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$(0; 0, 1)$ - INTERPOLATION ON LAGUERRE ABSCISSAS

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Abstract: In this paper, we have considered the explicit representation of the Pál type $(0; 0, 1)-$ interpolation when function values are prescribed on the zeros of Laguerre Polynomials $L_n^{(\alpha)}(x)$ and Hermite data is prescribed on the zeros of the derivative of Laguerre Polynomials $(L_n^{(\alpha)})'(x), \alpha > -1$ and vice-versa.

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INTRODUCTION

In 1975, L.G. P'al [4] introduced the following interpolation process. Let

$$-\infty < x_{n,n} < \cdots < x_{1,n} < \infty$$

be a system of distinct real points which are zeros of $W_n(x)$, i.e.,

$$W_n(x) = \prod_{i=1}^{n} (x - x_{i,a}).$$

The roots $y_{i,n}(i = 1, 2, \ldots, n-1)$ of $W_n^{'}(x)$ are interscaled between the roots of $W_n(x)$, i.e.,

$$-\infty < x_{n,n} < y_{n-1,n} < x_{n-1,n} \cdots < y_{1,n} < x_{1,n} < +\infty. \tag{1}$$

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Pál proved that for given arbitrary numbers $(\alpha_{i,n})_{i=1}^{n}$ and $(\beta_{i,n})_{i=1}^{n-1}$, there exists a unique polynomial of degree 2n-1 satisfying the conditions:

$$R_n(x_{i,n}) = \alpha_{i,n}, \ i = 1, 2, \ldots, n; \ R_n'(y_{i,n}) = \beta_{i,n-1}, \ i = 1, 2, \ldots, n-1,$$

and an initial condition $R_n(a) = 0$, where a is a given point, different from the nodal points (1). Szili [11] was the first to apply this method on infinite interval by taking the mixed of the Hermite polynomial $H_n(x)$ and its derivative $H_n'(x)$. Later I. Joo ([5],[6]) sharpened his results by improving the estimates of fundamental polynomials. Srivastava and Mathur [9] studied the problem of (0;0,1) - interpolation on the mixed zeros of $H_n(x)$ and its derivative. In another paper, Lenard [7] showed that a modified Pál type interpolation on Laguerre abscissas when $(x_i)_{i=1}^{n}$ and $(x_i^*)_{i=1}^{n}$ as the zeros of the Laguerre polynomials $L_n^k(x)$ and $L_n^{k-1}(x)$, respectively and $x_0 = 0$ is regular and there exists a polynomial $R_m(x)$ of degree $2n + k$ satisfying the following conditions:

$$R_m(x_i) = y_i, \quad R_m'(x_i^*) = y_i'(i = 1, 2, \ldots, n),$$

with Hermite type boundary conditions:

$$R_m^{(j)}(x_0) = y_0^{(j)}(j = 0, 1, \ldots, k),$$

where y_i, y_i' and $y_0^{(j)}$ are arbitrary real numbers.

In this paper, we have considered $\{x_k\}_{k=1}^{k=n}$ and $\{y_k\}_{k=1}^{n-1}$ as the zeros of Laguerre Polynomial $L_n^{(\alpha)}(x)$ and its derivative $(L_n^{(\alpha)})'(x)$ respectively which are interscaled as:

$$0 < x_1 < y_1 < x_2 < \cdots < x_{n-1} < y_{n-1} < x_n < \infty. \tag{2}$$

Then for an arbitrarily given set of real numbers:

$$\{\alpha_k, k = 1(1)n; \beta_k, k = 1(1)n-1; \gamma_k, k = 1(1)n-1\}, \tag{3}$$

we seek to determine a polynomial $R_n(x)$ of minimal possible degree $\leq 3n-3$ such that:

$$\begin{cases} R_n(x_k) = \alpha_k, & k = 1(1)n. \\ R_n(y_k) = \beta_k, & k = 1(1)n-1. \\ R_n^{'}(y_k) = \gamma_k, & k = 1(1)n-1. \end{cases} \tag{4}$$

The problem when $\{x_k\}_{k=1}^{k=n}$ and $\{y_k\}_{k=1}^{n-1}$ are interchanged i.e, when function values are prescribed on the zeros of $(L_n^{(\alpha)})'(x), \alpha > -1$ and Hermite data is prescribed on the zeros of $L_n^{(\alpha)}(x)$ has also been dealt with.

PRELIMINARIES

The differential equation of Laguerre Polynomial $L_n^{(\alpha)}(x)$ is given by

$$x(\frac{d^2y}{dx^2}) + (\alpha + 1 - x)(\frac{dy}{dx}) + ny = 0. \tag{5}$$

where n is a positive integer and $\alpha > -1$.The recurrence relations between Laguerre polynomial and its derivative are as follows:

$$(n+1)L_{n+1}^{(\alpha)}(x) = (2n+1-x)L_n^{(\alpha)}(x) - nL_{n-1}^{(\alpha)}(x). \tag{6}$$

$$x(L_n^{(\alpha)})'(x) = nL_n^{(\alpha)}(x) - nL_{n-1}^{(\alpha)}(x). \tag{7}$$

Let the fundamental polynomials of Lagrange interpolation on the nodes x_k and y_k be denoted by

$$l_k(x) = \frac{L_n^{(\alpha)}(x)}{(x-x_k)(L_n^{(\alpha)})'(x_k)}, \qquad k = 1(1)n \tag{8}$$

and

$$l_k^*(x) = \frac{(L_n^{(\alpha)})'(x)}{(x-y_k))(L_n^{(\alpha)})''(y_k)}, \qquad k = 1(1)n-1, \tag{9}$$

respectively.

EXPLICIT REPRESENTATION OF THE INTERPOLATORY POLYNOMIAL

Let $(2n-1)$ points in $(0,\infty)$ be given by (2). Then to the prescribed numbers $\{\alpha_k\}_{k=1}^n$, $\{\beta_k\}_{k=1}^{n-1}$ and $\{\gamma_k\}_{k=1}^{n-1}$, there exists a unique polynomial $R_n(x)$ of degree $\leq 3n-3$ satisfying the conditions (4).

The polynomial $R_n(x)$ is explicitly given by:

$$R_n(x) = \sum_{k=1}^{n} \alpha_k A_k(x) + \sum_{k=1}^{n-1} \beta_k B_k(x) + \sum_{k=1}^{n-1} \gamma_k C_k(x), \tag{10}$$

where $\{A_k(x)\}_{k=1}^n$, $\{B_k(x)\}_{k=1}^{n-1}$ and $\{C_k(x)\}_{k=1}^{n-1}$ are uniquely determined polynomials each of degree $\leq 3n-3$, satisfying the following conditions:
For $k = 1,\ldots,n$,

$$\begin{cases} A_k(x_j) = \delta_{jk}, & j = 1,2,\ldots,n \\ A_k(y_j) = 0, & j = 1,2,\ldots,n-1 \\ A_k^{'}(y_j) = 0, & j = 1,2,\ldots,n-1. \end{cases} \tag{11}$$

For $k = 1,\ldots,n-1$,

$$\begin{cases} B_k(x_j) = 0, & j = 1,2,\ldots,n \\ B_k(y_j) = \delta_{jk}, & j = 1,2,\ldots,n-1 \\ B_k^{'}(y_j) = 0, & j = 1,2,\ldots,n-1. \end{cases} \tag{12}$$

For $k = 1,\ldots,n-1$,

$$\begin{cases} C_k(x_j) = 0, & j = 1,2,\ldots,n \\ C_k(y_j) = 0, & j = 1,2,\ldots,n-1 \\ C_k^{'}(y_j) = \delta_{jk}, & j = 1,2,\ldots,n-1. \end{cases} \tag{13}$$

The explicit representation of the $A_k(x), k = 1,2,\ldots,n$; $B_k(x), k = 1,2,\ldots,n-1$ and $C_k(x), k = 1,2,\ldots,n-1$ are given in the following theorems.

Theorem 1. The fundamental polynomials $\{C_k(x)\}_{k=1}^{n-1}$ satisfying the conditions (13) can be explicitly represented as:

$$C_k(x) = -\frac{y_k}{n}\frac{L_n^{(\alpha)}(x)(L_n^{(\alpha)})'(x)}{[L_n^{(\alpha)}(y_k)]^2}l_k^*(x), \quad k = 1,2,\ldots,n-1 \tag{14}$$

where $l_k^*(x)$ are given by (9).

Proof. For $k = 1,2,\ldots,n-1$, consider

$$C_k(x) = c_k L_n^{(\alpha)}(x)(L_n^{(\alpha)})'(x)l_k^*(x), \tag{15}$$

where c_k are constants. Then obviously $C_k(x)$ are polynomials of $degree \leq 3n-3$ with $C_k(x_j) = 0, j = 1,2,\ldots,n$ and $C_k(y_j) = 0, j = 1,2,\ldots,n-1$. On differentiating (23) with respect to x and substituting $x = y_j$, we get

$$C_k^{'}(y_j) = c_k[L_n^{(\alpha)}(y_j)(L_n^{(\alpha)})''(y_j)l_k^*(y_j)]$$

which implies for $j \neq k$, $C_k^{'}(y_j) = 0$ and for $j = k$, $C_k^{'}(y_k) = 1$,which leads to

$$c_k = \frac{1}{[L_n^{(\alpha)}(y_k)(L_n^{(\alpha)})''(y_k)]}. \tag{16}$$

From (5), it follows

$$\frac{(L_n^{(\alpha)})''(y_k)}{(L_n^{(\alpha)})'(y_k)} = -\frac{n}{y_k}$$

Hence,

$$c_k = -\frac{y_k}{n[(L_n^{(\alpha)})'(y_k)]^2}. \tag{17}$$

Substituting c_k in (23), we get the required result. ■

Theorem 2. The fundamental polynomials $\{B_k(x)\}_{k=1}^{n-1}$ satisfying the conditions (12) can be explicitly represented as:

$$B_k(x) = \frac{L_n^{(\alpha)}(x)(l_k^*(x))^2}{L_n^{(\alpha)}(y_k)} + \frac{(y_k - 2)L_n^{(\alpha)}(x)(L_n^{(\alpha)})'(x)l_k^*(x)}{n[L_n^{(\alpha)}(y_k)]^2}, \tag{18}$$

where $l_k^*(x)$ are given by (9).

Proof. For $k = 1, 2, \ldots, n-1$, consider

$$B_k(x) = b_{k1}L_n^{(\alpha)}(x)(l_k^*(x))^2 + b_{k2}C_k(x), \tag{19}$$

where b_{k1} and b_{k2} are constants. Then obviously $B_k(x)$ are polynomials of $degree \leq 3n-3$ with $B_k(x_j) = 0, j = 1, 2, \ldots, n$. Also, due to (12), equation (19) implies that for $j \neq k$, $B_k(y_j) = 0$ and for $j = k$, $B_k(y_k) = 1$, which leads to

$$b_{k1} = \frac{1}{L_n^{(\alpha)}(y_k)}. \tag{20}$$

On differentiating (19) with respect to x and substituting $x = y_j$, we get

$$(B_k)'(y_j) = 2b_{k1}L_n^{(\alpha)}(y_j)l_k^*(y_j)(l_k^*)'(y_j) + b_{k2}C_k^{'}(y_j),$$

which owing to third condition of (12) and (13) for $j \neq k$, $(B_k)'(y_j) = 0$. For $j = k$, $(B_k)'(y_k) = 1$, $(C_k)'(y_k) = 1$ and using (20), we have

$$b_{k2} = -2(l_k^*)'(y_k). \tag{21}$$

Now, on differentiating (9) with respect to x and substituting $x = y_k$, we get

$$(l_k^*)'(y_k) = \frac{(L_n^{(\alpha)})'''(y_k)}{2(L_n^{(\alpha)})''(y_k)} \tag{22}$$

and on differentiating (5) with respect to x and substituting $x = y_k$, we get

$$\frac{(L_n^{(\alpha)})'''(y_k)}{(L_n^{(\alpha)})''(y_k)} = \frac{y_k - 2}{y_k}.$$

Hence,

$$b_{k2} = \frac{2 - y_k}{y_k}, \tag{23}$$

which completes the proof. ■

Theorem 3. The fundamental polynomials $\{A_k(x)\}_{k=1}^{n}$ satisfying the conditions (11) can be explicitly represented as:

$$A_k(x) = \frac{[(L_n^{(\alpha)})'(x)]^2 l_k(x)}{[(L_n^{(\alpha)})'(x_k)]^2}, \quad k = 1, 2, \ldots, n, \tag{24}$$

where $l_k(x)$ are given by (8). $k = 1, 2, \ldots, n-1,$

Proof. For $k = 1, 2, \ldots, n$ let $A_k(x)$ be defined as

$$A_k(x) = a_k[(L_n^{(\alpha)})'(x)]^2 l_k(x), \tag{25}$$

where a_k are constants. Obviously, $A_k(x)$ are polynomials of degree $\leq 3n-3$ with $A_k(y_j) = 0, j = 1, 2, \ldots, n$ and on differentiating (25) with respect to x and substituting $x = y_j$, we get $A_k^{'}(y_j) = 0$. Also, due to (11), equation (25) implies that for $j \neq k$, $A_k(x_j) = 0$ and for $j = k$, $A_k(x_k) = 1$, which leads to

$$a_k = \frac{1}{[(L_n^{(\alpha)})'(x_k)]^2}. \tag{26}$$

Substituting a_k in (25), we get the required result. ■

EXPLICIT REPRESENTATION OF THE INTERPOLATORY POLYNOMIAL WHEN NODES ARE INTECHANGESD

Let $(2n-1)$ points in $(0, \infty)$ be given by (2)then to the prescribed numbers $\{\alpha_k^*\}_{k=1}^{n}$, $\{\beta_k^*\}_{k=1}^{n}$ and $\{\gamma_k^*\}_{k=1}^{n-1}$, there exists a unique polynomial $R_n^*(x)$ of degree $\leq 3n-2$ satisfying the conditions

$$\begin{cases} R_n^*(x_k) = \alpha_k^*, & k = 1(1)n. \\ R_n^*(y_k) = \beta_k^*, & k = 1(1)n-1. \\ (R_n^*)^{'}(y_k) = \gamma_k^*, & k = 1(1)n-1. \end{cases} \tag{27}$$

The polynomial $R_n^*(x)$ can be explicitly represented as:

$$R_n^*(x) = \sum_{k=1}^{n} \alpha_k^* A_k(x) + \sum_{k=1}^{n} \beta_k^* B_k^*(x) + \sum_{k=1}^{n-1} \gamma_k^* C_k^*(x), \tag{28}$$

where $\{A_k^*(x)\}_{k=1}^n$, $\{B_k^*(x)\}_{k=1}^n$ and $\{C_k^*(x)\}_{k=1}^{n-1}$ are uniquely determined polynomials each of degree $\leq 3n-2$ and can be explicitly represented as

$$C_k^*(x) = \frac{[L_n^{(\alpha)}(x)]^2}{[L_n^{(\alpha)}(y_k)]^2} l_k^*(x), \quad k = 1, 2, \dots, n-1, \tag{29}$$

$$B_k^*(x) = \frac{L_n^{(\alpha)}(x)(L_n^{(\alpha)})'(x) l_k(x)}{[(L_n^{(\alpha)})'(x_k)]^2}, \qquad k = 1, 2, \dots, n, \tag{30}$$

$$\begin{aligned} A_k^*(x) = & \frac{(L_n^{(\alpha)})'(x)[l_k(x)]^2}{(L_n^{(\alpha)})'(x_k)} \\ & + \frac{2(1+\alpha-x_k)L_n^{\alpha}(x)(L_n^{\alpha})'(x) l_k(x)}{x_k[(L_n^{\alpha})'(x_k)]^2}, \\ & k = 1, 2, \dots, n, \end{aligned} \tag{31}$$

where $l_k(x)$ and $l_k^*(x)$ are given by (8) and (9) respectively.

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