq s[J(x,a,a) + J(a,y,z) - J(a,a,a)] + ((1-s)/3)[J(x,x,x) + J(y,y,y) + J(z,z,z)]$ for all $x,y,z,a \in X$ (rectangle inequality).

Then J is called a J-cone metric and (X,J) is called a J-cone metric space over Banach algebra A.

Since $s \geq 1$, so from J_4 we have the following inequality:

$$J(x,y,z) \preccurlyeq s[J(x,a,a) + J(a,y,z) - J(a,a,a)]$$

The J-cone metric space J is called symmetric if $J(x,x,y) = J(x,y,y)$ holds for all $x,y \in X$. Otherwise, J is an asymmetric J-cone metric over Banach algebra A.

Some examples of J-cone metric space over Banach algebra A are given below.

Example 7. ([13]) Let $X = [0,\infty)$ and let $G_{p_b} : X \times X \times X \to R^+$ be given by $G_{p_b}(x,y,z) = \left[\max\{x,y,z\}\right]^p$ where $p > 1$. Let $J : X \times X \times X \to A$ where A is a real commutative Banach algebra is given by

$$J(x,y,z) = \left(G_{p_b}(x,y,z), \alpha G_{p_b}(x,y,z)\right).$$

Obviously, (X, G_{p_b}) is not a G-metric space. Therefore (X,J) is a J-cone metric space over Banach algebra which is not a G-cone metric space [4].

The following examples show that a J-cone metric space over Banach algebra on X need not be a G_b-cone metric [23].

Example 8. ([13]) Let $X = [0,1]$ and A be the set of all real valued function on X which also have continuous derivatives on X with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$. Define multiplication in the usual way. Let $P = \{x \in A : x(t) \geq 0, t \in X\}$. It is clear that P is a nonnormal cone and A is a Banach algebra with a unit $e = 1$. Define a mapping $J : X \times X \times X \to A$ by

$$J(x,y,z)(t) = \big((\max\{x,y\}^2) + (\max\{y,z\}^2) + (\max\{z,x\}^2)\big)e^t$$

for all $x,y,z \in X$. This makes (X,J) into a J-cone metric space over Banach algebra A but it is not a G_b-cone metric space since $J(x,x,x)(t) = 3x^2e^t \neq \theta$.

Example 9. ([13]) Let $A = C^1_R[0,1]$ and define a norm on A by $\|x\| = \|x\|_\infty + \|x'\|_\infty$ for $x \in A$. Define multiplication in A as just point wise multiplication. Then A is a real unit Banach algebra with unit $e = 1$. Set $P = \{x \in A : x \geq 0\}$ is a cone in A. Moreover, P is not normal (see [29]). Let $X = [0,\infty)$ and $a > 0$ be any constant. Define a mapping $J : X \times X \times X \to A$ by

$$J(x,y,z)(t) = \big([\max\{x,y,z\}]^2 + a\big)e^t$$

for all $x,y,z \in X$. This (X,J) is a J-cone metric space over Banach algebra A but it is not a G_b-cone metric space since $J(x,x,x)(t) = (x^2+a)e^t \neq \theta$.

Example 10. ([13]) Let $A = C[a,b]$ be the set of continuous functions on the interval $[a,b]$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$. Define multiplication in the usual way. Then A is a Banach algebra with a unit 1. Set $P = \{x \in A : x(t) \geq 0, t \in [a,b]\}$ and $X = R^+$. Define a mapping $J : X \times X \times X \to A$ by

$$J(x,y,z)(t) = \big([\max\{x,y,z\}]^2 + |x-y|^2 + |y-z|^2 + |z-x|^2\big)e^t$$

for all $x,y,z \in X$. Then (X,J) is a J-cone metric space over Banach algebra A but it is not a G_b-cone metric space since $J(x,x,x)(t) = x^2e^t \neq \theta$.

Lemma 11. ([13]) Let (X,J) be a J-cone metric space over Banach algebra A. Then

(a) if $J(x,y,z) = 0$, then $x = y = z$
(b) if $x \neq y$, then $J(x,y,y) \succ \theta$.

Proof. The proof is obvious. ■

Definition 12. ([13]) Let (X,J) is a J-cone metric space over Banach algebra A. Then for an $x \in X$ and $c > \theta$, the J-balls with center x and radius $c > \theta$ is

$$B_J(x,c) = \{y \in X : J(x,x,y) \ll J(x,x,x) + c\}$$

TOPOLOGY ON J-CONE METRIC SPACE OVER BANACH ALGEBRA

Definition 13. ([13]) Let (X,J) be a J-cone metric space over Banach algebra A with coefficient $s \geq 1$. For each $x \in X$ and each $\theta \ll c$, put $B_J(x,c) = \{y \in X : J(x,x,y) \ll J(x,x,x) + c\}$ and put $\mathfrak{B} = \{B_J(x,c) : x \in X \text{ and } \theta \ll c\}$. Then $\mathfrak{B}$ is a subbase for some topology τ on X.

Remark 14. **([13])** Let (X, J) be a J-cone metric space over Banach algebra A. In this paper, τ denotes the topology on X, $\mathfrak{B}$ denotes a subbase for the topology on τ and $B_J(x, c)$ denotes the J-ball in (X, J), which are described in Definition 4.1. In addition $U^{\cdot}$ denotes the base generated by the subbase $\mathfrak{B}$.

Theorem 15. **([13])** Every J-cone metric space over Banach algebra A is a topological space.

Theorem 16. **([13])** Let (X, J) be a J-cone metric space over Banach algebra A and let P be a solid cone in a Banach algebra A where $k \in P$ is an arbitrarily given vector, then (X, J) is a T_0-space.

Definition 17. **([13])** Let (X, J) be a J-cone metric space over Banach algebra A. A sequence $\{x_n\}$ in (X, J) converges to a point $x \in X$ whenever for every $c \gg \theta$ there is a natural number N such that $J(x_n, x, x) \ll c$ for all $n \geq N$. We denote this by

$$\lim_{n\to\infty} x_n = x \ \text{ or } \ x_n \to x \ (n \to \infty).$$

Definition 18. **([13])** Let (X, J) be a J-cone metric space over Banach algebra A. A sequence $\{x_n\}$ in X is said to be a θ-Cauchy sequence in (X, J) if $\{J(x_n, x_m, x_m)\}$ is a c-sequence in A, i.e. if for every $c \gg \theta$ there exists $n_0 \in N$ such that, $J(x_n, x_m, x_m) \ll c$ for all $n, m \geq n_0$.

Definition 19. **([13])** Let (X, J) be a J-cone metric space over Banach algebra A. Then X is said to be θ-complete if every θ-Cauchy sequence $\{x_n\}$ in (X, J) converges to $x \in X$ such that $J(x, x, x) = \theta$.

Definition 20. **([13])** Let (X, J) and (X', J') be a J-cone metric space over Banach algebra A. Then a function $f : X \to X'$ is said to be continuous at a point $x \in X$ if and only if it is sequentially continuous at x, that is whenever $\{x_n\}$ is convergent to x we have $\{f x_n\}$ is convergent to $f(x)$.

Definition 21. **([13])** Let (X, J) be an J-cone metric space over Banach algebra A and P be a cone in A. A map $T : X \to X$ is said to be a generalized Lipschitz mapping if there exists a vector $k \in P$ with $\rho(k) < 1$ for all $x, y \in X$ such that

$$J(Tx, Ty, Tz) \preccurlyeq k \ J(x, y, z)$$

Example 22. Let Banach algebra A and cone P be the same ones as those in Example 9 and let $X = R^+$. Define a mapping $J : X \times X \times X \to A$ by

$$J(x, y, z)(t) = \big([\max\{x, y, z\}]^2 + |x - y|^2 + |y - z|^2 + |z - x|^2\big)e^t$$

for all $x, y, z \in X$. Then (X, J) into a J-cone metric space over Banach algebra A. Now define the mappings $T : X \to X$ by $T(x) = \ln\left(1 + \frac{x}{2}\right)$. Since $\ln(1 + u) \leq u$ for each $u \in [0, \infty)$, for all $x, y, z \in X$. Clearly, T is a generalized Lipschitz map in X.

Now we review some facts on c-sequence theory.

Definition 23. **([18])** Let P be a solid cone in a Banach space E. A sequence $\{u_n\} \subset P$ is said to be a c-sequence if for each $c \gg \theta$ there exists a natural number N such that $u_n \ll c$ for all $n > N$.

Lemma 24. **([15])** If E is a real Banach space with a solid cone P and $\{u_n\} \subset P$ be a sequence with $\|u_n\| \to 0$ $(n \to \infty)$, then u_n is a c-sequence.

Lemma 25. **([22])** Let A be a Banach algebra with a unit $e, k \in A$, then $\lim_{n\to\infty} \|k^n\|^{\frac{1}{n}}$ exists and the spectral radius $\rho(k)$ satisfies

$$\rho(k) = \lim_{n\to\infty} \left\|k^n\right\|^{\frac{1}{n}} = \inf \left\|k^n\right\|^{\frac{1}{n}}.$$

If $\rho(k) < |\lambda|$, then $(\lambda e - k)$ is invertible in A, moreover,

$$(\lambda e - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{\lambda^{i+1}}$$

where λ is a complex constant.

Lemma 26. **([22])** Let A be a Banach algebra with a unit $e, a, b \in A$. If a commutes with b, then

$$\rho(a + b) \leq \rho(a) + \rho(b), \quad \rho(ab) \leq \rho(a)\rho(b).$$

Lemma 27. **([17])** If E is a real Banach space with a solid cone P

(1) If $a, b, c \in E$ and $a \leq b \ll c$, then $a \ll c$.
(2) If $a \in P$ and $a \ll c$ for each $c \gg \theta$, then $a = \theta$.

Lemma 28. **([24])** Let P be a solid cone in a Banach algebra A. Suppose that $k \in P$ and $\{u_n\}$ is a c-sequence in P. Then $\{ku_n\}$ is a c-sequence.

Lemma 29. **([15])** Let A be a Banach algebra with a unit e and $k \in A$. If λ is a complex constant and $\rho(k) < |\lambda|$, then

$$\rho((\lambda e - k)^{-1}) \leq \frac{1}{|\lambda| - \rho(k)}.$$

Lemma 30. **([15])** Let A be a Banach algebra with a unit e and P be a solid cone in A. Let $a, k, l \in P$ hold $l \preccurlyeq k$ and $a \preccurlyeq la$. If $\rho(k) < 1$, then $a = \theta$.

Now, we can state our main results.

APPLICATIONS TO FIXED POINT THEORY

In this Section, we prove some fixed point theorems for satisfying generalized Lipschitz maps in the setting of J-cone metric space over Banach algebra A.

Theorem 31. Let (X, J) be a θ-complete symmetric J-cone metric space over Banach algebra A and P be a solid cone. Suppose $F : X \times X \to X$ be a mapping satisfying the following condition for all $x, y, x^*, y^* \in X$

$$\begin{aligned} &J(F(x, y), F(x^*, y^*), F(x^*, y^*)) \\ &\preccurlyeq \alpha\ J(x, x^*, x^*) + \beta J(y, y^*, y^*) \end{aligned} \quad (5.1)$$

where $\rho(\alpha + \beta) < \frac{1}{s}$. Then F has a unique coupled fixed point.

Proof. Let $x_0, y_0 \in X$ and set $x_1 = F(x_0, y_0)$, $y_1 = F(y_0, x_0), \ldots, x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n)$.

From (5.1), we have

$$\begin{aligned} J(x_n, x_{n+1}, x_{n+1}) &= J(F(x_{n-1}, y_{n-1}), F(x_n, y_n), F(x_n, y_n)) \\ &\preccurlyeq \alpha\ J(x_{n-1}, x_n, x_n) + \beta J(y_{n-1}, y_n, y_n) \end{aligned} \quad (5.2)$$

and

$$\begin{aligned} J(y_n, y_{n+1}, y_{n+1}) &= J(F(y_{n-1}, x_{n-1}), F(y_n, x_n), F(y_n, x_n)) \\ &\preccurlyeq \alpha\ J(y_{n-1}, y_n, y_n) + \beta J(x_{n-1}, x_n, x_n) \end{aligned} \quad (5.3)$$

Let $J_n = J(x_n, x_{n+1}, x_{n+1}) + J(y_n, y_{n+1}, y_{n+1})$.
From (5.2) and (5.3), we get

$$\begin{aligned} J_n &\preccurlyeq (\alpha + \beta) J(x_{n-1}, x_n, x_n) + (\alpha + \beta) J(y_{n-1}, y_n, y_n) \\ J_n &\preccurlyeq (\alpha + \beta)[J(x_{n-1}, x_n, x_n) + J(y_{n-1}, y_n, y_n)] \\ &\preccurlyeq \lambda J_{n-1} \end{aligned}$$

where $\lambda = \alpha + \beta$. Thus, for all n,

$$\theta \preccurlyeq J_n \preccurlyeq \lambda J_{n-1} \preccurlyeq \lambda^2 J_{n-2} \preccurlyeq ----\preccurlyeq \lambda^n J_0 \quad (5.4)$$

If $J_0 = 0$ then (x_0, y_0) is a coupled fixed point of F. Now, let $J_0 > 0$. If $m > n$, we have

$$\begin{aligned} &J(x_n, x_m, x_m) \\ &\preccurlyeq s\big[J(x_n, x_{n+1}, x_{n+1}) + J(x_{n+1}, x_m, x_m) \\ &\quad - J(x_{n+1}, x_{n+1}, x_{n+1})\big] \\ &\preccurlyeq s\ J(x_n, x_{n+1}, x_{n+1}) + s\ J(x_{n+1}, x_m, x_m) \\ &\preccurlyeq s\ J(x_n, x_{n+1}, x_{n+1}) + s^2\big[J(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + J(x_{n+2}, x_m, x_m) - J(x_{n+2}, x_{n+2}, x_{n+2})\big] \\ &\preccurlyeq s\ J(x_n, x_{n+1}, x_{n+1}) + s^2 J(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + s^2 J(x_{n+2}, x_m, x_m) \\ &\vdots \\ &\preccurlyeq s\ J(x_n, x_{n+1}, x_{n+1}) + s^2 J(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots \\ &\quad + s^{m-n} J(x_{m-1}, x_m, x_m) \end{aligned} \quad (5.5)$$

and similarly,

$$\begin{aligned} J(y_n, y_m, y_m) &\preccurlyeq s\ J(y_n, y_{n+1}, y_{n+1}) + s^2 J(y_{n+1}, y_{n+2}, y_{n+2}) \\ &\quad + \cdots + s^{m-n} J(y_{m-1}, y_m, y_m) \end{aligned} \quad (5.6)$$

Adding up (5.5) and (5.6) and using (5.4). Since

$$\begin{aligned} &J(x_n, x_m, x_m) + J(y_n, y_m, y_m) \\ &\preccurlyeq s J_n + s^2 J_{n+1} + \cdots + s^{m-n} J_{m-1} \\ &\preccurlyeq [s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n}\lambda^{m-1}] J_0 \\ &= s\lambda^n[e + s\lambda + (s\lambda)^2 + \cdots + (s\lambda)^{m-n-1}] J_0 \\ &= (e - s\lambda)^{-1} s\lambda^n J_0. \end{aligned}$$

In view of Remark 4, $\|(s\lambda)^n J_0\| \preccurlyeq \|(sk)^n\| \, \|J_0\| \to 0$ $(n \to \infty)$, by Lemma 24, we have $\{(s\lambda)^n J_0\}$ is a c-sequence. Next by using Lemma 27 and Lemma 28, we conclude that $\{x_n\}$ and $\{y_n\}$ is a θ-Cauchy sequence in X.

By the θ-completeness of X, there exists $x', y' \in X$ such that

$$\lim_{n\to\infty} J(x_n, x', x') = \lim_{n\to\infty} J(x_n, x_m, x_m) = J(x', x', x') = \theta. \quad (5.7)$$

and

$$\lim_{n\to\infty} J(y_n, y', y') = \lim_{n\to\infty} J(y_n, y_m, y_m) = J(y', y', y') = \theta. \quad (5.8)$$

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Now, we prove that $F(x', y') = x'$ and $F(y', x') = y'$. From triangle inequality and (5.1), we have

$$\begin{aligned}
&J(F(x',y'),x',x') \\
&\quad \preccurlyeq s[J(F(x',y'),x_{n+1},x_{n+1}) \\
&\qquad + J(x_{n+1},x',x') - J(x_{n+1},x_{n+1},x_{n+1})] \\
&\quad \preccurlyeq s[J(F(x',y'),x_{n+1},x_{n+1}) \\
&\qquad + J(x_{n+1},x',x')] \\
&\quad = sJ(F(x',y'),F(x_n,y_n),F(x_n,y_n)) \\
&\qquad + sJ(x_{n+1},x',x') \\
&\quad \preccurlyeq s\alpha J(x',x_n,x_n) + s\beta J(y',y_n,y_n) \\
&\qquad + sJ(x_{n+1},x',x')
\end{aligned}$$

Now that $\{J(F(x', x_n, x_n)\}$, $\{J(y', y_n, y_n)\}$ and $\{J(F(x_{n+1}, x', x')\}$ are c-sequences then by using Lemma 27 and Lemma 28, it concludes that $F(x', y') = x'$. Similarly, we can get $F(y', x') = y'$.

Therefore, (x', y') is a coupled fixed point of F. Now, if (x'', y'') is another coupled fixed point of F, then

$$\begin{aligned}
&J(x',x'',x'') + J(y',y'',y'') \\
&\quad = J(F(x',y'),F(x'',y''),F(x'',y'')) \\
&\qquad + J(F(y',x'),F(y'',x''),F(y'',x'')) \\
&\quad \preccurlyeq \alpha J(x',x'',x'') + \beta J(y',y'',y'') + \alpha J(y',y'',y'') \\
&\qquad + \beta J(x',x'',x'') \\
&\quad \preccurlyeq (\alpha+\beta)J(x',x'',x'') + (\alpha+\beta)J(y',y'',y'') \\
&\quad \preccurlyeq (\alpha+\beta)(J(x',x'',x'') + J(y',y'',y''))
\end{aligned}$$

$$[e-(\alpha+\beta)](J(x',x'',x'') + J(y',y'',y'')) \preccurlyeq \theta$$

we get, $J(x', x'', x'') + J(y', y'', y'') \preccurlyeq \theta$.
Thus, $J(x', x'', x'') + J(y', y'', y'') = \theta$.
Thus $(x', y') = (x'', y'')$. This completes the proof. ∎

Corollary 32. Let (X, J) be a θ-complete symmetric J-cone metric space over Banach algebra A and P be a solid cone. Suppose $F : X \times X \to X$ be a mapping satisfying the following condition for all $x, y, x^*, y^* \in X$

$$\begin{aligned}
&J(F(x,y),F(x^*,y^*),F(x^*,y^*)) \\
&\quad \preccurlyeq \frac{\gamma}{2}[\,J(x,x^*,x^*) + J(y,y^*,y^*)]
\end{aligned} \tag{5.9}$$

where $\rho(\gamma) < \frac{1}{s}$. Then F has a unique coupled fixed point.

Proof. Corollary 32 follows from Theorem 31 by setting $\alpha = \beta = \frac{\gamma}{2}$.

Theorem 33. Let (X, J) be a θ-complete symmetric J-cone metric space over Banach algebra A and P be a solid cone. Suppose $F : X \times X \to X$ be a mapping satisfying the following condition for all $x, y, x^*, y^* \in X$

$$\begin{aligned}
&J(F(x,y),F(x^*,y^*),F(x^*,y^*)) \\
&\quad \preccurlyeq \alpha\ J(F(x,y),x,x) + \beta J(F(x^*,y^*),x^*,x^*)
\end{aligned} \tag{5.10}$$

where $\rho(\lambda) < \frac{1}{s}$. Then F has a unique coupled fixed point.

Proof. Let $x_0, y_0 \in X$ and set $x_1 = F(x_0, y_0)$, $y_1 = F(y_0, x_0), \ldots, x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n)$.

From (5.10), we have

$$\begin{aligned}
&J(x_n,x_{n+1},x_{n+1}) \\
&\quad = J(F(x_{n-1},y_{n-1}),F(x_n,y_n),F(x_n,y_n)) \\
&\quad \preccurlyeq \alpha\ J(F(x_{n-1},y_{n-1}),x_{n-1},x_{n-1}) + \beta J(F(x_n,y_n),x_n,x_n) \\
&\quad \preccurlyeq \alpha\ J(x_n,x_{n-1},x_{n-1}) + \beta J(x_{n+1},x_n,x_n)
\end{aligned}$$

$$(e-\beta)J(x_n,x_{n+1},x_{n+1}) \preccurlyeq \alpha\ J(x_n,x_{n-1},x_{n-1}) \tag{5.11}$$

and

$$\begin{aligned}
&J(y_n,y_{n+1},y_{n+1}) \\
&\quad = J(F(y_{n-1},x_{n-1}),F(y_n,x_n),F(y_n,x_n)) \\
&\quad \preccurlyeq \alpha\ J(F(y_{n-1},x_{n-1}),y_{n-1},y_{n-1}) + \beta J(F(y_n,x_n),y_n,y_n) \\
&\quad \preccurlyeq \alpha\ J(y_n,y_{n-1},y_{n-1}) + \beta J(y_{n+1},y_n,y_n)
\end{aligned}$$

$$(e-\beta)J(y_n,y_{n+1},y_{n+1}) \preccurlyeq \alpha\ J(y_n,y_{n-1},y_{n-1}) \tag{5.12}$$

Since $s\ \rho(\alpha) + \rho(\beta) < 1$ leads to $\rho(\beta) < 1$, it concludes by Lemma 4.13 that $(e - \beta)$ is invertible. From (5.11) and (5.12), we get

$$J(x_n,x_{n+1},x_{n+1}) \preccurlyeq (e-\beta)^{-1}\alpha J(x_n,x_{n-1},x_{n-1}) \tag{5.13}$$

and

$$J(y_n,y_{n+1},y_{n+1}) \preccurlyeq (e-\beta)^{-1}\alpha\ J(y_n,y_{n-1},y_{n-1}) \tag{5.14}$$

Put $\lambda = (e - \beta)^{-1}\alpha$, it is evident that from 5.13 and 5.14

$$J(x_n,x_{n+1},x_{n+1}) \preccurlyeq \lambda J(x_n,x_{n-1},x_{n-1}) \tag{5.15}$$

and

$$J(y_n,y_{n+1},y_{n+1}) \preccurlyeq \lambda J(y_n,y_{n-1},y_{n-1}) \tag{5.16}$$

Note that by Lemma 26 and Lemma 30 that

$$\begin{aligned}\rho(\lambda) &= \rho\big[(e-\beta)^{-1}\alpha\big] \\ &\leq \rho\big((e-\beta)^{-1}\big)\rho(\alpha) \\ &\leq \frac{\rho(\alpha)}{1-\rho(\beta)} \\ &< \frac{1}{s}\end{aligned}$$

By the analogous arguments as in Theorem 31 we conclude that $\{x_n\}$ and $\{y_n\}$ is a θ-Cauchy sequence in X. By the θ-completeness of X, there exists $x^{'}, y^{'} \in X$ such that

$$\lim_{n\to\infty} J(x_n, x^{'}, x^{'}) = \lim_{n\to\infty} J(x_n, x_m, x_m) = J(x^{'}, x^{'}, x^{'}) = \theta. \tag{5.17}$$

and

$$\lim_{n\to\infty} J(y_n, y^{'}, y^{'}) = \lim_{n\to\infty} J(y_n, y_m, y_m) = J(y^{'}, y^{'}, y^{'}) = \theta. \tag{5.18}$$

Now, we prove that $F(x^{'}, y^{'}) = x^{'}$ and $F(y^{'}, x^{'}) = y^{'}$. From triangle inequality and (5.10), we have

$$\begin{aligned}&J(F(x^{'}, y^{'}), x^{'}, x^{'}) \\ &\quad\preccurlyeq s[J(F(x^{'}, y^{'}), x_{n+1}, x_{n+1}) + J(x_{n+1}, x^{'}, x^{'}) \\ &\qquad - J(x_{n+1}, x_{n+1}, x_{n+1})] \\ &\quad\preccurlyeq s[J(F(x^{'}, y^{'}), x_{n+1}, x_{n+1}) + J(x_{n+1}, x^{'}, x^{'})] \\ &\quad= sJ(F(x^{'}, y^{'}), F(x_n, y_n), F(x_n, y_n)) + sJ(x_{n+1}, x^{'}, x^{'}) \\ &\quad\preccurlyeq s\alpha J(F(x^{'}, y^{'}), x^{'}, x^{'}) + s\beta J(F(x_n, y_n), x_n, x_n) \\ &\qquad + sJ(x_{n+1}, x^{'}, x^{'})\end{aligned}$$

which implies that

$$\begin{aligned}&(e - s\alpha)J(F(x^{'}, y^{'}), x^{'}, x^{'}) \\ &\quad\preccurlyeq s\beta J(F(x_n, y_n), x_n, x_n) + sJ(x_{n+1}, x^{'}, x^{'})\end{aligned}$$

Now that $\{J(F(x_n, y_n), x_n, x_n)\}$ and $\{J(x_{n+1}, x^{'}, x^{'})\}$ are c-sequences then by using Lemma 4.15 and Lemma 4.16, it concludes that $F(x^{'}, y^{'}) = x^{'}$. Similarly, we can get $F(y^{'}, x^{'}) = y^{'}$.
Therefore, $(x^{'}, y^{'})$ is a coupled fixed point of F. Now, if $(x^{''}, y^{''})$ is another coupled fixed point of F, then

$$\begin{aligned}J(x^{'}, x^{''}, x^{''}) &\preccurlyeq J(F(x^{'}, y^{'}), F(x^{''}, y^{''}), F(x^{''}, y^{''})) \\ &\preccurlyeq \alpha J(F(x^{'}, y^{'}), x^{'}, x^{'}) \\ &\quad + \beta J(F(x^{''}, y^{''}), x^{''}, x^{''})\end{aligned}$$

Now that $\{J(F(x^{'}, y^{'}), x^{'}, x^{'})\}$ and $\{J(F(x^{''}, y^{''}), x^{''}, x^{''})\}$ are c-sequences then by using Lemma 4.15 and Lemma 4.16, it concludes that $x^{'} = x^{''}$. Similarly, we can get $y^{'} = y^{''}$. Thus, $(x^{'}, y^{'}) = (x^{''}, y^{''})$. This completes the proof. ■

Example 34. Let $A = C_1^R[0,1]$ and and define a norm on A by $\|x\| = \|x\|_\infty + \left\|x^{'}\right\|_\infty$ for $x \in A$. Define multiplication in A as just point wise multiplication. Then A is a real unit Banach algebra with unit $e = 1$. Set $P = \{x \in A : x \geq 0\}$ is a cone in A. Moreover, P is not normal (see [21]). Let $X = [0,\infty)$. Define a mapping $J : X^3 \to A$ by $J(x,y,z)(t) = (\max\{x,y,z\})^2 e^t$, for all $x, y, z \in X$. Then (X, J) is a complete J-cone metric space over Banach algebra A. Now define the mappings $F : X \times X \to X$ by $F(x,y) = \frac{x+y}{4}$, for all $x, y \in X$. Also,

$$\begin{aligned}&J\big(F(x,y), F(x^*,y^*), F(x^*,y^*)\big)(t) \\ &\quad= J\Big(\frac{x+y}{4}, \frac{x^*+y^*}{4}, \frac{x^*+y^*}{4}\Big)(t) \\ &\quad= \Big(\max\Big\{\frac{x+y}{4}, \frac{x^*+y^*}{4}, \frac{x^*+y^*}{4}\Big\}\Big)^2 e^t \\ &\quad= \Big(\max\Big\{\frac{x}{4}, \frac{x^*}{4}, \frac{x^*}{4}\Big\} + \max\Big\{\frac{y}{4}, \frac{y^*}{4}, \frac{y^*}{4}\Big\}\Big)^2 e^t \\ &\quad= \Big(\frac{1}{4}\big[\max\{x,x^*,x^*\} + \max\{y,y^*,y^*\}\big]\Big)^2 e^t \\ &\quad\preccurlyeq \frac{1}{16}\big[\max\{x,x^*,x^*\} + \max\{y,y^*,y^*\}\big]^2 e^t \\ &\quad\preccurlyeq \frac{1}{16}\big[J(x,x^*,x^*) + J(y,y^*,y^*)\big](t)\end{aligned}$$

Then the contractive condition (5.1) holds trivially good and θ is the unique fixed point of the map T. That is, conditions of Theorem 31 hold for this example.

CONCLUSION

In this paper, we have proved the existence and uniqueness of fixed point results for a generalized Lipschitz maps in J-cone metric space over Banach algebra A. An example is given to illustrate the results.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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