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A Study on Bessel Wavelet Transform

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Abstract:

Using Hankel transform, generalized Bessel wavelet transform is defined. Continuity and boundedness result for the generalized Bessel wavelet transform is obtained. Product of two Bessel wavelet transforms is investigated and a reconstruction formula is derived.

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INTRODUCTION

Let γ be a positive real number. Set

$$d\sigma(x) = \frac{x^{2\gamma}}{2^{\gamma-\frac{1}{2}}\Gamma(\gamma+\frac{1}{2})}dx \qquad (1)$$

and $j(x) = 2^{\gamma-\frac{1}{2}}\Gamma(\gamma+\frac{1}{2})\, x^{\frac{1}{2}-\gamma}\, J_{\gamma-\frac{1}{2}}$

where $J_{\gamma-\frac{1}{2}}$ denote the Bessel function of order $\gamma-\dfrac{1}{2}$.

We denote $L_{p,\sigma}(0,\infty),\ 1\le p\le\infty$ as the space of those real measurable function f on $(0,\infty)$ for which

$$\|f\|_{p,\sigma} = \left[\int_0^\infty |f(x)|^p\, d\sigma(x)\right]^{1/p} < \infty \qquad (2)$$

$$\|f\|_{\infty,\sigma} = \operatorname*{ess\,sup}_{0<x<\infty}|f(x)| < \infty \qquad (3)$$

From [1, 4], the Hankel translation for $f\in L_{p,\infty}(0,\infty),\ 1\le p\le\infty$, is defined by

$$\tau_y f(x) = f(x,y) = \int_0^\infty f(z)\, D(x,y,z)\, d\sigma(z), \quad 0<x,y<\infty \qquad (4)$$

where

$$D(x,y,z) = \int_0^\infty j(xt)\, j(yt)\, j(zt)\, d\sigma(t)$$

From [5], in terms of the Hankel translation τ_y, and dilation D_a defined by

$$D_a f(x,y) = f\left(\frac{x}{a},\frac{y}{a}\right) \qquad (5)$$

We define Bessel wavelet by

$$\phi_{b,a}(x) = D_a\, \tau_b\, \phi(x) = D_a\, \phi(b,x) = \phi\left(\frac{b}{a},\frac{x}{a}\right) \qquad (6)$$

$$= \int_0^\infty \phi(z)\, D\left(\frac{b}{a},\frac{x}{a},z\right) d\sigma(z). \qquad (7)$$

Using the wavelet $\phi_{b,a}$ we now define the Bessel wavelet transform for $f\in L_{p,\sigma}$

$$\left(B_\phi f\right)(b,a) = \left\langle f,\phi_{b,a}\right\rangle = \int_0^\infty f(t)\,\overline{\phi_{b,a}(t)}\, d\sigma(t) \qquad (8)$$

$$= \int_0^\infty\int_0^\infty f(t)\,\overline{\phi(z)}\, D\left(\frac{b}{a},\frac{t}{a},z\right) d\sigma(z)\, d\sigma(t)$$

$$= \int_0^\infty \left(\frac{bx}{a}\right)^{\frac{1}{2}} J_{\gamma-\frac{1}{2}}\left(\frac{bx}{a}\right) x^{-\gamma}\, b^{-\gamma}\, a^{2\gamma}$$

$$\times\left\{\int_0^\infty t^\gamma\, f(t)\left(\frac{tx}{a}\right)^{\frac{1}{2}} J_{\gamma-\frac{1}{2}}\left(\frac{tx}{a}\right) dt \int_0^\infty z^\gamma\, \overline{\phi(z)}\,(zx)^{\frac{1}{2}} J_{\gamma-\frac{1}{2}}(zx)\, dz\right\} dx. \qquad (9)$$

From [1], the Hankel transformation is defined by

$$\left(h_\mu f\right)(u) = \int_0^\infty (xu)^{\frac{1}{2}} J_\mu(xu)\, f(x)\, dx, \quad u\in R_+ = (0,\infty) \qquad (10)$$

From Eqt. (9) and Eqt. (10), we have

$$\left(B_{\phi}\, f\right)(b,a) = a^{2\gamma} b^{-\gamma} \int_0^\infty x^{-\gamma} \left(\frac{bx}{a}\right)^{\frac{1}{2}} J_{\gamma-\frac{1}{2}}\left(\frac{bx}{a}\right) h_{\gamma-\frac{1}{2}}\left(t^\gamma f(t)\right)\left(\frac{x}{a}\right) h_{\gamma-\frac{1}{2}}\left(z^\gamma \overline{\phi(z)}\right)(x)\, dx$$

$$= a^{\gamma+1} b^{-\gamma} \int_0^\infty (u)^{-\gamma} (bu)^{\frac{1}{2}} J_{\gamma-\frac{1}{2}}(bu)\, h_{\gamma-\frac{1}{2}}\left(t^\gamma f(t)\right) h_{\gamma-\frac{1}{2}}\left(z^\gamma \overline{\phi(z)}\right)(au)\, du$$

$$= a^{\mu+3/2}\, b^{-\mu-1/2} \int_0^\infty (u)^{-\mu-1/2} (bu)^{1/2}\, J_\mu(bu)\, h_\mu\left(t^{\mu+1/2} f(t)\right) h_\mu\left(z^{\mu+1/2}\overline{\phi(z)}\right)(au)\, du \text{ where } \gamma - \frac{1}{2} = \mu .$$

Let us set

$$u^{-\gamma}\, h_{\gamma-\frac{1}{2}}\left(t^\gamma f(t)\right) = (h_\mu f)(u)$$

$$z^\gamma \overline{\phi(z)} = \overline{\psi(z)}$$

and

$$\left(B_\psi f\right)(b,a) = b^{\mu+1/2}\, a^{-\mu-3/2} \left(B_\phi f\right)(b,a).$$

We get the following convenient form of Bessel wavelet transform

$$\left(B_\psi f\right)(b,a) = \int_0^\infty (bu)^{1/2} J_\mu(bu)\, \tilde{f}(u)\, \overline{\tilde{\psi}(au)}\, du \qquad (11)$$

where $\tilde{f}(u) = \left(h_\mu f\right)(u)$.

THE GENERALIZED BESSEL WAVELET TRANSFORM

In this section certain space of functions are introduced on which Bessel wavelet transform called generalized Bessel wavelet transform will be defined.

Let us recall the Zemanian space denoted by $Z_\mu(R_+)$, consist of all C^∞ functions ϕ on R_+ belongs to $Z_\mu(R_+)$ if

$$\xi^{\mu}_{k,m}(\phi) = \sup_{x\in R_+} \left| x^k \left(x^{-1}\frac{d}{dx}\right)^m x^{-\mu-1/2}\phi(x)\right| < \infty, \quad \forall k, m \in N_0$$

Next we define a space on $R\times R_+$ needed in our present investigation.

Definition

The set of all infinitely differentiable functions $W(b,a)$ on $R\times R_+$ satisfying the condition

$$\xi^{\sigma,\mu}_{k,q,l}(w) = \sup_{a,b} \left| \begin{matrix} (1+a^2)^{-\rho-k} b^k \left(b^{-1}\dfrac{d}{db}\right)^q \\ \left(a^{-1}\dfrac{d}{da}\right)^l b^{-\mu-1/2} W(b,a) \end{matrix} \right| < \infty, \quad \forall k, q, l \in N_0$$

is denoted by $Z_{\mu,\rho}(R\times R_+)$, where $\mu, \rho \in R$.

Let us assume that for any real numbers $\rho, \tilde{\psi}(x)$ satisfies

$$\left| \left(x^{-1}\frac{d}{dx}\right)^k \tilde{\psi}(x)\right| \le C_{k,\rho}(1+x^2)^{\rho-k}, \quad \forall k \in N_0 \qquad (12)$$

where $\tilde{\psi}$ denote the Hankel transform of the basic wavelet ψ.

In the proof of following theorem we shall use the Leibnitz-type formula:

$$\left(x^{-1}\frac{d}{dx}\right)^n \left(x^{-\mu-1/2}\phi\psi\right) = \sum_{r=0}^{n}\binom{n}{r}\left(x^{-1}\frac{d}{dx}\right)^r \left(x^{-\mu-1/2}\phi\right)\left(x^{-1}\frac{d}{dx}\right)^{n-r}\psi \qquad (13)$$

Theorem

Let Z_μ denote the Zemanian space. Then the generalized Bessel wavelet transform defined by

$$\left(B\phi\right)(b,a) = \Phi(b,a) = \int_0^\infty (xb)^{1/2} J_\mu(xb)\, \tilde{\phi}(x)\, \overline{\tilde{\psi}(ax)}\, dx$$

with respect to basic wavelet ψ, is a continuous linear mapping:

(i) from $Z_\mu(R_+)$ into $Z_{\mu,\rho}(R\times R_+)$, when $\rho \ge 0$

(ii) from $Z_\mu(R_+)$ into $Z_{\mu,\rho}(R\times R_+)$, when $\rho < 0$.

Proof: Using the technique used in proof of theorem 5.4 given by Zemanian (1968)

We can show that

$$(-1)^{k+q} b^k \left(b^{-1}\frac{d}{db}\right)^q b^{-\mu-1/2}\,\Phi(b,a)$$

$$=\int_0^\infty x^{2\mu+2q+k+1}\left\{\left(x^{-1}\frac{d}{dx}\right)^k \left(x^{-\mu-1/2}\,\tilde{\phi}(x)\overline{\tilde{\psi}(ax)}\right)\right\}(xb)^{-\mu-q}J_{\mu+k+q}(xb)\,dx$$

$$=\int_0^\infty x^{2\mu+2q+k+1}\left\{\left(x^{-1}\frac{d}{dx}\right)^k \left(x^{-\mu-1/2}\overline{\tilde{\psi}(ax)}\tilde{\phi}(x)\right)\right\}(xb)^{-\mu-q}J_{\mu+k+q}(xb)\,dx$$

$$=\int_0^\infty x^{2\mu+2q+k+1}\left\{\sum_{r=0}^{n}\binom{k}{r}\left(x^{-1}\frac{d}{dx}\right)^{k-r} x^{-\mu-1/2}\tilde{\phi}(x)\left(x^{-1}\frac{d}{dx}\right)^r \overline{\tilde{\psi}(ax)}\right\}(xb)^{-\mu-q}J_{\mu+k+q}(xb)\,dx.$$

Using Eqt. (13), therefore

$$\left|b^k\left(b^{-1}\frac{d}{db}\right)^q\left(a^{-1}\frac{d}{da}\right)^l b^{-\mu-1/2}\,\Phi(b,a)\right|$$

$$=\left|\int_0^\infty x^{2\mu+2q+k+1}\left\{\sum_{r=0}^{n}\binom{k}{r}\left(x^{-1}\frac{d}{dx}\right)^{k-r} x^{-\mu-1/2}\tilde{\phi}(x)\left(a^{-1}\frac{d}{da}\right)^l\left(x^{-1}\frac{d}{dx}\right)^r \overline{\tilde{\psi}(ax)}\right\}\right.$$

$$\left.\times(xb)^{-\mu-q}J_{\mu+k+q}(xb)\,dx\right|$$

$$\le\int_0^\infty x^{2\mu+2q+k+1}\sum_{r=0}^{\infty}\binom{k}{r}\left|\left(x^{-1}\frac{d}{dx}\right)^{k-r} x^{-\mu-1/2}\tilde{\phi}(x)\right|\left|\left(x^{-1}\frac{d}{dx}\right)^r\left(a^{-1}\frac{d}{da}\right)^l\overline{\tilde{\psi}(ax)}\right|$$

$$\times\left|(xb)^{-\mu-q}J_{\mu+k+q}(xb)\right|dx. \qquad (14)$$

By the asymptotic properties of the Bessel function, we have

$$\left|(xb)^{-\mu-q}J_{\mu+k+q}(xb)\right|\le B_{\mu,k,q}$$

where $B_{\mu,k,q}$ is positive constant.

Since $\phi(x)\in Z_\mu(R_+)$,

$$\left|\left(x^{-1}\frac{d}{dx}\right)^{k-r} x^{-\mu-1/2}\tilde{\phi}(x)\right|\le D_{\mu,k,r}(1+x^2)^{-p},\quad \forall p\in N_0.$$

In view of estimation, we have for $t = ax$,

$$\left|\left(x^{-1}\frac{d}{dx}\right)^r\left(a^{-1}\frac{d}{da}\right)^l\overline{\tilde{\psi}(ax)}\right|=\left|a^{2r}x^{2l}\left(t^{-1}\frac{d}{dt}\right)^{r+l}\overline{\tilde{\psi}(t)}\right|$$

$$\leq C_{r+l,\rho}(1+t^2)^{\rho-r-l} a^{2r} x^{2l}$$

$$\leq C_{r+l,\rho}(1+a^2x^2)^{\rho-r-l} a^{2r} x^{2l}.$$

Therefore Eqt. (14) becomes

$$\left| b^k \left(b^{-1}\frac{d}{db}\right)^q \left(a^{-1}\frac{d}{da}\right)^l b^{-\mu-1/2}\,\Phi(b,a)\right|$$

$$\leq \int_0^\infty x^{2\mu+2q+k+1} \sum_{r=0}^{n} \binom{k}{r} D_{\mu,k,r}(1+x^2)^{-p}\, C_{r+l,\rho}(1+a^2x^2)^{\rho-r-l} a^{2r} x^{2l} B_{\mu,k,q}\, dx$$

$$\leq \sum_{r=0}^{n} \binom{k}{r} D'_{\mu,k,q,l}(1+a^2)^k \int_0^\infty (1+x^2)^{-p}\,(1+a^2x^2)^{\rho-r-l} x^{2\mu+2q+k+2l+1} dx.$$

For $\rho \geq 0$ the above inequality gives

$$\left|(1+a^2)^{-\rho-k} b^k \left(b^{-1}\frac{d}{db}\right)^q \left(a^{-1}\frac{d}{da}\right)^l b^{-\mu-1/2}\,\Phi(b,a)\right|$$

$$\leq \sum_{r=0}^{\infty} \binom{k}{r} D'_{\mu,k,q,l,\rho} \int_0^\infty \frac{x^{2\mu+2q+k+2l+1}}{(1+x^2)^p}(1+x^2)^\rho\, dx\ . \qquad (15)$$

Since ρ may chosen large enough, integral in Eqt. (15) is convergent for $\mu \geq -\frac{1}{2}$.

Therefore

$$\left|(1+a^2)^{\rho-k} b^k \left(b^{-1}\frac{d}{db}\right)^q \left(a^{-1}\frac{d}{da}\right)^l b^{-\mu-1/2}\,\Phi(b,a)\right| < \infty,\ \forall k,q,l \in N_0 .$$

Hence $\Phi(b,a) \in Z_{\mu,\rho}(R\times R_+),\ \mu \geq -\frac{1}{2},\ \ \rho \geq 0$.

For $\rho < 0$ inequality in Eqt. (15) becomes

$$\left|(1+a^2)^{-k} b^k \left(b^{-1}\frac{d}{db}\right)^q \left(a^{-1}\frac{d}{da}\right)^l b^{-\mu-1/2}\Phi(b,a)\right|$$

$$\leq \sum_{r=0}^{\infty} \binom{k}{r} D'_{\mu,k,q,r,l} \int_0^\infty \frac{x^{2\mu+2q+2l+1}}{(1+x^2)^p} dx. \qquad (16)$$

The integral in Eqt. (16) can be made convergent by choosing *p* large enough and $\mu \geq -\frac{1}{2}$.

Therefore

$$\left|(1+a^2)^{-k} b^k \left(b^{-1}\frac{d}{db}\right)^q \left(a^{-1}\frac{d}{da}\right)^l b^{-\mu-1/2}\Phi(b,a)\right| < \infty .$$

Hence $\Phi(b,a) \in Z_{\mu,0}(R\times R_+),\ \mu \geq -\frac{1}{2},\ \ \rho < 0$.

This completes the proof.

PRODUCT OF GENERALIZED BESSEL WAVELET TRANSFORMS

In this section the product of the generalized Bessel wavelet transforms is defined when the basic wavelet ψ satisfy certain growth condition.

Definition

The space $Z_{\mu,\rho,\sigma}(R\times R\times R_+),\ \mu \geq -\frac{1}{2},\ \rho,\sigma \in R$, is defined to be the set of all infinitely differentiable functions $\Phi(b,a,c)$, which satisfy the following conditions:

$$\zeta_{k,l,m}^{\rho,\sigma,\mu}(\Phi)=\sup_{a,b,c}\left|\begin{matrix}(1+a^2)^{-\rho-k}(1+c^2)^{-\sigma-k}b^k\left(a^{-1}\frac{d}{da}\right)^l\\ \left(c^{-1}\frac{d}{dc}\right)^m b^{-\mu-1/2}\Phi(b,a,c)\end{matrix}\right|<\infty,$$

Let us assume that for any real number ρ and σ , the Hankel transform of basic wavelets Ψ_1,Ψ_2 satisfy

$$\left|\left(x^{-1}\frac{d}{dx}\right)^i\tilde{\Psi}_1(x)\right|\le C_{i,\rho}(1+x^2)^{\rho-i}$$

and

$$\left|\left(x^{-1}\frac{d}{dx}\right)^j\tilde{\Psi}_2(x)\right|\le C_{j,\rho}(1+x^2)^{\rho-j}.$$

Let Φ_1,Φ_2 be continuous generalized Bessel wavelet transforms of $\phi\in Z_\mu(R_+)$ defined as follows:

$$\left(B\Phi_1\right)(b,a)=\Phi_1(b,a)=\int_0^\infty(xb)^{\frac{1}{2}}J_\mu(xb)\tilde{\phi}(x)\overline{\tilde{\Psi}_1(ax)}\,dx$$

and

$$\left(B\Phi_2\right)(b,a)=\Phi_2(b,a)=\int_0^\infty(xb)^{\frac{1}{2}}J_\mu(xb)\tilde{\phi}(x)\overline{\tilde{\Psi}_2(ax)}\,dx$$

Then product $\Phi_1\circ\Phi_2$ is defined by

$$\Phi(b,a,c)=(\Phi_1\circ\Phi_2)(b,a,c)$$

$$=\int_0^\infty(xb)^{\frac{1}{2}}J_\mu(xb)\overline{\tilde{\Psi}_1(ax)}\,h_{\mu,d}(B_2\phi)(x,c)\,dx$$

$$=\int_0^\infty(xb)^{\frac{1}{2}}J_\mu(xb)\overline{\tilde{\Psi}_1(ax)\tilde{\Psi}_2(cx)}\,\tilde{\phi}(x)\,dx$$

$$=\int_0^\infty(xb)^{\frac{1}{2}}J_\mu(xb)\,\chi_{(a,c,x)}\tilde{\phi}(x)\,dx$$

where $h_{\mu,d}$ denote the Hankel transform with respect to the variable d and $\chi_{(a,c,x)}=\overline{\tilde{\Psi}_1(ax)\tilde{\Psi}_2(cx)}$ provided the integral are convergent.

Admissibility condition.

Let $\Psi_1\in L^2(R_+),\Psi_2\in L^2(R_+)$ be such that there exist a constant C_{Ψ_1,Ψ_2} with property that $0<C_{\Psi_1,\Psi_2}<\infty$ and for ξ almost everywhere on R_+

$$C_{\Psi_1,\Psi_2}=\int_0^\infty\int_0^\infty\left|\tilde{\Psi}_1(a\xi)\right|^2\left|\tilde{\Psi}_2(c\xi)\right|^2\frac{dadc}{ac}.$$

Theorem.

Let $\Psi_1\in L^2(R_+),\Psi_2\in L^1(R_+)$ and $\tilde{f}=(h_\mu f)\in L^2(R_+)$

Let the product of generalized Bessel wavelet transform of f defined by

$$\left(\Phi\, f\right)(b,a,c)=\int_0^\infty(xb)^{\frac{1}{2}}J_\mu(xb)\overline{\tilde{\Psi}_1(ax)\tilde{\Psi}_2(cx)}\,\tilde{f}(x)\,dx$$

Then for all $g\in L^2(R_+)$,

$$\int_0^\infty\int_0^\infty\int_0^\infty(\Phi\, f)(b,a,c)\ (\Phi\, g)(b,a,c)\,db\frac{dadc}{ac}=C_{\Psi_1,\Psi_2}\left\langle f,g\right\rangle.$$

Proof. Let us write

$$\Phi_{a,c}(x)=\tilde{\Psi}_1(ax)\tilde{\Psi}_2(cx)\,\overline{\tilde{f}(x)}$$

$$\Psi_{a,c}(x)=\tilde{\Psi}_1(ay)\tilde{\Psi}_2(cy)\,\overline{\tilde{g}(y)}.$$

Then by unitary property of Hankel transform, we have

$$\int_0^\infty\int_0^\infty\int_0^\infty(\Phi\, f)(b,a,c)\ \overline{(\Phi\, g)(b,a,c)}\,db\frac{dadc}{ac}$$

$$=\int_0^\infty db\int_0^\infty\int_0^\infty[\int_0^\infty(xb)^{\frac{1}{2}}J_\mu(xb)\overline{\tilde{\Psi}_1(ax)\tilde{\Psi}_2(cx)}\,\tilde{f}(x)\,dx$$

$$\times\int_0^\infty(yb)^{\frac{1}{2}}J_\mu(yb)\overline{\tilde{\Psi}_1(ay)\tilde{\Psi}_2(cy)}\,\overline{\tilde{g}(y)}\,dy]\frac{dadc}{ac}$$

$$=\int_0^\infty db\int_0^\infty\int_0^\infty h_\mu[\Phi_{a,c}(x)](b)\,h_\mu[\Psi_{a,c}(y)](b)\frac{dadc}{ac}$$

$$=\int_0^\infty\int_0^\infty[\int_0^\infty\overline{\Phi_{a,c}(\xi)}\,\Psi_{a,c}(\xi)\,d\xi]\frac{dadc}{ac}$$

$$=\int_0^\infty\int_0^\infty[\int_0^\infty\overline{\tilde{\Psi}_1(a\xi)}\,\overline{\tilde{\Psi}_2(c\xi)}\,\tilde{f}(\xi)\tilde{\Psi}_1(a\xi)\tilde{\Psi}_2(c\xi)\,\overline{\tilde{g}(\xi)}d\xi]\frac{dadc}{ac}$$

.

Now using Fubini's theorem and Parseval formula for Hankel transform, we get

$$\int_0^\infty\int_0^\infty\int_0^\infty(\Phi\, f)(b,a,c)\overline{(\Phi\, g)(b,a,c)}\,db\frac{dadc}{ac}$$

$$=\int_0^\infty f(\xi)\overline{g(\xi)}\left(\int_0^\infty\int_0^\infty\left|\tilde{\Psi}_1(a\xi)\right|^2\left|\tilde{\Psi}_2(c\xi)\right|^2\frac{dadc}{ac}\right)d\xi$$

$$=C_{\Psi_1,\Psi_2}\int_0^\infty f(\xi)\,\overline{g(\xi)}\,d\xi$$

$$=C_{\Psi_1,\Psi_2}\left\langle f,g\right\rangle.$$

This complete the proof.

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