DEFINITION AND APPROACHES OF MEASURING OPERATIONAL RISK

Before studying the various approaches of measuring operational risk, we give a definition of operational risk and discuss its segmentation and quantification.

Definition, Segmentation and quantification of operational risk

The operational risk is defined as "The risk of direct or indirect loss resulting from inadequate or failed internal processes, people and systems or from external events" (Basel Committee on Banking Supervision, 2006).

Because of the disastrous consequences of operational losses on the continuity of the activity of some banks, the Basel Committee has integrated operational risk into the calculation of the minimum capital requirement in the first pillar of the Basel II agreement.

The segmentation of operational risk

The operational risk situation

The operational risk situation is composed of three elements:

- *The generating factor of the risk (hazard):* it constitutes the factors that favour the occurrence of the risk incident as inexperienced personnel, the malfunction of control device.

- *The operational risk event (incident):* it constitutes the single incident whose occurrence can generates losses for the bank as internal fraud and external fraud.

-*The impact (loss*): it constitutes the amount of financial damage resulting from an event.

The operational risk category and the business lines

To normalize the identification of the situations of operational risks, the Basel Committee on Banking Supervision (2006) defines a generic mapping of operational risks within credit institutions, comprising 8 business lines and 7 categories of operational risks.

The operational risk categories ($RT_c, 1 \leq c \leq 7$) are:

- RT_1 : Execution, delivery and process management.
- RT_2 : Business disruption and system failures.
- RT_3 : Damage to physical assets.
- RT_4 : Clients, products and business practices.
- RT_5 : Employment practices and workplace safety.
- RT_6 : External Fraud.
- RT_7 : Internal Fraud.

The business lines $BL_i, 1 \leq i \leq 8$ are:

- BL_1: Corporate finance.
- BL_2: Trading and sales.
- BL_3: Retail banking.
- BL_4: Commercial banking.
- BL_5: Payment and settlement.
- BL_6: Agency services.
- BL_7: Asset management.
- BL_8: Asset management.

Approaches of measuring operational risk

In terms of quantification of operational risk, the Basel Committee provides three approaches for calculating the capital requirement K for operational risk.

. Basic indicator Approach (BI)

The Basic Indicator Approach (BI) is a simple approach. This approach considers that the operational risk of a credit institution is proportional to the gross annual incomes GI generated by this institution. The required regulatory capital represents a fixed percentage α of the average gross annual income of n years, whose the GI is positive over the last three years. To calculate the average, the years with the negative or null gross incomes should be excluded from the numerator and denominator. As a result, the operational risk capital, according to the BI approach, noted K_{BI}, is defined by the following expression:

$$K_{BI} = \alpha \times \frac{\sum_1^n GI_i}{n} \qquad (1)$$

α is set at 15% by the Basel committee, n represents the number of years over the past three years for which the gross incomes is positive and GI_i is the gross income of the year i.

Standardized approach STA

In the Standardized Approach, banks' activities are divided into eight business lines for each year j. In a given year j, the gross annual income GI_i^j of the business line i, is an indicator of exposure at the operational risk . The requirement into capital FL_i^j for each line i of the year j is calculated by multiplying GI_i^j by the factor, noted β_i, attributed to the line i (independent of the chosen year) :

$$FL_i^j = \beta_i \times GI_i^j$$

The capital required for all business lines in a given year j is deducted by:

$$FL^j = \sum_{i=1}^{8} FL_i^j$$

The total requirement of capital, noted K_{STA}, according to standardized approach may be expressed as:

$$K_{STA} = \frac{\sum_{j=1}^{3} Max(FL^j, 0)}{3} = \frac{\sum_{j=1}^{3} Max\left(\sum_{i=1}^{8} \beta_i \times GI_i^j, 0\right)}{3} \qquad (2)$$

The factors β_i are a fixed percentages determined by the Basel Committee as follows:

Table 1 : The factors β_i for standardized approach

BL_i	BL_1	BL_2	BL_3	BL_4	BL_5	BL_6	BL_7	BL_8
β_i	18 %	18 %	12 %	15 %	18 %	15 %	12 %	12 %

Advanced Measurement Approach (AMA)

In the Advanced Measurement Approach (AMA), the bank can use its own model to assess their operational risk exposure. The bank must demonstrate that the operational risk is assessed for a one-year holding period and for a high level of confidence (for example 99.9%).

In order to ensure the reliability of the valuation methodology the bank may, in addition to internal data, supplements their databases with external data (appropriately resized) and uses techniques such as factor analysis, stress testing and Bayesian method.

In 2001, three approaches were proposed by the Basel Committee in the framework of Advanced Measurement Approach: the Internal Measurement Approach (IMA), the Loss Distribution Approach (LDA) and the Scorecard Approach (SCA).

The regulatory capital guidelines of June 2006 do not give the names of the possible approach grouped under AMA, but allow banks to develop their own more sophisticated and robust AMA models that reflect their operational risk capital.

➢ **The Internal Measurement Approach (IMA)**

As part of the Internal Measurement Approach (IMA), for each of the 56 cells (7 $risk\ categories \times 8\ business\ lines$), we first define the expected loss EL_{ij} of the business line i and the risk category j by :

$$EL_{ij} = EI_i \times PE_{ij} \times LGE_{ij}$$

with:

- EI_i : The exposure indicator (EI_i) (such as the gross income of the business line i).

- PE_{ij} : The probability of occurrence of an operational risk event of the business line i and the risk category j.

- LGE_{ij} (loss given event): The amount of loss associated to the business line i and the risk category j.

The total requirement of capital K_{IMA} for a one-year horizon for 56 cells is calculated by:

$$K_{IMA} = \sum_{i=1}^{8}\sum_{j=1}^{7} \gamma_{ij} \times EL_{ij} = \sum_{i=1}^{8}\sum_{j=1}^{7} \gamma_{ij} \times EI_i \times PE_{ij} \times LGE_{ij} \qquad (3)$$

The γ_{ij} for each business line and risk category would be specified by banks (possibly via the consortia) and subject to acceptance by supervisors.

The hypotheses on which is based this approach are:

- A perfect correlation between combinations (business line and risk category).

- A linear relation between expected losses and unexpected losses.

Let us indicate that the inconveniences of this approach are its hypotheses which are too restrictive.

➢ **The Scorecard Approach (SCA)**

The Scorecard Approach (SCA) is an approach whereby banks determine an initial level of capital requirement for operational risk (as based on BIA or TSA) at the global level or at the business line level. The approach is prospective because it is designed to reflect improvements in the risk control environment, which would reduce both the frequency and the severity of future operational risk losses. The capital requirement, noted K_{SCA}, for a one-year horizon and for the eight business lines is calculate as:

$$K_{SCA} = \sum_{i=1}^{8} K_i \times R_i \qquad (4)$$

where R_i is a score of risk of the business line i, which revalues the initial requirement of capital K_i of this business line.

➢ **Loss Distribution Approach (LDA)**

The LDA approach uses distributions of the frequency and the severity of operational losses occurred to determine operational losses over a time horizon T.

A segmentation of losses by business lines or by categories of risk or by the combined business lines and categories of risk is suggested. On the basis of the possible segmentation, the distributions of the severity and the frequency are estimated. The bank calculates then the distribution of the aggregated operational loss.

The aggregated operational loss P for a time horizon T is the sum of the individual losses incurred and collected by the bank in a horizon T. Therefore this aggregate loss can be represented by the random variable P defined by $P = \sum_{i=1}^{N} X_i$. Where X_i is the individual loss of the operational risk incident and N is the number of incidents that occurred during the period T.

In this article we adopt two types of modelling: we first start with the modelling of the total database and then we model the loss by category of risk. We proceed in the same way for both classical LDA Approach and Pure Bayesian LDA Approach.

Classical LDA Approach

Presentation of the Classical LDA model

Mathematical formulation of the model

In the LDA approach, the operational loss in the horizon T is a random variable P_N defined by:

$$P_N = \sum_{i=1}^{N} X_i \tag{5}$$

with: - X_i : is the random variable that represents the individual impact of operational risk incidents.

- N : is the random variable that represents the number of occurrences on a horizon T.

The random variables X_i are independent and identically distributed. The random variable N is independent with the variables X_i.

Calculation of the moments of the operational loss P

We note $E(X) = E(X_i) = \mu$ and $Var(X) = Var(X_i) = \sigma^2$.

The mathematical expectation, the variance and the central moments of order 3 and 4 of the compound random variable P are defined as follows:

$E(P) = E(X) \times E(N)$	(6)
$Var(P) = E(N) \times Var(X) + Var(N) \times E(X)^2$	(7)
$E((P - E(P))^3) = E(N) \times E((X - E(X))^3) + 3 \times Var(N) \times Var(X) \times E(X) + E((N - E(N))^3) \times (E(X))^3$	(8)
$E((P - E(P))^4) = E(N) \times E((X - E(X))^4) + 4 \times Var(N) \times E((X - E(X))^3) \times E(X) + 3 \times \left(Var(N) + E(N) \times (E(N) - 1)\right) \times (Var(X)^2) + 6(E((X - E(X))^3) + E(N) \times Var(N)) \times E(X)^2 \times Var(X)$	(9)

Modelling of the individual severity X of losses

Several distributions can be used to represent the severity random variable X as the LogNormal distribution, the Beta distribution, the Weibull distribution or other distributions which are detailed in Chernoubai et al (2007). In our study, we limit ourselves to LogNormal distribution $LN(\mu, \sigma)$ and Weibull distribution $Weib(\alpha, \beta)$ defined as follows:

Table 2 : Definition of $LN(\mu, \sigma)$ and $Weib(\alpha, \beta)$ and estimation of their parameters

Distribution	Probability Density Function $f_X(x)$	Estimation of the parameters by the maximum likelihood method
$LN(\mu, \sigma)$	$\begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} e^{\frac{-(\ln(x)-\mu)^2}{2\sigma^2}} & if\ x > 0 \\ 0 & if\ not \end{cases}$	$\begin{cases} \hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} ln(x_i) \\ \hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{N}(ln(x_i) - \hat{\mu})^2 \end{cases}$ (10)
$Weib(\alpha, \beta)$ $\alpha > 0, \beta > 0$	$\begin{cases} \frac{\alpha}{\beta} x^{\alpha-1} e^{\frac{-x^\alpha}{\beta}} & if\ x \geq 0 \\ 0 & if\ not \end{cases}$	$\begin{cases} \frac{\sum_{i=1}^{n} {x_i}^{\hat{\alpha}} ln(x_i)}{\sum_{i=1}^{n} {x_i}^{\hat{\alpha}}} - \frac{1}{\hat{\alpha}} - \frac{1}{n}\sum_{i=1}^{n} \log(x_i) = 0 \\ \hat{\beta} = \frac{1}{n}\sum_{i=1}^{n} {x_i}^{\hat{\alpha}} \end{cases}$ (11)

Table 3 : Definition of $P(\lambda)$ et $BN(a, b)$ and estimation of their parameters

Distribution	Probability Density Function	Estimation of the parameters by the maximum likelihood method
$\mathcal{P}(\lambda)$	$P(N = n) = \frac{\lambda^n}{n!} e^{-\lambda}$	$\hat{\lambda} = \frac{1}{N}\sum_{i=1}^{N} n_i$ (12)
$BN(a, b)$	$P(N = n) = \frac{\Gamma(a + n)}{\Gamma(a)n!} \times \frac{b^a}{(b + 1)^{n+a}}$ with $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx$ $\Gamma(r) = (r - 1)!\quad si\quad r \in \mathbb{N}^*$	$\hat{a} = \frac{\bar{N}^2}{(\hat{\sigma}_N^2 - \bar{N})}$ $\hat{b} = \frac{\bar{X}}{(\hat{\sigma}_N^2 - \bar{N})}$ $\hat{\sigma}_N^2 = \frac{1}{N-1}\sum_{i=1}^{N}(n_i - \bar{N})^2$ (13)

Modelling of the frequency N of losses

With regard to the modelling of the loss frequency N, we use the Poisson distribution $P(\lambda)$ or the Negative Binomial distribution $BN(a,b)$ defined as follows :

Goodness-of-Fit tests of the modelling

To ensure the representativeness of the model chosen for the calculation of capital, we test the good-fit of the theoretical distribution of the severity and the frequency with the empirical data. In our work, we use the Kolmogorov-Smirnov (KS) test of goodness-of-fit of distribution for individual loss with the LogNormal distribution and the Weibull distribution and the Chi-Square χ_2 test of goodness-of-fit of distribution for periodic frequency.

The presentation of the Kolmogorov-Smirnov Goodness-of-fit test

Let X be the severity and F_X be the empirical cumulative distribution function of the random variable X. Let $x_1, x_2, \ldots, x_m$ be m orderly realizations in ascending order of X.

In the Kolmogorov-Smirnov test, the null hypothesis and the alternative hypothesis are formulated by:

$$\begin{cases} H_0 : F(x_i;\hat{\theta}) = F_X(x_i) \ \ for\ 1 \le i \le m \\ H_1 : F(x_i;\hat{\theta}) \ne F_X(x_i) \ \ for\ at\ least\ one\ i \end{cases}$$

with $\hat{\theta}$ designates the vector of estimated parameters.

The function $F(x_i;\hat{\theta})$ is the theoretical cumulative distribution function. The empirical cumulative distribution function F_X is given by:

$$F_X(x_i) = P(X \le x_i) = \frac{1}{m}\sum_{i=1}^{m} I(\mathrm{X} \le x_i) = \frac{number\ of\ empirical\ losses\ \le\ x_i}{total\ number\ of\ losses} = \frac{i}{m}$$

The model provides a good fit of the distribution if the hypothesis H_0 is not rejected.

The KS test consists of measuring the maximum absolute deviation between $F(x_i;\hat{\theta})$ and $F_X(x_i)$. The statistic used for the KS test is given by:

$$KS = \sqrt{m} \times \max_{1\le i\le m} \left| F(x_i;\hat{\theta}) - \frac{i}{m} \right| \tag{14}$$

This statistic is compared with a critical value (1.36 and 1.63 respectively for a threshold of 95% and 99%).

Goodness-of-fit test by slice of distribution

➢ **Construction of the theoretical distribution by slice $F_{\alpha,\beta,x_s,a}$**

The goodness-of-fit test by slice consists in testing the goodness-of-fit by segment of distribution. Indeed, this consists to cut the empirical observations into S segments and carry out the goodness-of-fit test for each segment with a theoretical distribution. For this, we construct the following algorithm to segment the empirical observations. Consequently, we construct a slice cumulative distribution function. Indeed, we have built the following segmentation algorithm, using the goodness-of-fit:

Let $(x_1, x_2, \ldots, x_m)$ be the m realizations of the operational losses and $(\tilde{x}_1, \tilde{x}_2, , \ldots, \tilde{x}_m)$ their ranking in ascending order.

Step 1: Select a first sample $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_{p_1})$ of size p_1 and test the adjustment of the empirical cumulative distribution function $F_X(\tilde{x}_i)$ with the cumulative distribution function of LogNormal (or the cumulative distribution function of Weibull) $F(\tilde{x}_i;\hat{\theta}_1)$ for $1 \le i \le p_1$.

➢ **If the sample does not adjust to the theoretical distribution then:**

1. Remove the last observation and repeat the test.

2. If the sample does not adjust, repeat 1. until an adjusted sample is obtained. Let $\tilde{x}_{s(1)}$ be the biggest element such that the sample $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{s(1)}$ adjusts.

3. Retain the sample $\tilde{x}_1, \tilde{x}_2, .. \tilde{x}_{s(1)}$ as a support for the first segment of the distribution.

➢ **If the sample does adjust to the theoretical distribution then:**

1. Add the following observation that does not belong to the sample and test the adjustment.

2. If the sample does adjust, repeat 1. until an unadjusted sample is obtained. Let $\tilde{x}_{s(1)}$ be the biggest element such that the sample $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{s(1)}$ adjusts.

3. Retain the sample $\tilde{x}_1, \tilde{x}_2, .. \tilde{x}_{s(1)}$ as a support for the first segment of the distribution.

Step 2: Select a sample of size p_2 such as the smallest element is $\tilde{x}_{s(1)}$. Let $(\tilde{x}_{s(1)}, \tilde{x}_{s(1)+1}, \ldots \tilde{x}_{s(1)+p_2-1})$ be this sample, remake the same process used in step 1 to determine the second sample that fits with the LogNormal (or the Weibull) $F(\tilde{x}_{s(1)+i};\hat{\theta}_2)$ for $0 \le i \le p_2 - 1$.

Let $(\tilde{x}_{s(1)}, \tilde{x}_{s(1)+1}, \ldots, \tilde{x}_{s(2)-1}, \tilde{x}_{s(2)})$ be the last adjusted sample which is chosen as a support for the second distribution segment.

Step 3: We continue the same process until the total segmenting of the initial sample into q adjusted segments:

$S_1 = (\tilde{x}_{s(0)}, \tilde{x}_2, \ldots, \tilde{x}_{s(1)}), \ldots, S_2 = (\tilde{x}_{s(1)}, \ldots, \tilde{x}_{s(2)}), \ldots, S_q = (\tilde{x}_{s(q-1)}, \ldots, \tilde{x}_{s(q)})$

with $s(0) = 1$ et $s(q) = m$.

Let consider the laws of theoretical distributions with which they fit:

$F(S_1;\hat{\theta}_1)$,$F(S_2;\hat{\theta}_2)$,…, $F(S_q;\hat{\theta}_q)$

We pose $\hat{\theta}_i = (\alpha_i, \beta_i)$ the parameters of the LogNormal law (or the Weibull law). We note:

$$F(S_i; \hat{\theta}_i) = K_{\alpha_i,\beta_i}$$

We pose $\alpha = (\alpha_1, \alpha_2, \dots \alpha_q)$; $\beta = (\beta_1, \beta_2, \dots \beta_q)$; $x_s = (x_{s(1)}, x_{s(2)}, \dots, x_{s(q)})$; $a = (a_1, a_2, \dots a_q)$

The theoretical cumulative distribution function $F_{\alpha,\beta,x_s,a}(x)$ by slice is defined as follows:

$$F_{\alpha,\beta,x_s,a}(x) = \begin{cases} K_{\alpha_1,\beta_1}(x) \times \frac{a_1}{a_q} & if\ x_{s(0)} \le x \le x_{s(1)} \\ K_{\alpha_2,\beta_2}(x) \times \frac{a_2 - a_1}{a_q} + \frac{a_1}{a_q} & if\ x_{s(1)} \le x \le x_{s(2)} \\ \vdots \\ K_{\alpha_i,\beta_i}(x) \times \frac{a_i - a_{i-1}}{a_q} + \frac{a_{i-1}}{a_q} & if\ x_{s(i-1)} \le x \le x_{s(i)} \\ \vdots \\ K_{\alpha_q,\beta_q}(x) \times \frac{a_q - a_{q-1}}{a_q} + \frac{a_{q-1}}{a_q} & if\ x_{s(q-1)} \le x \le x_{s(q)} \end{cases}$$

where a_i= number of observations less than $x_{s(i)}$ and $a_q = m$ (overall size of the initial sample).

The empirical function F_X satisfies:

$$\begin{cases} F_X(x) \le \frac{a_1}{a_q} & if\ x_{s(0)} \le x \le x_{s(1)} \\ \frac{a_1}{a_q} \le F_X(x) \le \frac{a_2}{a_q} & if\ x_{s(1)} \le x \le x_{s(2)} \\ \vdots \\ \frac{a_{i-1}}{a_q} \le F_X(x) \le \frac{a_i}{a_q} & if\ x_{s(i-1)} \le x \le x_{s(i)} \\ \vdots \\ \frac{a_{q-1}}{a_q} \le F_X(x) \le 1 & if\ x_{s(q-1)} \le x \le x_{s(q)} \end{cases}$$

- **Kolmogorov-Smirnov test of adjustment of the function $F_{\alpha,\beta,x_s,a}(x)$ with the empirical cumulative distribution function**

We demonstrate that the function $F_{\alpha,\beta,x_s,a}(x)$, adjusts with the overall sample. Indeed, we must show that:

$$KS_{(x_1,\dots,x_m)} = \sqrt{m} \times \max_{1\le i\le m} \left| F_{\alpha,\beta,x_s,a}(x_i) - \frac{i}{m} \right| \le ks_t$$

where ks_t is the value tabulated of Kolmogorov-Smirnov test at the threshold of 95%. Let's show that :

$$\max_{1\le i\le m} \left| F_{\alpha,\beta,x_s,a}(x_i) - \frac{i}{m} \right| \le \frac{ks_t}{\sqrt{m}}$$

For all x_i it exists a sample $A_j = (\tilde{x}_{s(j-1)}, \dots, \tilde{x}_{s(j)})$ who contains the observation x_i. Thus:

$$F_{\alpha,\beta,x_s,a}(x_i) = K_{\alpha_j,\beta_j}(x_i) \times \frac{a_j - a_{j-1}}{a_q} + \frac{a_{j-1}}{a_q}$$

with $card(A_j) = a_j - a_{j-1}$ and $a_q = m$. We deduct:

$$\left| F_{\alpha,\beta,x_s,a}(x_i) - \frac{i}{m} \right| = \left| K_{\alpha_j,\beta_j}(x_i) \times \frac{a_j - a_{j-1}}{a_q} + \frac{a_{j-1}}{a_q} - \frac{i}{m} \right|$$
$$= \left| K_{\alpha_j,\beta_j}(x_i) \times \frac{a_j - a_{j-1}}{m} + \frac{a_{j-1}}{m} - \left(\frac{i - a_{j-1}}{m} + \frac{a_{j-1}}{m} \right) \right|$$
$$= \left| K_{\alpha_j,\beta_j}(x_i) \times \frac{a_j - a_{j-1}}{m} - \frac{i - a_{j-1}}{m} \right|$$
$$\Leftrightarrow \left| F_{\alpha,\beta,x_s,a}(x_i) - \frac{i}{m} \right| = \frac{a_j - a_{j-1}}{m} \left| K_{\alpha_j,\beta_j}(x_i) - \frac{i - a_{j-1}}{a_j - a_{j-1}} \right|$$

We know that:

$$\frac{i - a_{j-1}}{a_j - a_{j-1}} = \frac{Numbre\ of\ losses \le x_i\ and\ belonging\ to\ (\tilde{x}_{s(j-1)}, \dots, \tilde{x}_{s(j)})}{size\ of\ the\ sample\ (\tilde{x}_{s(j-1)}, \dots, \tilde{x}_{s(j)}) = card(\tilde{x}_{s(j-1)}, \dots, \tilde{x}_{s(j)})}$$

is the empirical cumulative distribution function of the sample $(\tilde{x}_{s(j-1)}, \dots, \tilde{x}_{s(j)})$ which is noted $F_{(\tilde{x}_{s(j-1)},\dots,\tilde{x}_{s(j)})}(x_i)$. We deduct :

$$\left| F_{\alpha,\beta,x_s,a}(x_i) - \frac{i}{m} \right| = \frac{a_j - a_{j-1}}{m} \left| K_{\alpha_j,\beta_j}(x_i) - \frac{i - a_{j-1}}{a_j - a_{j-1}} \right| = \frac{a_j - a_{j-1}}{m} \left| K_{\alpha_j,\beta_j}(x_i) - F_{(\tilde{x}_{s(j-1)},\dots,\tilde{x}_{s(j)})}(x_i) \right|$$

We know that K_{α_j,β_j} adjusts with $F_{(\tilde{x}_{s(j-1)},\dots,\tilde{x}_{s(j)})}$. Thus:

$$\left| K_{\alpha_j,\beta_j}(x_i) - F_{(\tilde{x}_{s(j-1)},\dots,\tilde{x}_{s(j)})}(x_i) \right| \le \max_{x_i \in (\tilde{x}_{s(j-1)},\dots,\tilde{x}_{s(j)})} \left| K_{\alpha_j,\beta_j}(x_i) - \frac{i - a_{j-1}}{a_j - a_{j-1}} \right| = \frac{KS_{(\tilde{x}_{s(j-1)},\dots,\tilde{x}_{s(j)})}}{\sqrt{a_j - a_{j-1}}} \le \frac{ks_t}{\sqrt{a_j - a_{j-1}}}$$

Hence

$$\left| K_{\alpha_j,\beta_j}(x_i) - F_{(\tilde{x}_{s(j-1)},\dots,\tilde{x}_{s(j)})} \right| \le \frac{ks_t}{\sqrt{a_j - a_{j-1}}}$$

Thus

$$\frac{a_j - a_{j-1}}{m} \left| K_{\alpha_j,\beta_j}(x_i) - F_{(\tilde{x}_{s(j-1)},\dots,\tilde{x}_{s(j)})} \right| \le \frac{a_j - a_{j-1}}{m} \times \frac{ks_t}{\sqrt{a_j - a_{j-1}}} \Leftrightarrow$$

$$\frac{a_j - a_{j-1}}{m} \left| K_{\alpha_j,\beta_j}(x_i) - F_{(\tilde{x}_{s(j-1)},\dots,\tilde{x}_{s(j)})} \right| \le \frac{\sqrt{a_j - a_{j-1}}}{m} \times ks_t \Leftrightarrow$$

$$\frac{a_j - a_{j-1}}{m} \left| K_{\alpha_j,\beta_j}(x_i) - F_{(\tilde{x}_{s(j-1)},\dots,\tilde{x}_{s(j)})} \right| \le \sqrt{\frac{a_j - a_{j-1}}{m}} \times \frac{ks_t}{\sqrt{m}} \le \frac{ks_t}{\sqrt{m}} \Leftrightarrow \left| F_{\alpha,\beta,x_s,a}(x_i) - \frac{i}{m} \right| \le \frac{ks_t}{\sqrt{m}}$$

Hence

$$\max_{1\le i\le m}\left|F_{\alpha,\beta,x_s,a}(x_i)-\frac{i}{m}\right|\le\frac{ks_t}{\sqrt{m}} \qquad (15)$$

Chi-Square Goodness-of-fit test

For the frequency adjustment test, we will used the test of chi-square χ^2. This test requires that data to be grouped into classes.

We note $P_N(n)$ the empirical law of the frequency N for a sample of size n segmented into k groups $g_1, g_2, \dots, g_k$. The size of the group g_i is noted n_i. We then have $n=\sum_{i=1}^{k} n_i$.

Let $P_{\hat{\theta}}$ be the theoretical law of the frequency.

The null hypothesis and the alternative hypothesis are formulated by:

$$\begin{cases} H_0: P_N(n)\ adjusts\ with\ the\ distribution\ P_{\hat{\theta}} \\ H_1: P_N(n)\ does\ not\ adjust\ with\ the\ distribution\ P_{\hat{\theta}} \end{cases}$$

Klugman, Panjer and Willmot (1998) define the chi-square test statistic as follow:

$$F=\sum_{i=1}^{k}\frac{(n_i-E_i)^2}{E_i} \qquad (16)$$

with : n_i : The number of observations in the group g_i, for $i = 1 \dots . k$.

E_i : The number of observations expected in each group g_i. It is calculated by the expression:

$$E_i = n \times P_{\hat{\theta}}\,(N \in groupe\ g_i)$$

where $P_{\hat{\theta}}$ is the theoretical law defined by the parameter estimated $\hat{\theta}$ which to be tested. If the statistic F is lower to the tabulated value of χ^2 to the threshold 5% then the null hypothesis H_0 does not rejected.

In the case where the frequency does not adjusted with the two theoretical distributions ($P(\lambda)$ and $BN(a,b)$), the simulation of the frequency will be done by the bootstrap technique to reproduce the past realizations.

The Pure Bayesian *LDA* Approach

In the pure Bayesian LDA approach, the parameters of the distributions of the frequency N and the individual loss X_i are considered as random variables with a probability density function.

The pure Bayesian approach considers the parameters (μ,σ) and λ of the density functions of X_i and N as the random variables whose the density are respectively π_μ, π_σ and π_λ.

Description of the Pure Bayesian *LDA* Approach

Let $Y=(Y_1,\dots Y_m)$ be a vector of random variables independent and identically distributed (i.i.d). Let $(y_1,\dots y_m)$ be a realization of the vector Y and let $\theta=(\theta_1,\theta_2,\dots,\theta_p)$ be a vector of the random variables of the parameters of the density of the vector Y.

The density function $f(Y,\theta)$ of the vector $(Y,\theta)=(Y_1,\dots Y_m,\theta_1,\theta_2,\dots,\theta_p)$ is defined by :

$$f(Y,\theta)=f(Y/\theta)\pi(\theta)=\pi(\theta/Y)f(Y) \qquad (17)$$

where :

- $\pi(\theta)$ is the probability density of the parameter θ, called “prior density function”.

- $\pi(\theta/Y)$ is the conditional probability density function of the parameter θ knowing Y, called “posterior density”;

- $f(Y,\theta)$ is a probability density function of the couple (Y,θ) ;

- $f(Y/\theta)$ is the conditional density function of Y knowing θ, it is the likelihood function $f(Y/\theta)=\prod_{i=1}^{m} f_i(Y_i/\theta)$

with $f_i(Y_i/\theta)$ is the conditional probability density function of Y_i.

- $f(Y)$) is a marginal density of Y that can be written as $\int f(Y/\theta)\pi(\theta)d\theta$.

The Bayes formula allows to determine $\pi(\theta/Y)$ of the parameter θ knowing Y as follows:

$$\pi(\theta/Y)=\frac{f(Y/\theta)\pi(\theta)}{f(Y)} \qquad (18)$$

Hence

$$\pi(\theta/Y)\propto f(Y/\theta)\pi(\theta) \qquad (19)$$

$f(Y)$ is a normalization constant and the posterior distribution $\pi(\theta/Y)$ can be viewed as a combination of a prior knowledge $\pi(\theta)$ with a likelihood function $f(Y/\theta)$ for observed data. Since $f(Y)$ is a normalization constant, the posterior distribution is often written with the form (19) where the symbol $\propto$ signified “is proportional” with a constant of proportionality independent of the parameter θ.

The estimate of Bayesian posterior mean $\hat{\theta}_{Bay}$ of θ is defined as follows:

- The parameter $\boldsymbol{\theta}$ is univariate :

The estimate of the Bayesian posterior mean of θ noted $\hat{\theta}_{Bay}$ is a conditional expectation of θ knowing Y:

$$\begin{aligned}\hat{\theta}_{Bay}=E(\theta/Y)&=\int\theta\times\pi(\theta/Y)d\theta \\ &=\frac{\int\theta\times f(Y/\theta)\pi(\theta)d\theta}{f(Y)}\end{aligned} \qquad (20)$$

- The parameter $\boldsymbol{\theta}$ is multivariate:

In a multidimensional context where $\theta=(\theta_1,\theta_2,\dots,\theta_p)$, the estimate of the Bayesian posterior mean of θ noted $\hat{\theta}_{Bay}$ is a conditional expectation of the vector θ knowing Y defined by :

$$\begin{aligned}\hat{\theta}_{Bay}&=E(\theta/Y) \\ &=\left(E(\theta_1/Y),E(\theta_2/Y),\dots,E(\theta_p/Y)\right)\end{aligned} \qquad (21)$$

$$= \left(\int \theta_1 \times \pi(\theta_1/X) d\theta_1 , \int \theta_2 \times \pi(\theta_2/X) d\theta_2 , \ldots , \int \theta_p \times \pi(\theta_p/X) d\theta_p \right)$$

Calculation of the estimate of the Bayesian posterior mean

To determine the estimate of Bayesian posterior means defined by the formulas (20) and (21), we must determine the prior law and the posterior law of the random variable θ.

In fact, we will limit our study to the Lognormal distribution for loss severity $X_i \backsim LN(\mu,\sigma)$,$1 \leq i \leq m$ and to the Poisson distribution for the frequency of the losses $N \backsim P(\lambda)$. The parameters μ, σ and λ are considered random variables.

Therefore, we have to determine the following estimate of the Bayesian posterior mean:

$$\hat{\theta}_{Bay} = (\hat{\mu}, \hat{\sigma}) = E(\mu, \sigma / X_1, \ldots X_n) \qquad (22)$$

$$\hat{\theta}_{Bay} = \hat{\lambda} = E(\lambda / N) \qquad (23)$$

Determination of the prior law of the parameters

The Bayesian approach depends on the accuracy of the information provided by experts on the parameters of the prior law. In fact, we will present below the approach adopted:

❖ **The prior law of the parameter λ with $N \backsim P(\lambda)$**

The choice of the prior distribution of the parameter λ depends on the description of the characteristics of the random variable given by the experts.

If the experts consider that the variable $\lambda(t)$ is concentrated around a value λ_0 with exceptional realizations distant from λ_0. The Gaussian distribution in this case is the adequate distribution to represent $\lambda(t)$.

Moreover, if the experts consider that the random variable $\lambda(t)$ is a positive support and that it represents the number of incidents occurring on heterogeneous spaces of time then in this case the gamma distribution with two parameters is the adequate distribution.

In our study, we will retain the second description and we will consider that the prior law is a gamma distribution Γ with the parameters (a, b) to be determined by the experts. Indeed, the Gamma distribution is defined as follows:

Table 4 : Definition of the gamma distribution $\Gamma(a, b)$

Probability density function	$E(\lambda)$	$V(\lambda)$	a	b
$\Gamma(\lambda) = \frac{\left(\frac{\lambda}{b}\right)^{a-1}}{\Gamma(a) \times b} \times e^{-\frac{\lambda}{b}}$	$a \times b$	$a \times b^2$	$\frac{E(\lambda)^2}{V(\lambda)}$	$\frac{V(\lambda)}{E(\lambda)}$

To calculate (a, b), the experts must determine the linear relation between $V(\lambda)$ and $E(\lambda)$ and the linear relation between $V(\lambda)$ and $E(\lambda)^2$.

❖ **The prior law of μ et σ with $X_i \backsim LN(\mu, \sigma)$**

In this paper we limit ourselves to the case where μ is a gaussian random variable $\mu \backsim N(\mu_0, \sigma_0)$ and σ a known constant.

However, Schevchenko PV (2011), represented σ^2 by the inverse Chi-square distribution (Inv.Chi.Sq) of parameters (α, β) whose the probability density function is defined by :

$$f(\sigma^2) = \frac{\left(\frac{\sigma^2}{\beta}\right)^{-1-\frac{\alpha}{2}}}{2^{\frac{\alpha}{2}} \times \Gamma\left(\frac{\alpha}{2}\right) \times \beta} \times e^{-\frac{\beta}{2\sigma^2}} \qquad (24)$$

Determination of the posterior law of the parameters λ and μ

The posterior distribution is determined from the likelihood function and the prior distribution by the formula (19). Thereby, we will calculate the posterior law of frequency and severity:

❖ **The posterior law of the parameter λ with $N \backsim P(\lambda)$**

Let $N = (N_1, \ldots N_l)$ be a vector of random variables of the frequency. Let $(n_1, \ldots n_l)$ be a realization of the vector N. We suppose that $N_j \backsim P(\lambda)$ and we consider that $\lambda \backsim \Gamma(a, b)$.

The posterior law conjugated at the prior law λ is defined by:

$$\pi(\lambda/N) \propto f(N/\lambda)\pi(\lambda) \propto f(N/\lambda) \frac{\left(\frac{\lambda}{b}\right)^{a-1}}{\Gamma(a) \times b} \times e^{-\frac{\lambda}{b}}$$

We have :

$$f(N/\lambda) = \prod_{j=1}^{l} f_j(N_j/\lambda) = \prod_{j=1}^{l} \frac{\lambda^{n_j}}{n_j!} e^{-\lambda}$$

Thus

$$\pi(\lambda/N) \propto \prod_{j=1}^{l} \frac{\lambda^{n_j}}{n_j!} e^{-\lambda} \times \frac{\left(\frac{\lambda}{b}\right)^{a-1}}{\Gamma(a) \times b} \times e^{-\frac{\lambda}{b}}$$
$$\propto \frac{\left(\frac{\lambda}{b}\right)^{a-1}}{\Gamma(a) \times b} \times e^{-\frac{\lambda}{b}} \times \prod_{j=1}^{l} e^{-\lambda} \frac{\lambda^{n_j}}{n_j!}$$
$$\propto \frac{\left(\frac{\lambda}{b}\right)^{a-1}}{\Gamma(a) \times b} e^{-\frac{\lambda}{b}} \prod_{j=1}^{l} \left(e^{-\lambda} \frac{\lambda^{n_j}}{n_j!}\right)$$
$$\propto \frac{\left(\frac{\lambda}{b}\right)^{a-1}}{\Gamma(a) \times b} \prod_{j=1}^{l} \frac{\lambda^{n_j}}{n_j!} \times \left(e^{-l\times\lambda} e^{-\frac{\lambda}{b}}\right)$$
$$\propto \frac{\left(\frac{\lambda}{b}\right)^{a-1}}{\Gamma(a) \times b} \prod_{j=1}^{l} \frac{\lambda^{n_j}}{n_j!} \times \left(e^{-l\times\lambda} e^{-\frac{\lambda}{b}}\right)$$
$$\propto \frac{\left(\frac{\lambda}{b}\right)^{a-1}}{\Gamma(a) \times b} \prod_{j=1}^{l} \frac{\lambda^{n_j}}{n_j!} \times e^{-\lambda\left(\frac{1}{b}+l\right)}$$
$$\propto \lambda^{a-1} \lambda^{\sum_{j=1}^{l} n_j} \times e^{-\lambda\left(\frac{1}{b}+l\right)}$$
$$\propto \lambda^{(a+\sum_{j=1}^{l} n_j)-1} \times e^{-\lambda\left(\frac{1+b\times l}{b}\right)}$$

We pose $a_l = a + \sum_{j=1}^{l} n_j$ et $b_l = \frac{b}{1+b\times l}$. Thus

$$\pi(\lambda/N) \propto \lambda^{a_l-1} e^{-\frac{\lambda}{b_l}} \qquad (25)$$

From the formula (25) we deduct that the posterior law is a gamma law $\Gamma(a_l, b_l)$.

❖ **The posterior law of the parameter $\mu \leadsto \mathcal{N}(\mu_0, \sigma_0)$ with σ a constant and $X_i \leadsto LN(\mu, \sigma)$:**

Let $x_1, \dots, x_m$ be the realizations of random variables $X_1, \dots, X_m$ representing the collected losses. We suppose here for the Bayesian modelling of the severity that $\mu \leadsto \mathcal{N}(\mu_0, \sigma_0)$ and σ a constant, which we estimate from the sample by the maximum likelihood method. We pose $Z_i = ln(X_i)$. Thus $Z_i \leadsto \mathcal{N}(\mu, \sigma)$.

We consider the random vector $Z = (Z_1, \dots, Z_m)$.The prior distribution of μ is given by:

$$\pi(\mu) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{(\mu-\mu_0)^2}{2\sigma_0{}^2}}$$

The conditional distribution of the random vector Z is given by:

$$f(Z/\mu, \sigma) = \prod_{i=1}^{m} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(Z_i-\mu)^2}{2\sigma^2}}$$

Hence the posterior law of μ:

$$\pi(\mu/Z) \propto f(Z/\mu)\pi(\mu)$$

$$\pi(\mu/Z = (z_1, \dots, z_m)) \propto \prod_{i=1}^{m} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(z_i-\mu)^2}{2\sigma^2}} \times \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{(\mu-\mu_0)^2}{2\sigma_0{}^2}}$$
$$\propto \prod_{i=1}^{m} e^{-\frac{(z_i-\mu)^2}{2\sigma^2}} \times e^{-\frac{(\mu-\mu_0)^2}{2\sigma_0{}^2}}$$

$$\pi(\mu/Z = (z_1, \dots, z_m)) \propto e^{-\sum_{i=1}^{m} \frac{(z_i-\mu)^2}{2\sigma^2}} e^{-\frac{(\mu-\mu_0)^2}{2\sigma_0{}^2}} \qquad (26)$$

$$\pi(\mu/Z = (z_1, \dots, z_m)) \propto e^{-\frac{(\mu-\mu_{0m})^2}{2\sigma_{0m}{}^2}} \qquad (27)$$

with :

$$\mu_{0m} = \frac{\mu_0 + m \times \varepsilon \times \bar{Z}}{1 + m \times \varepsilon} \qquad \sigma_{0m}^2 = \frac{\sigma_0^2}{1 + m \times \varepsilon}$$

where

$$\bar{Z} = \frac{1}{m} \sum_{l=1}^{m} z_i \qquad \varepsilon = \frac{\sigma_0^2}{\sigma^2}$$

The formula (27) shows that the posterior law of μ is a gaussian law $\mathcal{N}(\mu_{0m}, \sigma_{0m})$.

Calculation of the Bayesian estimator $\hat{\mu}_{Bay}$ and $\hat{\lambda}_{Bay}$:

❖ **The Bayesian estimator $\hat{\lambda}_{Bay}$ of the parameter λ :**

The Bayesian estimator $\hat{\lambda}_{Bay}$ is given by $\hat{\lambda}_{Bay} = E(\lambda/N)$.

The result (25) shows that the posterior law of λ is a $\Gamma(a_l, b_l)$ distribution with $(a_l, b_l) = \left(a + \sum_{j=1}^{l} n_j, \frac{b}{1+b\times l}\right)$. Consequently, the estimator $\hat{\lambda}_{Bay}$ is the mathematical expectation of the posterior law of λ:

$$\hat{\lambda}_{Bay} = a_l \times b_l = \left(a + \sum_{j=1}^{l} n_j\right) \times \frac{b}{1 + b \times l}$$

$$= \frac{a \times b + b \times l \times \left(\frac{\sum_{j=1}^{l} n_j}{l}\right)}{1 + b \times l} = \frac{\lambda_0 + b \times l \times \left(\frac{\sum_{j=1}^{l} n_j}{l}\right)}{1 + b \times l}$$
$$= \frac{\lambda_0 + b \times l \times (\bar{N})}{1 + b \times l}$$

$$\hat{\lambda}_{Bay} = \varepsilon_0 \times \lambda_0 + (1 - \varepsilon_0) \times \bar{N} = \varepsilon_0 \times \lambda_0 + (1 - \varepsilon_0) \times \lambda_{observed} \qquad (28)$$

with $\varepsilon_0 = \frac{1}{1+b\times l}, \lambda_{observed} = \bar{N} = \frac{\sum_{j=1}^{l} n_j}{l}$ *and* $\lambda_0 = E(\lambda)$. The parameter λ_0 is estimated by the experts.

❖ **The Bayesian estimator $\hat{\mu}_{Bay}$ of the parameter μ:**

The Bayesian estimator $\hat{\mu}_{Bay}$ is given by $\hat{\mu}_{Bay} = E(\mu/(\sigma; X_1, \dots, X_m)) = E(\mu/(\sigma; x_1, \dots, x_m))$.

with σ is a constant and $x_1, \dots, x_m$ are realizations of the random variables $X_1, \dots, X_m$.

The result (27), shows that the posterior law of μ is a gaussian distribution $\mathcal{N}(\mu_{0m}, \sigma_{0m})$. Consequently, the estimator $\hat{\mu}_{Bay}$ is the mathematical expectation of the posterior law of μ. Thus:

$$\hat{\mu}_{Bay} = \mu_{0m} = \frac{\mu_0 + m \times \varepsilon \times \bar{Z}}{1 + m \times \varepsilon}$$

which can be written:

$$\hat{\mu}_{Bay} = \varepsilon_2 \times \mu_0 + (1 - \varepsilon_2) \times \bar{Z} = \varepsilon_2 \times \mu_0 + (1 - \varepsilon_2) \times \mu_{observed} \quad (29)$$

With

$$\varepsilon_2 = \frac{1}{1 + m \times \varepsilon} \quad \varepsilon = \frac{\sigma_0^2}{\sigma^2} \quad \bar{Z} = \frac{1}{m}\sum_{l=1}^{m} z_i = \mu_{observed} \quad z_i = ln(x_i) \quad \mu_0 = E(\mu)$$

The parameter μ_0 is estimated by the experts. Consequently, the parameters of the LogNormal law used in the simulation are $\hat{\mu}_{Bay}$ and σ.

Calculation of the Capital at Risk (VaR)

Definition of the capital at operational risk

We consider the aggregated loss $P_N = \sum_{i=1}^{N} X_i$ in a given horizon T. We fix the level of confidence $1 - \alpha = 99.9\%$.

The requirement of capital to cover the operational risk is measured by the Value At Risk (VaR). The VaR is the quantile of order $1 - \alpha$ of the aggregated loss P_N defined by:

$$F_{P_N}(VaR) = F_{\sum_{i=1}^{N} X_i}(VaR) = P(P_N \le VaR) = 1 - \alpha \quad (30)$$

Where F_{P_N} is the cumulative distribution function of P_N. The VaR is given by:

$$VaR = F_{P_N}^{-1}(1 - \alpha) \quad (31)$$

Analytical calculation of the capital at operational risk

The formula (31) can be written by using the conditional probability as follows:

$$P(P_N \le VaR) = P\left(\sum_{i=1}^{N} X_i \le VaR\right) = \sum_{i=1}^{n} P\left(\sum_{i=1}^{n} X_i \le VaR\right) \times P(N = n) \quad (32)$$

We pose $P_n = \sum_{i=1}^{n} X_i$. Thus :

$$P\left(\sum_{i=1}^{N} X_i \le VaR\right) = \sum_{i=1}^{n} P(P_n \le VaR) \times P(N = n) \quad (33)$$

$$P\left(\sum_{i=1}^{N} X_i \le VaR\right) = \sum_{i=1}^{n} F_{P_n}(VaR) \times P(N = n) \quad (34)$$

Where F_{P_n} is the cumulative distribution function of the variable P_n.

Let denote by f the probability density function of the aggregated loss $P_n = \sum_{i=1}^{n} X_i$ (sum of n random variables) and by f_i the probability density function of X_i. The probability density of function f is a convolution product of the functions f_i, $1 \le i \le n$:

$$f(y) = f_1 * f_2 * \cdots * f_n(y) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_1\left(y - \sum_{i=2}^{n} x_i\right) f_2(x_2) \cdots f_n(x_n) dx_2 \cdots dx_n$$

We deduct the analytical formula of the calculation of the VaR :

$$F_{P_n}(VaR) = \int_{-\infty}^{VaR} f(y)dy = \int_{-\infty}^{VaR} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_1\left(y - \sum_{i=2}^{n} x_i\right) f_2(x_2) \cdots f_n(x_n) dx_2 \cdots dx_n dy$$

Due to the complexity of the cumulative distribution function of the random variable P_n, the analytical calculation of the VaR is impossible and consequently the recourse to numerical methods or to simulation techniques is indispensable.

In our study, we use the Monte Carlo simulation for the calculation of the VaR.

The Monte Carlo simulation of the Capital At Risk

To simulate the losses, we use the appropriate estimator. For the classical LDA approach, we use the maximum likelihood estimator $\hat{\lambda}$ of the parameter of the Poisson law, $(\hat{a}, \hat{b})$ of the parameters of the Negative Binomial law and $(\hat{\alpha}_i, \hat{\beta}_i)$ of the parameters of the LogNormal law (or the Weibull law). For the pure Bayesian approach we use the Bayesian estimators $(\hat{\lambda}_{Bay}, \hat{\mu}_{Bay} \hat{\sigma}_{Bay})$.

Monte Carlo simulation by the inverse cumulative distribution function

The Monte Carlo Method consists of simulating an important sample of realizations p_j of size $J = 100000$ in the following manner: for $1 \leq j \leq J$

– Simulate a realization n_j of the frequency N from the law of frequency chosen ($P(\lambda)$ or $BN(a, b)$)

– Simulate n_j realizations x_i, $1 \leq i \leq n_j$, of the severity X, from the law of severity chosen ($LN(\alpha, \beta)$ or $Wei(\alpha, \beta)$)

– Calculate $p_j = \sum_{i=1}^{n_j} x_i$ which constitutes a realization of the loss $P_N = \sum_{i=1}^{N} X_i$.

In the following paragraph, we present the used algorithms to simulate the law $P(\lambda)$ or $BN(a, b)$ of the frequency and the law $LN(\alpha, \beta)$ or $Wei(\alpha, \beta)$ of the severity.

Before presenting the simulation by Monte Carlo method, we cite firstly the theorem of the inverse cumulative function that allows the simulation of the continuous random variables.

Theorem: If U is uniform random variable on the interval $[0,1]$ and F a cumulative distribution function continuous and strictly increasing. Let Y be the random variable defined from the inverse cumulative distribution function F^{-1} by $Y = F^{-1}(U)$. Then the cumulative distribution function of Y is F.

Consequently, for simulating a realization y_i of the random variable Y which has F as a cumulative distribution function, it suffices to:

– Simulate a realization u_i of the Uniform distribution $U[0,1]$.

– Calculate the inverse cumulative distribution function $y_i = F^{-1}(u_i)$. Then y_i is considered as a realization of Y.

Simulation of the realizations n_j for $1 \leq j \leq 100000$

To simulate the realizations of the frequency N, we use the Poisson distribution $P(\lambda)$ or the gamma distribution $\Gamma(a, b)$.

❖ **Simulation the Poisson distribution**

***Propriety*:** Let $(V_i)_{i\geq 1}$ be a sequence of exponential random variables of parameter λ. Then the random variable defined by:

$M = Sup\{k \in \mathbb{N}^*/\sum_{i=1}^{k} V_i\}$ and $M = 0$ if $V_1 > 1$

is a Poisson random variable of parameter λ.

To simulate the realizations of the Poisson's law of the parameter λ we use the following algorithm.

Step 1 : Simulation of n_1

To simulate the realization n_1 of the frequency we proceed as follows :

1. Simulate a realization v_1 of the law $Exp(\lambda)$ by the inverse cumulative distribution function. For that we must :

– Simulate a realization u_1 of the Uniform law $U[0,1]$.

– The cumulative distribution function of the exponential law $Exp(\lambda)$ is defined by $F^{-1}(u) = -\frac{ln(1-u)}{\lambda}$. We deduct $v_1 = F^{-1}(u_1) = -\frac{ln(1-u_1)}{\lambda}$

2. If $v_1 > 1$ then $n_1 = 0$.

If not, simulate a second realization v_2 of the exponential law $Exp(\lambda)$ according to the procedure 1. If $v_1 + v_2 > 1$ then $n_1 = 1$ is a realization of Poisson of the parameter λ, otherwise, simulate the k realizations v_i, $1 \leq i \leq k$ until that $\sum_{i=1}^{k} v_i \leq 1$ and $\sum_{i=1}^{k+1} V_i > 1$. The value k that verify the last two inequalities is the realization $n_1 = k$ of the frequency.

Step j : Simulation of $n_j, 2 \leq j \leq 100000$

We redo 100000 times the step 1. We obtain thus 100000 realization n_j.

❖ **Simulation of the Negative binomial law $BN(a, b)$**

The $BN(a, b)$ distribution is in reality the Poisson-gamma mixture of parameters (a, b). Indeed, the negative binomial distribution can be viewed as a Poisson distribution where the Poisson parameter λ is a random variable, distributed according to a Gamma distribution of parameters (a, b). To simulate the realizations of the law $BN(a, b)$ we perform the following steps:

Step 1: Simulation of n_1

1. Simulate a realization λ_1 of the variable λ distributed according to $\Gamma(a, b)$ whose the cumulative distribution function is noted $\Gamma_{(a,b)}(.)$:

To simulate a realization λ_1, we proceed thus as:

- Simulate a realization $u_1$of the Uniform distribution $U[0,1]$.

- Calculate $\lambda_1 = \Gamma_{(a,b)}^{-1}(u_1)$.

2. Simulate a realization n_1 of the Poisson's distribution $P(\lambda_1)$. We use the algorithm defined above for the simulation of Poisson's law with the inverse cumulative distribution function method of the exponential law. The realization n_1 is a realization of the law $BN(a, b)$).

Simulation of $n_j, 2 \leq j \leq 100\,000$

We simulate the 100000 realizations n_j of the law $BN(a, b)$ by repeating 100000 times the step 1.

Simulation of the losses

❖ **Simulation of the laws $LN(\alpha, \beta)$ and $Wei(\alpha, \beta)$**

To simulate the laws $LN(\alpha, \beta)$ and $Wei(\alpha, \beta)$ we use the inverse cumulative distribution function method as follows:

- Simulation of the law $Wei(\alpha, \beta)$:

1. Simulate a realization u_i of the Uniform law $U[0,1]$.

2. Calculate $x_i = Wei^{-1}_{(\alpha,\beta)}(u_i)$ where $Wei_{(\alpha,\beta)}$ is a cumulative distribution function of law $Wei(\alpha,\beta)$ defined by $Wei_{(\alpha,\beta)}(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$. We deduct $x_i = \beta \times \left(-ln(1-u_i)\right)^{1/\alpha}$.

- Simulation of the law $LN(\alpha,\beta)$:

1. Simulate a realization u_i of the Uniform law $U[0,1]$.

2. Calculate $x_i = F^{-1}_{(\alpha,\beta)}(u_i)$ where $F_{(\alpha,\beta)}$ is a cumulative distribution function of the law $LN(\alpha,\beta)$. As $F^{-1}_{(\alpha,\beta)}(u_i)$ has not analytical expression we simulate numerically x_i.

❖ **Simulation of the realizations of the law defined by the cumulative function $F_{\alpha,\beta,x_s,a}$**

To simulate the realizations of $F_{\alpha,\beta,x_s,a}$ we use the method of inverse cumulative distribution function:

1. Simulate a realization u_i of the Uniform law $U[0,1]$.

2. It exists a_i that $\frac{a_{i-1}}{a_q} \le u_i \le \frac{a_i}{a_q}$. The realization x_i satisfies

$$F_{\alpha,\beta,x_s,a}(x_i) = u_i \Leftrightarrow K_{\alpha_i,\beta_i}(x_i) \times \frac{a_i - a_{i-1}}{a_q} + \frac{a_{i-1}}{a_q} = u_i$$

$$\Leftrightarrow K_{\alpha_i,\beta_i}(x_i) = \left(u_i - \frac{a_{i-1}}{a_q}\right)\frac{a_q}{a_i - a_{i-1}}$$

We deduct $x_i = K^{-1}_{\alpha_i,\beta_i}\left(\left(u_i - \frac{a_{i-1}}{a_q}\right)\frac{a_q}{a_i - a_{i-1}}\right)$.

Calculation of the Capital at operational risk

The capital at operational risk is calculated by the determination of the percentile 99.9% of the empirical distribution of the losses $p_j = \sum_{i=1}^{n_j} x_i$, for $1 \le j \le 100000$, simulated by Monte Carlo.

Let F_P be the empirical cumulative distribution function of the loss P determined from the simulated realizations p_j. The function F_P is given by :

$$F_P(y) = \frac{numbre\ of\ p_j \le y}{number\ of\ p_j} \qquad (34)$$

The value at risk VaR is expressed by the formula:

$$VaR = Inf\{y/F_P(y) \ge 99.9\%\}$$
$$= Inf\left\{y/\frac{number\ of\ p_j \le y}{number\ of\ p_j} \ge 99.9\%\right\}$$

In this paper, the modelling of the frequency is made for a horizon of one year $T = 12\ month$ or by dividing the year T into k sub-horizons $T_k = \frac{T}{k}$ for k an integer $2 \le k \le 12$.

The annual VaR for the global database without segmentation

❖ **The modelling of the Loss frequency for an annual horizon**

The horizon chosen is a year $T = 12\ month$ and the level of confidence is $1 - \alpha = 99.9\%$.

Let $F_{P_{annual}}$ be the empirical cumulative distribution function of the annual loss P defined by the formula (34) and determined from the simulated realizations $p_j = \sum_{i=1}^{n_j} x_i$ where n_j is a realization of the annual frequency:

$$F_{P_{annual}}(y) = \frac{number\ of\ p_j \le y}{number\ of\ p_j}$$

The capital at operational risk VaR is:

$$VaR = Inf\{y/F_{P_{annual}}(y) \ge 99.9\%\} \qquad (35)$$

$$VaR = Inf\left\{y/\frac{numbre\ of\ p_j \le y}{numbre\ of\ p_j} \ge 99.9\%\right\} \qquad (36)$$

❖ **The modelling of the Losses frequency for the sub-horizon $T_k = \frac{T}{k}$**

Let F_{PT} be the empirical distribution function of the operational loss P on the horizon T_k defined by the formula (34) and determined from the simulated realization p_j where n_j is a realization of the frequency on the sub-horizon T_k.

The cumulative distribution function F_{PT} is simulated k times on the horizon T, let F_{PT_i} be the ith simulation. The capital at operational risk VaR on a horizon of one year (on the horizon $T = 12\ month$) is the sum of $VaR_i, 1 \le i \le k$, with VaR_i is the ith capital at risk, determined from the ith simulation of the losses:

$$VaR_i = Inf\{y/F_{PT_i}(y) \ge 99.9\%\} \qquad (37)$$

We have then:

$$VaR = \sum_{i=1}^{k} VaR_i \qquad (38)$$

The annual VaR with segmentation of the database by risk category

The second approach consists in segmenting of the database by risk category RT_c, $1 \le c \le 7$. The operational loss P_c of risk category RT_c is a random variable defined by $P_c = \sum_{i=1}^{N_c} X_{ci}$. With :

– N_c : the random variable that represents the frequency of losses of the risk category RT_c

– X_{ci} : the random variable, for $1 \le i \le N_c$, that represents the severity of the losses of the risk category RT_c

Let n_{jc}, $1 \le j \le 100000$, the annual frequency of the losses collected for the risk category RT_c and let x_{ci} be the simulated realizations of the losses of the risk category RT_c.

The realizations $p_{jc} = \sum_{i=1}^{n_{jc}} x_{ci}$, $1 \le j \le 100000$ permit to calculate the capital at risk VaR_c for each risk category RT_c. The annual VaR is the sum of the VaR_c because it is supposed that the risk categories are independent.The modelling of the frequency of the loss is made for a horizon of one year $T = 12\,month$ or by dividing the year T into k sub-horizons $T_k = \frac{T}{k}$ for k an integer $2 \leq k \leq 12$.

❖ **The modelling of the Loss frequency for an annual horizon**

The horizon chosen is a year $T = 12\,month$ and the level of confidence is $1 - \alpha = 99.9\%$.

The empirical cumulative distribution function F_{P_c} of the losses for the risk category RT_c is defined by:

$$F_{P_c}(y) = \frac{number\ of\ p_{jc} \leq y}{number\ of\ p_{jc}} \quad (39)$$

The capital of operational risk for the category risk RT_c is:

$$VaR_c = Inf\{y / F_{P_c}(y) \geq 99.9\%\} \quad (40)$$

The capital at risk on the annual horizon is the sum of the VaR_c:

$$VaR = \sum_{c=1}^{7} VaR_c \quad (41)$$

❖ **The modelling of the Losses frequency for the sub-horizon $T_k = \frac{T}{k}$, $2 \leq k \leq 12$:**

Let F_{PT_c} be the empirical cumulative distribution function of the operational risk of a given risk category RT_c for the horizon $T = 1\,year$ defined by the formula (44) and determined from the simulate realizations p_{jc} with n_{jc} is a realization of the frequency of losses on the horizon T.

The cumulative distribution function F_{PT_c} is simulated k times on the horizon T. Let $F_{PT_{ci}}$ be the ith simulation. The capital at operational risk VaR_c on an annual horizon is the sum of the $VaR_{ci}, 1 \leq i \leq k$, VaR_{ci} is the ith capital at operational risk determined from the ith simulation of the losses.

The capital at risk on the annual horizon is the sum of the $VaR_{ci},\ 1 \leq i \leq k$:

$$VaR_c = \sum_{i=1}^{k} VaR_{ci} \quad (42)$$

The capital at risk on the annual horizon is the sum of the VaR_c :

$$VaR = \sum_{c=1}^{7} \sum_{i=1}^{k} VaR_{ci} \quad (43)$$

EMPIRICAL RESULTS

Description of database

We present the global data and their repartition by risk category.

Description of the global database

In this study, we use a database of the incidents of losses of a Moroccan Bank. The data are constituted from losses collected by the bank since the 80s along with the reports and missions of the audit relating to the internal and external fraud. However, it should be noted that since 2007, a device of collecting of the operational losses has been set up to constitute the database. This device permits the declaration of the incidents, assess their impact and the classification of the losses by Basel category.

The database is composed of 3547 losses and the number of the distinct amounts of losses is 3296. The statistical of the distinct amounts of losses are summarized in the table below:

Table 5 : Statistics of losses (amount)

Mean	Standard deviation	Skewness	Kurtosis
2 466 422.60	86 035 957.13	53.13	2 935.78

Description of the data by risk category of Basel

The distribution of data by risk category of Basel shows that the losses of the category "RT_1 : Execution, delivery and process management" represents 38% of the collected losses, followed in the second position by "RT_3 : Damage to physical assets" that represented 24%. In the third position, we finds the category "RT_6 : External Fraud" with 15% followed by the category "RT_7 : Internal Fraud" with 10% and the other categories represent 13%. The statistical characteristics of the losses amounts by risk category are summarized in the following table:

Table 6 : The statistical characteristics of the losses by risk category (amounts)

Risk category	% of the losses	Mean	St. deviation	Skewness	Kurtosis
RT_1	38%	95 333.67	448 986.93	9.056	99.80
RT_2	7%	1 197 938.91	1 872 9780.54	15.94	251.97
RT_3	24%	43 992.56	405 500.62	18.38	379.29
RT_4	5%	184 359.36	816 593.78	6.80	49.71
RT_5	1%	297 863.36	715 487.64	5.94	33.84
RT_6	15%	164 517.37	1 235 314.13	12.33	175.75
RT_7	10%	2 671 271.03	20 380 162.30	15.84	271.39

The description of the data shows that the severity distribution cannot be modelled by a Gaussian distribution.

To determine the frequency of the losses we segment the database by a given time horizon. Indeed, we proceed as follows:

- ***Frequency of losses of the global database:***

For the global database, we have fixed a semi-annual time horizon, hence the horizon of modelling is $T_2 = \frac{12\ mois}{2}$. The empirical frequency of occurrence is a number of incidents that occurs during each semester of the period of collection. The statistical characteristics of the frequency of losses for a semi-annual horizon are summarized in the following table:

Table 7 : Statistical characteristics of the frequency of losses for a semiannual horizon (global database)

Mean	Standard deviation
185.66	50.13

- ***Frequency of the losses by risk category RT_c, $1 \leq c \leq 7$:***

The choice of the horizon for each risk category is made on the basis of the data available for modelling, which must be superior to 30 observations. Indeed, the horizon chosen for each risk category is summarized by the following table:

Table 8 : Choice of horizon for each risk category

Category	RT_1	RT_2	RT_3	RT_4	RT_5	RT_6	RT_7
Horizon	Monthly (T_{12})	Monthly (T_{12})	Semi-annual (T_2)	Monthly (T_{12})	Monthly (T_{12})	Monthly (T_{12})	Annual (T)

The statistical characteristics of the frequency of losses by the risk category, on the horizon defined by the Table 8, are given by the following table:

Table 9: The statistical characteristics of the frequency of losses by the risk category

Risk category	Mean	St. deviation
RT_1	25.240	28.062
RT_2	6.250	6.194
RT_3	68.44	12.277
RT_4	1.139	3.559
RT_5	2.122	3.587
RT_6	22.326	15.377
RT_7	23.682	13.162

The data description shows that the frequency distributions are not equi-distributed because the mean and the variance are different.

The classical LDA approach

In this paragraph we present the estimated parameters of the severity distribution and of the frequency distribution and the results of the adjustment tests.

Modelling of the losses based on the global data

For modelling the global data of the losses, we have tested the adjustment of the amounts of the losses with the LogNormal and Weibull laws of parameters (α, β) and the adjustment of the frequency on the semi-annual horizon (T_2) with the Poisson law of parameter λ and Negative Binomial law of parameters (a, b).

Parameters estimation

The estimation of the parameters (α, β) of the laws $Wei(\alpha, \beta)$ and $LN(\alpha, \beta)$, and of the parameters λ and (a, b) of the laws $P(\lambda\ )$ and $BN(a, b)$ is made by the maximum likelihood method whose the results are summarized in the following table :

Table 10 : Estimation of the parameters (α, β) of the $Wei(\alpha, \beta)$ et $LN(\alpha, \beta)$, and of the parameters λ et (a, b) of the laws $P(\lambda\ )$ and $BN(a, b)$

$Wei(\alpha, \beta)$		$LN(\alpha, \beta)$		$\mathcal{P}(\lambda)$	$BN(a, b)$	
$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$
0.327	27 346.921	8.826	2.702	185.66	0.062	148.963

Adjustment of theoretical laws of the severity and of the frequency with the observed data

– **Adjustment of the laws $Wei(\alpha, \beta)$ and $LN(\alpha, \beta)$ with the observed data of the severity:**

The goodness-of-fit test of Kolmogorov Smirnov (KS) has rejected the adjustment of the data with the two laws $Wei(0.327; 27346.921)$ and $LN(8.826; 2.702)$ because the ks statistics of test of the two laws respectively 0.10 and 0.029 are superior to the critical value $vc = 0.023$.

The cumulative distribution function by slice $F_{\alpha,\beta,x_s,a}$ $\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i, \beta_i)\right)$ and $F_{\alpha,\beta,x_s,a}$ $\left(K_{\alpha_i,\beta_i} = LN(\alpha_i, \beta_i)\right)$ based respectively on the laws Weibull $Wei(\alpha_i, \beta_i)$ and LogNormale $LN(\alpha_i, \beta_i)$ which adjusts with the amounts of the losses are defined by the following estimated parameters:

Table 11: Estimation of the parameters $\hat{\alpha}_i$, $\hat{\beta}_i$, a_i for $1 \le i \le 20$ of $F_{\alpha,\beta,x_s,a}$ $\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i,\beta_i)\right)$

i	1	2	3	4	5	6	7	8	9	10
$\hat{\alpha}_i$	1.53	4.41	109.58	6.14	91.39	6.71	20.59	7.87	8	10.76
$\hat{\beta}_i$	115.28	430.13	555.78	927.6	1 121.91	1 623.47	2 265.25	3 338.28	5 537.85	8 062.72
a_i	312	552	620	867	933	1144	1239	1437	1682	1848
i	11	12	13	14	15	16	17	18	19	20
$\hat{\alpha}_i$	9.31	6.58	6.99	5.45	16.32	4.62	3.63	2.22	4.05	0.53
$\hat{\beta}_i$	11 682.33	201 48.7	35 343.54	67 234.96	100 073.49	185 600.51	465 237.42	1 834 339.39	6 290 724.03	102 002 253.6
a_i	2 032	2 320	2 536	2 715	2 799	3 014	3 143	3 237	3 267	3 297

Table 12: Estimation of the parameters $\hat{\alpha}_i$, $\hat{\beta}_i$, a_i for $1 \le i \le 17$ of $F_{\alpha,\beta,x_s,a}$ $\left(K_{\alpha_i,\beta_i} = LN(\alpha_i,\beta_i)\right)$

i	1	2	3	4	5	6	7	8	9
$\hat{\alpha}_i$	2.55	3.96	5.01	5.83	6.29	6.29	6.99	7.07	7.63
$\hat{\beta}_i$	0.67	0.36	0.26	0.24	0.01	0.01	0.01	0.05	0.26
a_i	35	157	300	510	581	817	904	982	1350
i	10	11	12	13	14	15	16	17	
$\hat{\alpha}_i$	8.5	9.11	9.64	10.27	11.35	12.37	13.86	16.86	
$\hat{\beta}_i$	0.23	0.13	0.16	0.22	0.37	0.26	0.71	1.52	
a_i	1 740	1 960	2 215	2 540	2 900	3 080	3 248	3 297	

– ***Adjustment of the laws $P(\lambda)$ and $BN(a,b)$ with the observed data of the frequency:***

The chi-square test has rejected the adjustment of the two theoretical laws with the observed frequency, because the observed values of χ^2 (5924.67 for $P(185.66)$ and 49.164 for $BN(0.062; 148.963)$) are superior to the critical values respectively $vc = 3.841$ and $vc = 14.067$.

Data does not fit with the two candidate distributions. Consequently, the simulation of the loss frequency will be done by the bootstrap technique.

The modelling of the losses based on the segmentation of data by risk category

In this paragraph, we determine the theoretical distribution of the severity of losses and of the frequency of losses that adjust with the observed data RT_c, $1 \le c \le 7$.

For each risk category, we test the adjustment of the amounts of the losses with the theoretical laws Weibull and LogNormale of parameters (α, β) and the adjustment of the frequency on the appropriate horizon T_k, as presented by Table 8, with the law of Poisson of parameter λ and the negative binomial law of parameters (a, b).

Risk category "Execution, delivery and process management: RT_1"

➢ **Parameters estimation**

The estimation of the parameters (α, β) of the laws $Wei(\alpha, \beta)$ and $LN(\alpha, \beta)$, and of the parameters λ and (a, b) of the laws (λ) and $BN(a, b)$ is made by the maximum likelihood method whose the result are summarized in the following table:

Table 13: Estimation of the parameters (α, β) of the $Wei(\alpha, \beta)$ and $LN(\alpha, \beta)$, and of the parameters λ and (a, b) of the laws $P(\lambda)$ and $BN(a, b)$

$Wei(\alpha,\beta)$		$LN(\alpha,\beta)$		$P(\lambda)$	$BN(a,b)$	
$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$
0.17	14 623.71	7.944	2.775	25.24	0.053	7.583

➢ **Adjustment of theoretical laws of the severity and of the frequency with the observed data**

– ***Adjustment of the laws $Wei(\alpha,\beta)$ and $LN(\alpha,\beta)$ with the observed data of the severity:***

The goodness-of-fit test of Kolmogorov Smirnov (KS) has rejected the adjustment of the data with the two laws $Wei(0.17; 14\,623.71)$ and $LN(7.944; 2.775)$ because the ks statistics of test of two laws respectively 0.515 and 0.059 are superior to the critical value $vc = 0.036$.

The cumulative distribution function by slice $F_{\alpha,\beta,x_s,a}$ $\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i,\beta_i)\right)$ and $F_{\alpha,\beta,x_s,a}$ $\left(K_{\alpha_i,\beta_i} = LN(\alpha_i,\beta_i)\right)$ based respectively on the laws Weibull $Wei(\alpha_i,\beta_i)$ and LogNormale $LN(\alpha_i,\beta_i)$ that adjusts with the amounts of the losses are defined by the following estimated parameters:

Table 14: Estimation of the parameters $\hat{\alpha}_i$, $\hat{\beta}_i$, a_i for $1 \le i \le 10$ of $F_{\alpha,\beta,x_s,a}$ $\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i,\beta_i)\right)$

i	1	2	3	4	5	6	7	8	9	10
$\hat{\alpha}_i$	1.28	5.39	6.46	3.79	3.27	3.21	2.59	2.08	3.54	0.42
$\hat{\beta}_i$	162.64	600.68	1 167.05	2 680.4	8544.49	27 101.76	104 219.68	422 100.82	1 761 359.22	136 042 489
a_i	344	467	622	781	977	1 097	1 233	1 318	1 341	1 361

Table 15: Estimation of the parameters $\hat{\alpha}_i$, $\hat{\beta}_i$, a_i for $1 \le i \le 7$ of $F_{\alpha,\beta,x_s,a}$ $\left(K_{\alpha_i,\beta_i} = LN(\alpha_i,\beta_i)\right)$

i	1	2	3	4	5	6	7
$\hat{\alpha}_i$	3.99	5.57	6.5	7.51	9.08	16.06	11.66
$\hat{\beta}_i$	0.76	0.28	0.3	0.37	0.49	2.07	1.05
a_i	204	350	540	774	1 040	1 331	1 361

– **Adjustment of the laws $P(\lambda)$ and $BN(a,b)$ with the observed data of the frequency:**

The chi-square test has rejected the adjustment of the two theoretical laws with the observed frequency on a horizon T_{12}, because the observed values of χ^2 (52 606 657 516.47 for $P(25.240)$ and 53.113 for $BN(0.053; 7.583)$) are superior to the critical values respectively $vc = 9.488$ and $vc = 14.067$.

Data does not fit with the two candidate distributions. Consequently, the simulation of the loss frequency will be done by the bootstrap technique.

Risk category "Business disruption and system failures: RT_2"

➢ **Parameters estimation**

The estimation of the parameters (α,β) of the laws $Wei(\alpha,\beta)$ and $LN(\alpha,\beta)$, and of the parameters λ and (a,b) of the laws (λ) and $BN(a,b)$ is made by the maximum likelihood method whose the result are summarized in the following table:

Table 16 : Estimation of the parameters (α,β) of the $Wei(\alpha,\beta)$ and $LN(\alpha,\beta)$, and of the parameters λ and (a,b) of the laws $P(\lambda)$ and $BN(a,b)$

$Wei(\alpha,\beta)$		$LN(\alpha,\beta)$		$P(\lambda)$	$BN(a,b)$	
$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$
0.30	3 688.27	7.691	2.062	6.250	1.53	4.06

➢ **Adjustment of theoretical laws of the severity and of the frequency with the observed data**

– ***Adjustment of the laws $Wei(\alpha,\beta)$ and $LN(\alpha,\beta)$ with the observed data of the severity:***

The goodness-of-fit test of Kolmogorov Smirnov (KS) has accepted the adjustment of the data with the law $LN(7.691; 2.062)$ ($ks = 0.083$ is inferior to critical value $vc = 0.084$). However, the test has rejected the adjustment of the data with the law $Wei(0.30, 3688.27)$ ($ks = 0.257$ is superior to critical value $vc = 0.084$).

The cumulative distribution function by slice $F_{\alpha,\beta,x_s,a}$ $\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i,\beta_i)\right)$ based on the law Weibull $Wei(\alpha_i,\beta_i)$ that adjusts with the amounts of the losses are defined by the following estimated parameters:

Table 17: Estimation of the parameters $\hat{\alpha}_i$, $\hat{\beta}_i$, a_i for $1 \le i \le 2$ of $F_{\alpha,\beta,x_s,a}$ $\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i,\beta_i)\right)$

i	1	2
$\hat{\alpha}_i$	0.89	0.67
$\hat{\beta}_i$	2 259.58	141 646.74
a_i	219	257

– ***Adjustment of the laws $P(\lambda)$ and $BN(a,b)$ with the observed data of the frequency:***

The chi-square test has accepted the adjustment of the monthly frequency with the law $BN(1.53; 4.06)$ ($\chi^2_{observed} = 11.234$ $is\ inferior$ to the critical value $vc = 14.067$).

However, the test has rejected the adjustment of the Poisson law $P(6.250)$ with the observed frequency to theoretical law with the observed frequency ($\chi^2_{observed} = 258\ 396\ 650.62\ is\ superior\ than\ vc = 15507$)

The monthly frequency of the losses will be simulated by the law $BN(1.53; 4.06)$.

Risk category "Damage to physical assets: RT_3"

➢ **Parameters estimation**

The estimation of the parameters (α, β) of the laws $Wei(\alpha, \beta)$ and $LN(\alpha, \beta)$, and of the parameters λ and (a, b) of the laws $P(\lambda)$ and $BN(a, b)$ is made by the maximum likelihood method whose the result are summarized in the following table:

Table 18 : Estimation of the parameters (α, β) of the $Wei(\alpha, \beta)$ and $LN(\alpha, \beta)$, and of the parameters λ et (a, b) of the laws $P(\lambda)$ and $BN(a, b)$

$Wei(\alpha,\beta)$		$LN(\alpha,\beta)$		$P(\lambda)$	$BN(a,b)$	
$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$
0.25	2 207.72	8.97	1.39	68.448	0.821	83.351

➢ **Adjustment of theoretical laws of the severity and of the frequency with the observed data**

– ***Adjustment of the laws $Wei(\alpha, \beta)$ and $LN(\alpha, \beta)$ with the observed data of the severity:***

The goodness-of-fit test of Kolmogorov Smirnov (KS) has accepted the adjustment of the data with the law $LN(8.97; 1.39)$ ($ks = 0.041$ is inferior to critical value $vc = 0.046$). However, the test has rejected the adjustment of the data with the law $Wei(0.25, 2\ 207.72)$ ($ks = 0.497$ is superior to critical value $vc = 0.046$).

The cumulative distribution function by slice $F_{\alpha,\beta,x_s,a}$ $\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i, \beta_i)\right)$ based on the laws Weibull $Wei(\alpha_i, \beta_i)$ that adjusts with the amounts of the losses are defined by the following estimated parameters:

Table 19: Estimation of the parameters $\hat{\alpha}_i, \hat{\beta}_i$, a_i for $1 \leq i \leq 5$ of $F_{\alpha,\beta,x_s,a}$ $\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i, \beta_i)\right)$

i	1	2	3	4	5
$\hat{\alpha}_i$	1.4	4.73	6.73	8.59	0.64
$\hat{\beta}_i$	5 530.09	22 046.7	40 463.74	68 165.26	78 0623.93
a_i	550	738	822	840	866

– ***Adjustment of the laws $P(\lambda)$ and $BN(a, b)$ with the observed data of the frequency:***

The chi-square test has accepted the adjustment of the monthly frequency with the law $BN(0.821; 83.351)$ ($\chi^2_{observed} = 11.652\ is\ inferior$ to the critical value $vc = 14.067$).

However, the test has rejected the adjustment of the Poisson law $P(68.448)$ with the observed frequency to theoretical laws with the observed frequency ($\chi^2_{observed} = 10\ 611\ 160\ 304.04\ is\ superior\ to\ vc = 9.488$)

The semi-annual frequency of the losses will be simulated by the law $BN(0.821; 83.351)$.

Risk category "Clients, products and business practices: RT_4"

➢ **Parameters estimation**

The estimation of the parameters (α, β) of the laws $Wei(\alpha, \beta)$ and $LN(\alpha, \beta)$, and of the parameters λ and (a, b) of the laws (λ) and $BN(a, b)$ is made by the maximum likelihood method whose the result are summarized in the following table:

Table 20 : estimation of the parameters (α, β) of the $Wei(\alpha, \beta)$ and $LN(\alpha, \beta)$, and of the parameters λ and (a, b) of the laws $P(\lambda)$ and $BN(a, b)$

$Wei(\alpha,\beta)$		$LN(\alpha,\beta)$		$P(\lambda)$	$BN(a,b)$	
$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$
0.416	37 078.55	9.335	2.383	1.139	1.342	0.849

➢ **Adjustment of the theoretical laws of the severity and of the frequency with the observed data**

– ***Adjustment of the laws $Wei(\alpha, \beta)$ and $LN(\alpha, \beta)$ with the observed data of the severity:***

The goodness-of-fit test of Kolmogorov Smirnov (KS) has accepted the adjustment of the data with the two laws $Wei(9.335; 2.383)$ and $LN(0.416; 37\ 078.55)$. The ks statistics of test of two laws respectively 0.103 and 0.048 are superior to the critical value $vc = 0.107$.

– ***Adjustment of the laws $P(\lambda)$ and $BN(a, b)$ with the observed data of the frequency:***

The chi-square test has accepted the adjustment of the monthly frequency with the law $BN(1.342; 0.849)$ ($\chi^2_{observed} = 10.917\ is\ inferior$ to the critical value $vc = 14.067$).

However, the test has rejected the adjustment of the Poisson law $P(1.139)$ with the observed frequency to theoretical laws with the observed frequency ($\chi^2_{observed} = 192.176\ is\ superior\ to\ vc = 15507$)

The monthly frequency of the losses will be simulated by the law $BN(1.342; 0.849)$.

Category risk "Employment practices and workplace safety: RT_5"

➢ **Parameters estimation**

The estimation of the parameters (α,β) of the laws $Wei(\alpha,\beta)$ and $LN(\alpha,\beta)$, and of the parameters λ and (a,b) of the laws $P(\lambda)$ and $BN(a,b)$ is made by the maximum likelihood method whose the result are summarized in the following table:

Table 21 : Estimation of the parameters (α,β) of the $Wei(\alpha,\beta)$ and $LN(\alpha,\beta)$, and of the parameters λ and (a,b) of the laws $P(\lambda)$ and $BN(a,b)$

$Wei(\alpha,\beta)$		$LN(\alpha,\beta)$		$P(\lambda)$	$BN(a,b)$	
$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$
0.905	283 031.476	12.143	0.704	2.122	2.003	1.058

➢ **Adjustment of theoretical laws of the severity and of the frequency with the observed data**

– ***Adjustment of the laws $Wei(\alpha,\beta)$ and $LN(\alpha,\beta)$ with the observed data of the severity :***

The goodness-of-fit test of Kolmogorov Smirnov (KS) has accepted the adjustment of the data with the law $LN(12.143; 0.704)$ ($ks = 0.211$ is inferior to critical value $vc = 0.217$).

However, the test has rejected the adjustment of the data with the law $Wei(0.905; 283031.476)$ ($ks = 0.287$ is superior to critical value $vc = 0.217$).

The cumulative distribution function by slice $F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i,\beta_i)\right)$ based on the laws Weibull $Wei(\alpha_i,\beta_i)$ that adjusts with the amounts of the losses are defined by the following estimated parameters:

Table 22: Estimation of the parameters $\hat{\alpha}_i$, $\hat{\beta}_i$, a_i for $1 \le i \le 2$ of $F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i,\beta_i)\right)$

i	1	2
$\hat{\alpha}_i$	3.303	0.825
$\hat{\beta}_i$	166 912.397	688 784.635
a_i	30	40

– ***Adjustment of the laws $P(\lambda)$ and $BN(a,b)$ with the observed data of the frequency:***

The chi-square test has accepted the adjustment of the monthly frequency with the law $P(2.122)$ ($\chi^2_{observed} = 11.922$ $is\ inferior$ to the critical value $vc = 12.59$).

However, the test has rejected the adjustment of the Poisson law $BN(2.732; 8.536)$ with the observed frequency to theoretical laws with the observed frequency ($\chi^2_{observed} = 19214.85$ $is\ superior\ to\ vc = 14.067$)

The monthly frequency of the losses will be simulated by the law $P(2.122)$.

Risk category "External Fraud: RT_6"

➢ **Parameters estimation**

The estimation of the parameters (α,β) of the laws $Wei(\alpha,\beta)$ and $LN(\alpha,\beta)$, and of the parameters λ and (a,b) of the laws (λ) and $BN(a,b)$ is made by the maximum likelihood method whose the result are summarized in the following table:

Table 23: Estimation of the parameters (α,β) of the $Wei(\alpha,\beta)$ and $LN(\alpha,\beta)$, and of the parameters λ and (a,b) of the laws $P(\lambda)$ and $BN(a,b)$

$Wei(\alpha,\beta)$		$LN(\alpha,\beta)$		$P(\lambda)$	$BN(a,b)$	
$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$
0.31	20 425.73	8.054	2.415	23.326	2.732	8.536

➢ **Adjustment of theoretical laws of the severity and of the frequency with the observed data**

– ***Adjustment of the laws $Wei(\alpha,\beta)$ and $LN(\alpha,\beta)$ with the observed data of the severity:***

The goodness-of-fit test of Kolmogorov Smirnov (KS) has rejected the adjustment of the data with the two laws $Wei(0.31; 20425.73)$ and $LN(8.054; 2.415)$ because the ks statistics of test of two laws respectively 0.190 and 0.208 are superior to the critical value $vc = 0.059$.

The cumulative distribution function by slice $F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i,\beta_i)\right)$ and $F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i} = LN(\alpha_i,\beta_i)\right)$ based respectively on the laws Weibull $Wei(\alpha_i,\beta_i)$ and LogNormale $LN(\alpha_i,\beta_i)$ that adjusts with the amounts of the losses are defined by the following estimated parameters:

Table 24: Estimation of the parameters $\hat{\alpha}_i$, $\hat{\beta}_i$, a_i for $1 \le i \le 9$ of $F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i,\beta_i)\right)$

i	1	2	3	4	5	6	7	8	9
$\hat{\alpha}_i$	2.78	168.48	5.16	88.22	6.38	106.95	1.15	2.88	0.64
$\hat{\beta}_i$	176.42	223.17	482.51	555.16	950.1	1 115.75	7 060.34	98 690.12	1 717 521.31
a_i	37	52	91	159	219	271	401	485	515

Table 25: Estimation of the parameters $\hat{\alpha}_i$, $\hat{\beta}_i$, a_i for $1 \le i \le 9$ of $F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i} = LN(\alpha_i,\beta_i)\right)$

i	1	2	3	4	5	6	7	8	9
$\hat{\alpha}_i$	4.93	5.68	6.3	6.56	6.99	7.33	9.71	11.65	13.57
$\hat{\beta}_i$	0.58	0.3	0.01	0.19	0.01	0.3	1.14	0.17	1.5
a_i	37	74	142	200	249	319	449	485	515

– ***Adjustment of the laws $P(\lambda)$ and $BN(a,b)$ with the observed data of the frequency:***

The chi-square test has accepted the adjustment of the monthly frequency with the law $BN(2.732; 8.536)$ ($\chi^2_{observed} = 8.300$ $is\ inferior$ to the critical value $vc = 14.067$).

However, the test has rejected the adjustment of the Poisson law $P(23.326)$ with the observed frequency to theoretical laws with the observed frequency ($\chi^2_{observed} = 4983042463{,}92$ $is\ superior\ to\ vc = 15.507$)

The monthly frequency of the losses will be simulated by the law $BN(2.732; 8.536)$.

Risk Category "Internal Fraud: RT_7"

➢ **Parameters estimation**

The estimation of the parameters (α, β) of the laws $Wei(\alpha, \beta)$ and $LN(\alpha, \beta)$, and of the parameters λ and (a, b) of the laws (λ) and $BN(a, b)$ is made by the maximum likelihood method whose the result are summarized in the following table:

Table 26 : Estimation of the parameters (α, β) of the $Wei(\alpha, \beta)$ and $LN(\alpha, \beta)$, and of the parameters λ and (a, b) of the laws $P(\lambda)$ and $BN(a, b)$

$Wei(\alpha,\beta)$		$LN(\alpha,\beta)$		$P(\lambda)$	$BN(a,b)$	
$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$
0.309	331 965.757	11.400	2.358	23.682	1.223	19.363

➢ **Adjustment of theoretical laws of the severity and of the frequency with the observed data**

– ***Adjustment of the laws $Wei(\alpha, \beta)$ and $LN(\alpha, \beta)$ with the observed data of the severity:***

The goodness-of-fit test of Kolmogorov Smirnov (KS) has accepted the adjustment of the data with the law $LN(11400{,}2.358)$ $(ks = 0.024$ is inferior to critical value $vc = 0.073)$. However, the test has rejected the adjustment of the data with the law $Wei(0.309{,}331965.757)$ $(ks = 0.148$ is superior to critical value $vc = 0.073)$.

The cumulative distribution function by slice $F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i, \beta_i)\right)$ based on the laws Weibull $Wei(\alpha_i, \beta_i)$ that adjusts with the amounts of the losses are defined by the following estimated parameters:

Table 27: Estimation of the parameters $\hat{\alpha}_i$, $\hat{\beta}_i$, a_i for $1 \leq i \leq 3$ of $F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i, \beta_i)\right)$

i	1	2	3
$\hat{\alpha}_i$	0.74	1.897	0.75
$\hat{\beta}_i$	95 059.95	1 439 533.66	21 686 360.81
a_i	264	317	347

– ***Adjustment of the laws $P(\lambda)$ and $BN(a, b)$ with the observed data of the frequency :***

The chi-square test has accepted the adjustment of the monthly frequency with the law $BN(1.223; 19.363)$ $(\chi^2_{observed} = 8.225$ $is\ inferior$ to the critical value $vc = 14.067)$.

However, the test has rejected the adjustment of the Poisson law $P(23.682)$ with the observed frequency to theoretical laws with the observed frequency $(\chi^2_{observed} = 2839296354.04$ $is\ superior\ to\ vc = 15.507)$

The monthly frequency of the losses will be simulated by the law $BN(1.223; 19.363)$.

3.2.2.8. Summary

The study of the adjustment of the data with the theoretical laws shows that several models can be used in the LDA approach. The number of models that can represent the losses P is summarized below, under the hypothesis of independence of risk categories:

➢ **The modelling of the global data:**

The two models presented in the following table can model the losses:

Table 28 : The two models that can model the losses

Models of the severity X_i	Models of the frequency N	Number of models
$- F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i, \beta_i)\right)$ $- F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i} = LN(\alpha_i, \beta_i)\right)$	Bootstrap technique	2

➢ **The modelling of data by risk category RT_c**

The models presented in the following table can model the losses by risk category RT_c:

Table 29 : The models that can model the losses by risk category RT_c

Risk category	Models of the severity X_i	Models of the frequency N	Number of models
RT_1	$- F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i, \beta_i)\right)$ $- F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i} = LN(\alpha_i, \beta_i)\right)$	Bootstrap technique	2
RT_2	$- F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i}\right.$: Weibull$\left.\right)$ $- LN(\alpha_i, \beta_i)$	Negative-Binomial	2
RT_3	$- F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i, \beta_i)\right)$ $- LN(\alpha, \beta)$	Negative-Binomial	2
RT_4	$- Wei(\alpha, \beta)$ $- LN(\alpha, \beta)$	Negative-Binomial	2
RT_5	$- F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i, \beta_i)\right)$ $- LN(\alpha, \beta)$	Poisson	2
RT_6	$- F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i, \beta_i)\right)$ $- F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i}\right.$: LogNormale$\left.\right)$	Negative-Binomial	2
RT_7	$- F_{\alpha,\beta,x_s,a}\left(K_{\alpha_i,\beta_i} = Wei(\alpha_i, \beta_i)\right)$ $- LN(\alpha, \beta)$	Negative-Binomial	2

The number of possible distribution combination to calculate the capital requirement by the VaR is 2^7, which gives the possibility of combination of 130 models, including the models of the global data, to calculate 130 values of VaR.

The Pure Bayesian Approach

To determine the Bayesian estimator of the parameters of the distribution of the severity and the frequency of losses, we use the expert opinion.

➢ **The Bayesian Estimator of the parameter of the losses frequency**

The formula (28) defines the Bayesian Estimator as follows:

$\hat{\lambda}_{Bay} = \varepsilon_1 \times \lambda_0 + (1 - \varepsilon_1) \times \lambda_{observed}$

In this formula, we deduct that the Bayesian Estimator is a weighted mean of the estimation by the likelihood method and the estimation by experts λ_0.

Therefore, we have to estimate with the experts the parameter λ_0 and we have to estimate the weighting coefficient of the experts' opinion ε_1.

In our study, we estimate λ_0 with the experts, whereas for ε_1 we use a weighting of 50% and 25%.

➢ **The severity**

In our study, the parameter σ is considered constant, estimated by the maximum likelihood method from the observed data $\hat{\sigma}$.

With regard to the Bayesian Estimator $\hat{\mu}_{Bay}$, we use the formula (29) which defines $\hat{\mu}_{Bay}$ as follows :

$\hat{\mu}_{Bay} = \varepsilon_2 \times \mu_0 + (1 - \varepsilon_2) \times \mu_{observed}$

In this formula, we deduct that the Bayesian Estimator is a weighted mean of the estimation by the likelihood method and the estimation by experts μ_0.

Therefore, we have to estimate with the experts the parameter μ_0 and we have to estimate the weighting coefficient of the experts' opinion ε_2.

In our study we estimate μ_0 with the experts, whereas for ε_2, we use a weighting of 50% and 25%.

The modelling of the losses from the global data

According to the Bayesian approach adopted, we determine the two models presented in the following table, for modelling the losses from the global data:

Table 30 : The two Bayesian models for modelled the global data

λ_0	$\hat{\lambda}$	$\hat{\lambda}_{Bay}$ $\varepsilon_1 = 25\%$	$\hat{\lambda}_{Bay}$ $\varepsilon_1 = 50\%$	μ_0	$\hat{\mu}$	$\hat{\mu}_{Bay}$ $\varepsilon_2 = 25\%$	$\hat{\mu}_{Bay}$ $\varepsilon_2 = 50\%$	$\hat{\sigma}$
150	185.66	176.745	167.83	7	8.826	8.3695	7.913	2.702

The modelling of losses by risk category RT_c

The Bayesian models presented in the following table can model the losses by risk category RT_c:

Table 31 : The Bayesian models by risk category RT_c

Risk type	λ_0	$\hat{\lambda}$	$\hat{\lambda}_{Bay}$ $\varepsilon_1 = 25\%$	$\hat{\lambda}_{Bay}$ $\varepsilon_1 = 50\%$	μ_0	$\hat{\mu}$	$\hat{\mu}_{Bay}$ $\varepsilon_2 = 25\%$	$\hat{\mu}_{Bay}$ $\varepsilon_2 = 50\%$	$\hat{\sigma}$
RT_1	20	25.24	23.93	22.62	6	7.944	7.083	6.972	2.775
RT_2	5	6.25	5.9375	5.625	5	7.691	7.01825	6.3455	2.062
RT_3	50	68.448	63.836	59.224	7.5	8.97	8.6025	8.235	1.39
RT_4	1	1.139	1.10425	1.0695	8	9.335	9.00125	8.6675	2.383
RT_5	2	2.122	2.0915	2.061	10	12.143	11.60725	11.0715	0.704
RT_6	15	23.326	21.2445	19.163	5	8.054	7.2905	6.527	2.415
RT_7	10	23.68	20.26	16.84	9	11.400	10.80	10.2	2.358

For each risk category, there are two possible models associated with the weighting coefficients ε_1 and ε_2 which must take on the same value because the estimates are made with the same expert.

The number of possible distribution combinations to calculate the capital requirement by the VaR is 2^7, which gives the possibility of combination of 130 models, including the models of the global data, for calculating 130 values of VaR.

Calculation the Capital at Risk and comparison of the risk profiles

To calculate the VaR and compare the risk profiles, we limit ourselves to the following models:

The Choice of the models

Choice of model for the global data

For the modelling of the global data, we will retain the following models:

- **The classical LDA approach**

For the classical LDA approach, we retain the following models:

Table 32 : The choice of models of the Classical LDA approach

	Model 1	Model 2
Severity	$F_{\alpha,\beta,x_s,a}$ $(K_{(\alpha_i;\beta_i)} = Wei(\alpha_i;\beta_i))$	$F_{\alpha,\beta,x_s,a}$ $(K_{(\alpha_i;\beta_i)} = LN(\alpha_i;\beta_i))$
Frequency	*Bootstrap*	*Bootstrap*

- **The Pure Bayesian approach**

For the Pure Bayesian Approach, we retain the two models based on the law LogNormal for the severity and the Poisson's law for the frequency, whose the parameters are presented in the following table:

Table 33 : The parameters of the two laws $LN(\mu,\sigma)$ and $P(\lambda)$ retained by the Bayesian approach

Model 3			Model 4		
$\hat{\lambda}_{Bay}$ $\varepsilon_1 = 25\%$	$\hat{\mu}_{Bay}$ $\varepsilon_2 = 25\%$	$\hat{\sigma}$	$\hat{\lambda}_{Bay}$ $\varepsilon_1 = 50\%$	$\hat{\mu}_{Bay}$ $\varepsilon_2 = 50\%$	$\hat{\sigma}$
176.745	8.3695	2.702	167.83	7.913	2.702

Choice of models by risk category

For modelling of the losses by risk category, we retain the following models:

- **The classical LDA approach**

For the classical LDA approach, we retain the following models:

Table 34 : The choice of models of the Classical LDA approach by risk category

Risk category	Model 5		Model 6	
	Severity	Frequency	Severity	Frequency
RT_1	$F_{\alpha,\beta,x_s,a}$ $(K_{(\alpha_i;\beta_i)} = Wei(\alpha_i;\beta_i))$	Bootstrap	$F_{\alpha,\beta,x_s,a}$ $(K_{(\alpha_i;\beta_i)} = LN(\alpha_i;\beta_i))$	Bootstrap
RT_2	$F_{\alpha,\beta,x_s,a}$ $(K_{(\alpha_i;\beta_i)} = Wei(\alpha_i;\beta_i))$	$BN(a,b)$	$LN(7.691\,;2.06$	$BN(a,b)$
RT_3	$F_{\alpha,\beta,x_s,a}$ $(K_{(\alpha_i;\beta_i)} = Wei(\alpha_i;\beta_i))$	$BN(a,b)$	$LN(8.97\,;1.39)$	$BN(a,b)$
RT_4	$Wei(0{,}416\,;37078$	$BN(a,b)$	$LN(9.335\,;2.38$	$BN(a,b)$
RT_5	$F_{\alpha,\beta,x_s,a}$ $(K_{(\alpha_i;\beta_i)} = Wei(\alpha_i;\beta_i))$	$P(\lambda)$	$LN(12.143\,;0.7$	$P(\lambda)$
RT_6	$F_{\alpha,\beta,x_s,a}$ $(K_{(\alpha_i;\beta_i)} = Wei(\alpha_i;\beta_i))$	$BN(a,b)$	$F_{\alpha,\beta,x_s,a}$ $(K_{(\alpha_i;\beta_i)} = LN(\alpha_i;\beta_i))$	$BN(a,b)$
RT_7	$F_{\alpha,\beta,x_s,a}$ $(K_{(\alpha_i;\beta_i)} = Wei(\alpha_i;\beta_i))$	$BN(a,b)$	$LN(11.400\,;2.3$	$BN(a,b)$

- **The Pure Bayesian approach**

For the Pure Bayesian Approach, we retain by risk category, the two models based on the law LogNormal for the severity and the Poisson's law for the frequency, whose the parameters are presented in the following table:

Table 35 : The parameters of the laws $LN(\mu,\sigma)$ and $P(\lambda)$ by risk category in the Bayesian approach

Risk category	*Model 7*			*Model 8*		
	$\hat{\lambda}_{Bay}$ $\varepsilon_1 = 25\%$	$\hat{\mu}_{Bay}$ $\varepsilon_2 = 25\%$	$\hat{\sigma}$	$\hat{\lambda}_{Bay}$ $\varepsilon_1 = 50\%$	$\hat{\mu}_{Bay}$ $\varepsilon_2 = 50\%$	$\hat{\sigma}$
RT_1	23.93	7.083	2.775	22.62	6.972	2.775
RT_2	5.9375	7.01825	2.062	5.625	6.3455	2.062
RT_3	63.836	8.9775	1.39	59.224	8.985	1.39
RT_4	1.10425	9.00125	2.383	1.0695	8.6675	2.383
RT_5	2.0915	11.60725	0.704	2.061	11.0715	0.704
RT_6	21.2445	7.2905	2.415	19.163	6.527	2.415
RT_7	20.26	10.80	2.493	16.84	10.2	2.358

Calculation of the capital at Risk (VaR) and comparison of the risk profiles

Calculation of the capital at Risk (VaR)

With the simulation method of the VaR, presented previously, we have calculated the VaR by each model as follows:

Table 36 : Calculation of the VaR according to the eight models

Model 1	*Model 2*	*Model 3*	*Model 4*
1.237 Billion MAD	1.952 Billion MAD	1.414 Billion MAD	0.833 Billion MAD
Model 5	*Model 6*	*Model 7*	*Model 8*
1.190 Billion MAD	1.644 Billion MAD	1.846 Billion MAD	1.174 Billion MAD

. Comparison of the Risk profiles

The comparison of the VaR for each model, will be made in reference at the capital requirement according to the basic indicator approach. Indeed, for the bank of this study, the average of gross incomes positive over the previous three years is 15.232 Billion MAD, which give the capital requirement K_{IB} equal to 2.282 Billion MAD. The results of the comparison of the risk profiles are presented as:

Table 37: Comparison of the risk profiles of the 8 models

Model 1	*Model 2*	*Model 3*	*Model 4*
54% K_{IB}	85.5% K_{IB}	62% K_{IB}	36% K_{IB}
Model 5	*Model 6*	*Model 7*	*Model 8*
52% K_{IB}	72% K_{IB}	81% K_{IB}	52% K_{IB}

CONCLUSION

The purpose of this article is to show that the internal models of quantification of operational risk generate the model risk. This is due to the imprecision and uncertainty of the calculation of the capital as each model determines a different level of capital requirement. Indeed, our objective is to verify the hypothesis that it is impossible to compare the risk profiles of banks on one hand, and to assess the evolution of the risk profile of a bank over time on the other hand, because of the divergence and difference between the models used.

To validate this hypothesis, we apply the classical LDA approach and the pure Bayesian LDA approach to assess the risk profiles of a Moroccan bank. We first give a definition of a bank's operational risk and we introduce the categories of risk and business lines according to the Basel Accord which we consider on our study. In the theoretical methodology section we develop the different versions of the LDA approach. In the section devoted to empirical results we first analyze and present the statistics of the data used in our work. We then present and interpret the empirical results of the application of the Classical LDA approach and pure Bayesian LDA approach to the Moroccan bank.

For the classical LDA approach, we use the Kolmogorov-Smirnov test to test the adjustment of the severity of the global data and of the data by risk category with the theoretical laws LogNormal and Weibull. If the adjustment is rejected, we determine the cumulative distribution function that adjusts with the data by slice, by use of an algorithm that we have constructed based on the adjustment of the losses recorded by slice with the Weibull and LogNormal law. For the frequency, we use the Chi-Square test to study the adjustment of the data with the theoretical laws Poisson and Negative-Binomial. If the adjustment is rejected, we opt for the use of the bootstrap technique.

We conclude for the classical LDA approach that the LogNormal and the Weibull law adjust with the global data and with the data by risk category with the total support and by slice. For the frequency, the Negative Binomial and Poisson laws adjust with a greater number of risk categories. After determining the distributions that adjust with the severity and the frequency, we show that 130 models can be used for the aggregation of the operational loss.

Concerning the pure Bayesian approach, we use the Lognormal-Poisson models to aggregate the loss variable. However, this aggregation is dependent on the Bayesian estimators. The expression of Bayesian estimator is a linear combination of the estimation provided by observed data and the estimation provided by experts. After determining the Bayesian Estimator, we show that 130 models can be used for aggregation of the operational loss.

We chose eight models to calculate the Capital at Risk (VaR) with the Monte Carlo method and we compare the results of the eight models with the capital requirement of the basic indicator approach. As a result, we infer seven different risk profiles from the same database for the same bank.

We emphasize that we limit our study to two distributions for modelling the severity while other distributions can be used in the classical LDA and pure Bayesian LDA approaches. Also, we use two restrictive hypotheses: for the Bayesian LDA approach, we assume that the standard deviation is constant and for the calculation of the VaR we assume that the risk categories are independent.

It is obvious that the number of models that can be used increases if: - other distributions are used: - the correlation between risk categories is integrated; all the parameters are considered random variables; the Markov Monte Carlo Chain Method (MCMC) is used to determine the Bayesian Estimator.

The multitude of models increases the uncertainty in the calculation of the capital requirements. As a result, the uncertainty about the risk profile of banks is increasing and consequently the regulator cant guarantee the transparency and the equality between banks, which creates an climate of doubt and instability over time.

References

[1] Alexander, C. (2003), "Operational Risk: Regulation, Analysis and Management". London: FT Prentice Hall.

[2] Bee, M. (2006). "Estimating the Parameters in the Loss Distribution Approach: How Can We Deal with Truncated Data". The Advanced Measurement Approach to Operational Risk. London: E. Davis, Risk Books.

[3] Chernobai, A., Menn, C., Truck, S. and Rachev, ST. (2005), "A Note on the Estimation of the Frequency and Severity Distribution of Operational Losses". Mathematical Scientist 30(2):87-97.

[4] Chernobai, A.S., Rachev, S.T. and Fabozzi, F.J. (2007), "Operational Risk". New Jersey: John Wiley & Sons, Inc.

[5] Cruz, M.G. (2002), "Modelling, measuring and hedging operational Risk". New Jersey: John Wiley & Sons, Inc.

[6] Cruz, M.G., Peters, G.W. and Shevchenko, P.V. (2015), "Fundamental Aspects of Operational Risk and Insurance Analytics: A Handbook of Operational Risk". New Jersey: John Wiley & Sons, Inc.

[7] Dahen, H. (2006), "La quantification du risque opérationnel des institutions bancaires". PhD thesis, HEC Montreal.

[8] Frachot, A., Goergis P. and Roncalli, T. (2001), "Loss Distribution Approach for operational risk. France: groupe de recherche opérationnel". Credit lyonnais.

[9] Frachot, A., Moudoulaud, O. and Roncalli, T. (2003), "Loss Distribution Approach in Practice". In The Basel Handbook: A Guide for Financial Practioners, Chicago: Micheal Ong, Risk Books.

[10] Gentle, J. (2003), "Random Number Generation and Monte Carlo Methods". Springer

[11] Hogg, R., Klugman, S.A. (1984), "Loss Distribution". New Jersey: John Wiley & Sons, Inc.

[12] Jorion, P. (2001), "Value at Risk". McGraw Hill, New York.

[13] King, J.L. (2001), "Operational Risk, Measurement and Modelling". New York: Wiley Finance.

[14] Panjer, H. and Willmot, G. (1992), "Insurance Risk Models". Chicago: Society of Actuaries.
[15] Perreault, L. and Bobée B. (1992), « Loi Weibull à deux paramètres, Propriétés mathématiques et statistiques, Estimation des paramètres et des quantiles X_T de période de retour ». Rapport scientifique n° 351. Université du Québec.
[16] Shevchenko, P.V. (2008), "Estimation of operational risk capital charge under parameter uncertainty". The Journal of Operational Risk 3 (1), 51–63
[17] Shevchenko, P.V. and Temnov, G. (2009), "Modeling operational risk data reported above a time-varying threshold". The Journal of Operational Risk 4(2), 19–42.
[18] Shevchenko, P.V. (2010). Implementing Loss Distribution Approach for Operational Risk. Applied Stochastic Models in Business and Industry 26(3), pp. 277-307
[19] Schevchenko, P.V. (2011), "Modelling operational risk using Bayesian inference". Springer.