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A New Application of G'/G-Expansion Method for Travelling Wave Solutions of Fractional PDEs

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Abstract: In this work, we construct travelling wave solutions for fractional partial differential equations by using the logarithmic derivative method, also called as the $(G^{'}/G)$-expansion method, where $G = G(\xi)$ satisfies a second order linear ordinary differential equation.The travelling wave solutions are expressed in terms of hyperbolic, trigonometric and rational functions. When applied the method to physical problems, it is observed that the exact solutions can be obtained just by solving a system of linear or nonlinear algebraic equations.

AMS subject classification: 35C07, 35C08, 35Q53.
Keywords: Travelling wave solution, $(G^{'}/G)$-expansion method, homogeneous balancing.

INTRODUCTION

The field of fractional calculus is almost as old as calculus itself, but over the last few decades the usefulness of this mathematical theory in various fields such as physics, biology, control theory, ecology, finance, signal processing etc. has become more and more evident [1,23]. Numerous attempts to solve equations involving non-integer order derivatives can be found in the literature. Scientists have solved fractional differential equations by using Homotopy Perturbation method [2], Adomian Decomposition Method [3], Variational Iteration Method [4], Differential Transform Method [6], Adam-Bashforth predictor method [5], various collocation method [7], [8] etc. But finding the exact solutions of nonlinear fractional differential equatios(FDEs) were very difficult until Li and He [10], [11] proposed a fractional complex transform to convert FDEs into ordinary differential equations (ODEs), so that all analytical methods applied to solve ODEs can be used to solve FDEs. Then to find exact solutions of FDEs many methods have been proposed. Along with fractional exp-function [12], [13], tanh-function [14], projective Riccati equation method [15], the fractional integral method [16], the fractional sub-equation method [17], the fractional modified trial equation method [18], the fractional simplest equation method [19] etc., $\frac{G^{'}}{G}$ [20], [21] is also one of the powerful method to solve FDEs. One of the advantages of this method is that, we can find the exact solution of FDEs without a initial or boundary conditions.

In section 2, Preliminaries and $\frac{G^{'}}{G}$-expansion method for fractional PDEs are described. The next three ensuing sections are devoted to compute solitary wave solutions of three non linear fractional PDEs, namely fractional Harry Dym equation and modified Zakharov- Kuznetsov equation respectively. The last section has concluding remarks.

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PRELIMINARIES AND $\frac{G'}{G}$-EXPANSION METHOD FOR FRACTIONAL PDES

There are several different definitions of the concept of a fractional derivatives [23]. Some of these are Riemann-Liouville, Grunwald-Letnikow, Caputo, and modifi'ed Riemann-Liouville definitive. The most commonly used defi'nitions are the Riemann-Liouville and Caputo derivatives. In this paper, we find the exact solution of three very important nonlinear partial FDEs [24], [33], [34] by $\frac{G'}{G}$ expansion method which was first proposed by the Chinese mathematician Wang et al. [22], with fractional derivative in Caputo sense.

Some of the basic definitions of fractional calculus are presented here [23]

Definition 1. A real valued function $f(t), t > 0$ is said to be in the space $C_\mu, \mu\epsilon\mathbb{R}$ if there exist a real number $p > \mu$ such that $f(t) = t_p f_1(t)$ where $f_1(t)\epsilon C(0,\infty)$ and it is said to be in space C_n if and only if $f^{(n)}\epsilon C_\mu, n\epsilon N$.

Definition 2. The Riemann-Liouville fractional integral operator J_t^β of order β is defined as

$$J_t^\beta(t) = \frac{1}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1} f(\tau)d\tau, \ \ (\beta > 0, t > 0)$$

and the fractional derivative of order α in Caputo sense with $n-1 < \alpha < n$ of $f(t), t > 0$ is defined by

$$D^\alpha f(t) = J^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{(\alpha-n+1)}}d\tau$$

following are some of the important properties of fractional derivative and integration: [6-7]
Let $\beta, \gamma\epsilon\mathbb{R}$ and $\alpha\epsilon(0,1)$. Than

1. $J_a^\beta : L^1 -> L^1$, and if $f(x)\epsilon L^1$, then $J_a^\gamma J_a^\beta f(x) = J_a^{\gamma+\beta} f(x)$.
2. $\lim_{\beta\to n} J_a^\beta f(x) = J_a^n f(x)$ uniformly on $[a,b], n = 1, 2, 3...$ where $J_a^1 f(x) = \int_a^x f(s)ds$
3. If $f(x)$ is absolutely continuous on $[a,b]$, then $\lim_{\alpha\to 1} D^\alpha f(x) = \frac{df(x)}{dx}$.
4. If $f(x) = k \neq 0$, k is constant, then $D^\alpha k = 0$.
5. $D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{(\beta-\alpha)}$
6. Chain Rule: if $V = V(x,y)$ and $x = x(t), y = y(t)$ then $\frac{d^\alpha V}{dt^\alpha} = \frac{\partial V}{\partial x}\frac{d^\alpha x}{dt^\alpha} + \frac{\partial V}{\partial y}\frac{d^\alpha y}{dt^\alpha}$

In this section we also apply a technique to find travelling wave solution of nonlinear PDEs, called $\frac{G\prime}{G}$ or logarithmic derivative method. We assume a nonlinear fractional partial differential equation(FPDE) for $u(x,t)$ in the form

$$f(u, D_t^\alpha u, D_x^\alpha u, D_x^\beta u, D_x^\alpha D_x^\beta u, D_t^\alpha D_t^\beta u, \dots) \quad (1)$$

Where f is a polynomial in its arguments. The $(\frac{G'}{G})$-expansion method can be presented in the following steps:
Step 1. Suppose that

$$u(x,t) = u(\xi), \quad \xi = \xi(x,t) = \frac{kx^{k\beta}}{\Gamma(1+\beta)} \pm \frac{kt^{c\alpha}}{\Gamma(1+\alpha)} \quad (2)$$

c and k are constants. Using the chain rule of fractional derivatives, the travelling wave variable 2 allow us to reduce 1 to an ordinary differential equation for $u = u(\xi)$ in the form :

$$F(u, u', u'', \dots) = 0 \quad (3)$$

where $' = d/d\xi$

Step 2. Suppose that the solution of 3 can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^n a_i \left(\frac{G'}{G}\right)^i \quad (4)$$

While $G = G(\xi)$ satisfies the second order linear differential equation in the form:

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \quad (5)$$

Where $a_i (i = 0, 1, \dots, n), \lambda \quad and \quad \mu$ are constants to be determined later.

Step3. Determine n by balancing the highest order nonlinear term(s) and the highest order derivative as follows:
If we define the degree of $u(\xi)$ as $D[u(\xi)] = n$ then the degree of the other expression is defined by

$$D[u^r \left(\frac{d^q u}{d\xi^q}\right)^s] = nr + s(q+n)$$

Substituting equation these along with equation 5 into equation 3 so that equation 3 is converted into a polynomial in $\frac{G'}{G}$ and all terms with the same order of $\frac{G'}{G}$. Then set each coefficient to zero to derive a set of algebraic equations for λ, μ, a_0, a_i.

Step 4. Solve the system of algebraic equations obtained in step 3 for λ, μ, a_0, a_i.
Solving the system of equations yields three cases

Case 1: $\lambda^2 - 4\mu > 0$

$$\left(\frac{G'}{G}\right) = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\frac{c_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right) + c_2 \sinh\left(\frac{1}{2}\right)\sqrt{\lambda^2-4\mu}\xi}{c_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right) + c_2 \cosh\left(\frac{1}{2}\right)\sqrt{\lambda^2-4\mu}\xi} - \frac{\lambda}{2}$$

Where c_1 and c_2 are constants.

Case 2: $\lambda^2 - 4\mu < 0$

$$\left(\frac{G'}{G}\right) = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\frac{-c_1 \sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right) + c_2 \cos\left(\frac{1}{2}\right)\sqrt{4\mu-\lambda^2}\xi}{c_1 \cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right) + c_2 \sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)} - \frac{\lambda}{2}$$

Where c_1 and c_2 are constants.

Case 3: $\lambda^2 - 4\mu = 0$

$$\frac{G'}{G} = \frac{c_2}{c_1 + c_2\xi} - \frac{\lambda}{2}$$

Where c_1 and c_2 are constants.

FRACTIONAL HARRY-DYM EQUATION

Harry Dym equation in its classical form is given as

$$u_t = u^3 u_{xxx} \tag{6}$$

The exact solution of the classical Harry Dym equation is given by [9]

$$u(x,t) = \left(a - \frac{3\sqrt{b}}{2}(x+bt)\right)^{2/3}$$

It is a nonlinear partial differential equation(PDE) that has nonlinearity and dispersion coupled together, found by Harry Dym while trying to transfer [24] some results about isospectral flow to the string equation. The relationship between the HD and the classical string problem, with variables elastic parameter, was pointed out in 1979 by Sabatier [25]. Since then some valuable investigations have been carried out about this equation. Its Hamiltonian formulation, its complete integrability, its infinitely many conservation laws, its infinitely many symmetries, applicability of the spectral gradient method, its reciprocal Backlund Transformation(BT) and so on are quite interesting.

HD equation is a completely integrable nonlinear equation which can be solved by the inverse scattering Transformation (IST) [26]. Dijkhuis and Drohm [27] and Calogero and Degasperies [28] discussed the HD equation as a special case of a new broad class of nonlinear PDEs. However HD equation does not possess the Painleve' property which indicate that the Painleve' property is sufficient but not necessary for integrability.

Ibragimov [29], [30], Calogero and Degasperies [28] and Weiss [31], [32] are the first researchers who investigated the connection between the HD and the Korteweg de Vries(KdV) equations. W Hereman, P P Barnerjee and M R Chatterjee [24] rederived the implicit solution to the HD equation without the machinery like IST.

In this section, we consider the following nonlinear time fractional Harry Dym equation of the form

$$D_t^\alpha u(x,t) = u^3(x,t)u_{xxx}(x,t), 0 < \alpha \leq 1 \tag{7}$$

with initial condition $u(x,0) = (a - \frac{3\sqrt{b}}{2}x)^{2/3}$ where α is parameter describing the order of the fractional derivative and $u(x,t)$ is a function of x and t. The fractional derivative is understood in the Caputo sense. For $\alpha = 1$, the fractional Harry Dym equation reduces to the classical nonlinear Harry Dym equation. Guided by Hereman et al. in [24] we employ the transformation

$$X = \int_{-\infty}^{x} \frac{ds}{u(s,t)}, T = -\frac{t^\alpha}{\Gamma(1+\alpha)}$$

with $W(X,T) = u(x(X,T), t(X,T))$ where $x = x(X,T), t = t(X,T)$ which represents new dependent variable. As Dym equation is a result of string equation, we shall employ the fact that $u(x,t)$ and its spatial derivative tends to zero as $|x| \to \infty$. Then

$$\begin{aligned}\frac{\partial^\alpha}{\partial t^\alpha} &= \frac{\partial}{\partial X}\frac{\partial^\alpha X}{\partial t^\alpha} + \frac{\partial}{\partial T}\frac{\partial^\alpha T}{\partial t^\alpha} \\ &= -\frac{\partial}{\partial T} - \left(\frac{WW_{XX} - \frac{3}{2}W_X^2}{W^2}\right)\frac{\partial}{\partial X}\end{aligned}$$

Here we have used the HD equation 7.

Also $\frac{\partial}{\partial x} = \frac{1}{W(X,T)}\frac{\partial}{\partial X}$

Hence the equation 7 can be expressed as

$$W_T + \frac{W_{XXX}W^2 - 3W_{XX}W_XW + \frac{3}{2}W_X^3}{W^2} = 0 \quad (8)$$

Now let us take the transformation

$$v(X,T) = \frac{W_X}{W} \quad (9)$$

From 8 and 9 we obtain the KdV equation

$$v_T - \frac{3}{2}v^2v_X + v_{XXX} = 0 \quad (10)$$

Let the travelling wave solution of eqiation 10 be

$$v(X,T) = v(\xi), \quad \xi = \xi(X,T) = X - cT \quad (11)$$

c and k are constants. Substituting 11 in 10 we obtain the ODE

$$-cv' - \frac{3}{2}v^2v' + v''' = 0 \quad (12)$$

On Integration of equation 12 we get

$$-cv - \frac{1}{2}v^3 + v'' = 0 \quad (13)$$

Suppose the solution of the ODE 13 can be expressed by the polynomial given in 4. Considering the homogeneous balance v'' and v^3 in the equation 13 we obtain $3n = n + 2$ or $n = 1$

$$v(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), \quad a_1 \neq 0 \quad (14)$$

Where a_0 and a_1 are constant to be determined later. Substituting equation 14 with equation 5 into equation 13 and collecting all terms with the same power of $\frac{G'}{G}$ and then equating each coefficients of this polynomial to zero, we get the following system of algebraic equations:

$$\left(\frac{G'}{G}\right)^0 : -a_0c + a_1\lambda\mu - \frac{a_0^3}{2} = 0$$

$$\left(\frac{G'}{G}\right)^1 : -a_1c + a_1\lambda^2 + 2a_1\mu - \frac{3}{2}a_0^2a_1 = 0$$

$$\left(\frac{G'}{G}\right)^2 : 3a_1\lambda - \frac{3}{2}a_0a_1^2 = 0$$

$$\left(\frac{G'}{G}\right)^3 : 2a_1 - \frac{a_1^3}{2} = 0 \quad (15)$$

Solving the system of nonlinear algebraic equation 15 we have

$$a_1 = \pm 2, \quad a_0 = \pm\lambda, \quad c = -\frac{1}{2}(\lambda^2 - 4\mu) \quad (16)$$

Substituting 16 into equation 14 we obtain

$$v(\xi) = \pm\lambda \pm 2\left(\frac{G'}{G}\right)$$

Refering to the Section , three types of travelling wave solutions to HD equation may be obtained

Case 1: $\lambda^2 - 4\mu > 0$

Then we have the solution of the HD equation as

$$v_1(x,t) = \pm\sqrt{2c}\,\frac{ic_1\cos\left(\sqrt{\frac{c}{2}}(x - \frac{ct^\alpha}{\Gamma(1+\alpha)})\right) + c_2\sin\left(\sqrt{\frac{c}{2}}(x - \frac{ct^\alpha}{\Gamma(1+\alpha)})\right)}{-ic_1\sin\left(\sqrt{\frac{c}{2}}(x - \frac{ct^\alpha}{\Gamma(1+\alpha)})\right) + c_2\cos\left(\sqrt{\frac{c}{2}}(x - \frac{ct^\alpha}{\Gamma(1+\alpha)})\right)}$$

In particular if $c_2 = 0$ and $c_1 \neq 0$ we have

$$v_1(x,t) = \mp\sqrt{2c}\cot\left(\sqrt{\frac{c}{2}}(x - \frac{ct^\alpha}{\Gamma(1+\alpha)})\right)$$

Case 2: $\lambda^2 - 4\mu < 0$

Then we have the solution of HD equation as

$$v_2(x,t) = \pm i\sqrt{2c}\,\frac{-c_1\sin\left(\sqrt{\frac{c}{2}}(x - \frac{ct^\alpha}{\Gamma(1+\alpha)})\right) + c_2\cos\left(\sqrt{\frac{c}{2}}(x - \frac{ct^\alpha}{\Gamma(1+\alpha)})\right)}{c_1\cos\left(\sqrt{\frac{c}{2}}(x - \frac{ct^\alpha}{\Gamma(1+\alpha)})\right) + c_2\sin\left(\sqrt{\frac{c}{2}}(x - \frac{ct^\alpha}{\Gamma(1+\alpha)})\right)}$$

In particular if $c_2 = 0$ and $c_1 \neq 0$ we have

$$v_2(x,t) = \mp i\sqrt{2c}\tan\left(\sqrt{\frac{c}{2}}(x - \frac{ct^\alpha}{\Gamma(1+\alpha)})\right)$$

Case 3: $\lambda^2 - 4\mu = 0$

We obtain the rational function solution of the HD equation as follows:

$$v_3(x,t) = \pm\frac{c_2}{c_1 + c_2(x - \frac{ct^\alpha}{\Gamma(1+\alpha)})}$$

MODIFIED ZAKHAROV-KUZNETSOV EQUATION

Zakharov-Kuznetsov(ZK) equation is a nonlinear model describing two-dimensional modulation of a KdV soliton

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[35], [36]. If a magnetic field is directed along the x-axix, the ZK equation in renormalized variables [37] takes the form

$$u_t + auu_x + \nabla^2 u_x = 0 \tag{17}$$

$\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_x^2$ is isotropic Laplacian. The ZK equation governs the behaviour of weakly nonlinear ion-acoustic waves in plasma comprising of cold ions and hot isothermal electrons in the presence of a uniform magnetic field [35]. The Two dimensional ZK equation was first derived for describing weakly nonlinear ion-acoustic waves in a strongly magnetized lossless plasma in two dimensions [38].

On the other hand, Kakutani and Ono [39] established that the equation describing the propagation of Alfv́en waves at a critical angle to the undisturbed magnetic field was the modified Korteweg-de Vries (mKdV) equation,

$$u_t + u^2 u_x + u_{xxx} = 0, \quad x, y \epsilon \mathbb{R}$$

The two-dimensional equation in this physical situation is the modified Zakharov-Kuznetsov equation

$$u_t + u^2 u_x + u_{xxx} + u_{xyy} = 0 \quad (x, y) \epsilon \mathbb{R}^2$$

appearing in [40]. In [41] Faminskii considered the IVP associated to the equation 17. He showed local and global well-posedness for initial data in $H^m(\mathbb{R}^2)$, $m \geq 1$ integer. Also in context of plasma physics, Scamel [42] derived the $(2+1)$ dimensional equation

$$u_t + u^{\frac{1}{2}} + b(u_{xx} + u_{yy})_x = 0$$

that describes ion-acoustic waves in a cold ion plasma where the electrons donot behave isothermally during their passage of the wave. Motivated by the widely studied literature available for ZK equation, in this paper we consider the $(2+1)$-dimensional modifi'ed fractional Zakharov Kuznetsov equations [34]:

$$u_t^\alpha + u^2 u_x + u_{xxx} + u_{xyy} = 0 \tag{18}$$

The solitary wave solutions of modiï¬'ed Zakharov-Kuznetsov equations is constructed in [34]. We shall use the travelling wave variable $\xi = x + y - \frac{ct^\alpha}{\Gamma(\alpha+1)}$ to transform 18 to an ODE

$$-cu' + u^2 u' + 2u''' = 0 \tag{19}$$

On integration 19 becomes

$$-cu + \frac{1}{3}u^3 + 2u'' = 0 \tag{20}$$

We find the solution of the ODE 20 in terms of the polynomial given in 4.By the homogeneous balance of the terms v'' and v^3 in the equation 20 we obtain $n = 1$ and the solution as

$$v(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), \quad a_1 \neq 0 \tag{21}$$

Where a_0 and a_1 are constant to be determined later. On substitution of 21 along with 5 yeild the following system of equations:

$$\begin{aligned}
\left(\frac{G'}{G}\right)^0 &: -a_0 c + 2a_1\lambda\mu + \frac{a_0^3}{3} = 0 \\
\left(\frac{G'}{G}\right)^1 &: -a_1 c + 2a_1\lambda^2 + 4a_1\mu + a_0^2 a_1 = 0 \\
\left(\frac{G'}{G}\right)^2 &: 6a_1\lambda + a_0 a_1^2 = 0 \\
\left(\frac{G'}{G}\right)^3 &: a_1^3 + 4a_1 = 0
\end{aligned} \tag{22}$$

Solving the system of nonlinear algebraic equation 22 we get

$$\begin{aligned}
&a_1 = \pm 2i, \lambda = 0 \quad \text{or} \quad \lambda = \pm\sqrt{\frac{2}{3}}\sqrt{\mu}, \\
&c = 4\mu - 7\lambda^2, a_0 = \frac{3a_1\lambda}{2}
\end{aligned}$$

Refering to the Section **??**, three types of travelling wave solutions may be obtained

Case 1: $\lambda^2 - 4\mu > 0$

$$v_{1,1}(x,y,t) = \mp 2i\sqrt{\mu}\,\frac{\begin{array}{c}-ic_1\cos\left(\sqrt{\mu}(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)})\right)\\ -c_2\sin\left(\sqrt{\mu}(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)})\right)\end{array}}{\begin{array}{c}-ic_1\sin\left(\sqrt{\mu}(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)})\right)\\ +c_2\cos\left(\sqrt{\mu}(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)})\right)\end{array}}$$

$$v_{1,2}(x,y,t) = \pm 2i\sqrt{\frac{8\mu}{3}} \mp i\sqrt{\frac{10\mu}{3}}\,\frac{-ic_1\cos\left(\sqrt{\frac{10\mu}{12}}(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)})\right) -c_2\sin\left(\sqrt{\frac{10\mu}{12}}(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)}{-ic_1\sin\left(\sqrt{\frac{10\mu}{12}}(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)})\right) +c_2\cos\left(\sqrt{\frac{10\mu}{12}}(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)}$$

In particular if $c_2 = 0$ and $c_1 \neq 0$ we have

$$v_{1,1}(x,t) = \mp 2i\sqrt{\mu}\cot\left(\sqrt{\mu}(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)})\right)$$

$$v_{1,2}(x,y,t) = \pm 2i\sqrt{\frac{8\mu}{3}} \mp i\sqrt{\frac{10\mu}{3}}\cot\left(\sqrt{\frac{10\mu}{12}}(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)})\right)$$

Case 2: $\lambda^2 - 4\mu < 0$
Then we have trigonometric solution as

$$v_{2,1}(x,y,t) = \mp 2i\sqrt{\mu}\,\frac{-c_1\sin\left(\sqrt{\mu}(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)})\right) +c_2\cos\left(\sqrt{\mu}(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)}{c_1\cos\left(\sqrt{\mu}(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)})\right) +c_2\sin\left(\sqrt{\mu}(x-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)}$$

$$v_{2,2}(x,y,t) = \pm 2i\sqrt{\frac{8\mu}{3}} \mp i\sqrt{\frac{10\mu}{3}}\,\frac{-c_1\sin\left(\sqrt{\frac{10\mu}{12}}(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)})\right) +c_2\cos\left(\sqrt{\frac{10\mu}{12}}(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)}{c_1\cos\left(\sqrt{\frac{10\mu}{12}}(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)})\right) +c_2\sin\left(\sqrt{\frac{10\mu}{12}}(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)}$$

In particular if $c_1 = 0$ and $c_2 \neq 0$ we have

$$v_{2,1}(x,y,t) = v_{1,1}(x,y,t), v_{2,2}(x,y,t) = v_{1,2}(x,y,t)$$

Case 3: $\lambda^2 - 4\mu = 0$
We obtain the rational function solution as follows:

$$v_3(x,t) = \pm\frac{c_2}{c_1 + c_2(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)})}$$

CONCLUSION

In this work, we have seen that three types of travelling wave solutions in terms of hyperbolic, trigonometric and rational functions are found based on the nature of $\lambda^2 - 4\mu$ for three fractional partial differential equations whose integer order derivative forms are highly popular in the literature. From our results obtained in this paper, we conclude that the $\frac{G'}{G}$-expansion method is powerful, effective and convenient method to find the exact solutions of nonlinear fractional PDEs. The performance of this method is reliable, simple and gives many new solutions. Moreover, this method provides a mathematical tool to obtain more general exact solutions of many nonlinear PDEs in mathematical physics even in absence of initial and boundary conditions.

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