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Generalized semi regular closed sets in bitopological spaces

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Abstract

In this paper, we introduce a new type of closed sets in bitopological space (X,τ_1,τ_2), used it to construct new types of normality, and introduce new forms of continuous function between bitopological spaces. Finally, we proved that the our new normality properties are preserved under some types of continuous functions between bitopological spaces.

INTRODUCTION AND PRELIMINARIES

The concepts of regular closed, generalized closed (briefly, g -closed), semiopen, regular generalized closed (briefly, rgclosed), and generalized semiclosed (briefly, gs -closed) sets have been introduced and investigated in [1-5]. The concepts of semiopen sets and regular open sets have been extended to bitopological spaces [6] called ij -semiopen and ij -regular open respectively. The mild normality and almost normality have been introduced in [7]. A weak form of normal spaces has been introduced in [8] called mildly normal spaces. In [9], the author used the semiopen sets to define seminormal spaces, recently, in [10] the author have continued the study of further properties of prenormal spaces and also defined and investigated mildly s-normal (resp. almost s -normal) spaces which are generalization of both mildly normal (resp. almost normal) spaces and s -normal spaces. The concept of generalized semiregular closed (briefly, gsr -closed) sets has been introduced in [11]. The concept of binormal spaces has been introduced in [12]. In [13,14] extended the concepts of g -closed, gs -closed and rg -closed sets, mildly normal and almost normal spaces to bitopological spaces. In this paper, we extend the concept of gsr -closed sets to bitopological spaces (X,τ_1,τ_2) called $ij-gsr$ -closed sets. Also, we construct a new types of normality in bitopological spaces based on ij -semiopen sets calledsemibinormal, almostsemibinormal and mildly semibinormal. We use the class of $ij-gsr$ -closed sets to characterization these types of normality and construct new types of continuous functions. We prove that the introduced binormality properties are preserved under some types of continuous functions. Throughout this paper, the following abbreviations will be adopted: Let A be a subset of a bitopological space (X,τ_1,τ_2), the interior (resp. closure) of A with respect to topology τ_i $(i=1,2)$ will be denoted by $\operatorname{int}^i(A)$ (resp. $cl^i(A)$). We denote the set of all closed sets with respect to the topology τ_i by $i-C(X)$.

In what follows, let $i,j\in\{1,2\}$ and $i\neq j$.

Definition 1.1 [6]. A subset A of a bitopological space (X,τ_1,τ_2) is said to be

(1) ij -semi open if $A\subseteq cl^j(\operatorname{int}^i(A))$.

(2) ij -regular open if $A=\operatorname{int}^i(cl^j(A))$.

The complement of ij -semi-open (resp. ij -regular open) set is called ij -semi-closed (resp. ij -regular closed) set. We denote the set of all ij -semi-open (resp. ij -semi-closed, ij -regular open and $ij-$ regular closed) sets by $ij-O^s(X)$ (resp. $ij-C^s(X)$, $ij-O^R(X)$ and $ij-C^R(X)$).

Definition 1.2 [6]. For any bitopological space (X,τ_1,τ_2) and $A\subseteq X$, ij -semi-interior (resp. ij -semi-closure) of A is denoted by $ij-\operatorname{int}^s(A)$ (resp. $ij-cl^s(A)$) and defined as

$$ij-\operatorname{int}^s(A)=\cup\{F\subseteq X\ :\ F\in ij-O^s(X),\quad F\subseteq A\}$$

$$\left(\text{resp}\, ij-cl^s(A)=\cap\{F\subseteq X\ :\ F\in ij-C^s(X), F\supseteq A\}\right).$$

Definition 1.3. A subset A of a bitopological space (X,τ_1,τ_2) is said to be

(1) ij -generalized closed [14] (briefly, ij - g -closed) if $A \subseteq U$, $U \in \tau_i \Rightarrow cl^j(A) \subseteq U$.

(2) ij -regular generalized closed [13] (briefly, ij - rg -closed) if $A \subseteq U$, $U \in ij-O^R(X) \Rightarrow cl^j(A) \subseteq U$.

(3) ij -generalized semi-closed [14] (briefly, ij - gs – closed) if $A \subseteq U$, $U \in \tau_i \Rightarrow ji-cl^s(A) \subseteq U$.

Definition 1.4. A subset A of a bitopological space (X,τ_1,τ_2) is said to be ij -generalized semi-regular closed (briefly, ij -gsr-closed) if $A \subseteq U$, $U \in ij-O^R(X) \Rightarrow ji-cl^s(A) \subseteq U$.

The complement of ij - g -closed (resp. ij - rg -closed, $ij-gs-$ closed and ij - gsr -closed) set is called ij - g -open (resp. $ij-rg$ -open, $ij-gs$ -open and ij - gsr -open) set and defined in the following lemma. Definition 1.4 is a particular case of Definition 8 from Noiri [15]. From Proposition 4 in [15], we obtain the following lemma.

Lemma 1.1. A subset A of a bitopological space (X,τ_1,τ_2) is:

(1) $ij-g$ -open iff $A \supseteq F$, $F \in i-C(X) \Rightarrow j-\text{int}(A) \supseteq F$

(2) $ij-rg$ -open iff $A \supseteq F$, $F \in ij-C^R(X) \Rightarrow j-\text{int}(A) \supseteq F$

(3) $ij-gs$ -open iff $A \supseteq F$, $F \in i-C(X) \Rightarrow ji-\text{int}^s(A) \supseteq F$

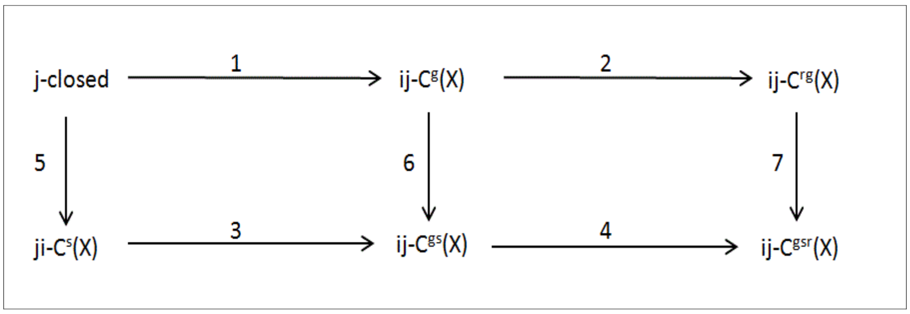
(4) $ij-gsr$ -open iff $A \supseteq F$, $F \in ij-C^R(X) \Rightarrow ji-\text{int}^s(A) \supseteq F$

We denote the set of all $ij-g$ -closed (resp. $ij-g$ -open, $ij-rg-$closed, $ij-rg$ -open, $ij-gs$ -closed, $ij-gs$ -open $ij-gsr$ -closed and $ij-gsr$ -open) sets by $ij-C^g(X)$ (resp. $ij-O^g(X)$, $ij-C^{rg}(X)$, $ij-O^{rg}(X)$, $ij-C^{gs}(X)$, $ij-O^{gs}(X)$, $ij-C^{gsr}(X)$ and $ij-O^{gsr}(X)$).

The arbitrary union of $ij-gsr$ -closed sets is an $ij-gsr-$ closed set. But the intersection of two of $ij-gsr$ -closed sets need not be an $ij-gsr$ -closed set as shown by the following example.

Example 1.1. Let $X=\{a,b,c,d\}$,
$\tau_1=\{x,\phi,\{a\},\{b\},\{a,b\}\}, \tau_{2=}\{x,\phi,\{a,b\},\{c,d\}\}$
We have $\{a,c\},\{a,d\} \in 21-C^{gsr}(X)$ but $\{a,c\} \cap \{a,d\}=\{a\} \notin 21-C^{gsr}(X)$.

Proposition 1.1. The following diagram shows the relationship between the above different types of closed sets.



Where none of these implications is reversible as shown by the following example.

Example 1.2. Let $X=\{a,b,c,d\}$, $\tau_1=\{x,\phi,\{a\},\{b\},\{a,b\}\}$ and $\tau_2=\{x,\phi,\{a,b\},\{c,d\}\}$.

(Arrows 1, 5) $\{d\} \in 12-C^g(X) \cap 21-C^s(X)$, but $\{d\} \notin 2-C(X)$.

(Arrows 2, 6) $\{a\} \in 12-C^{rg}(X) \cap 12-C^{gs}(X)$ but $\{a\} \notin 12-C^g(X)$.

(Arrow 3) $\{a,d\} \in 12-C^{gs}(X)$ but $\{a,d\} \notin 12-C^s(X)$

(Arrow 4)

Example 1.3. In Example 1.2. Let $X=\{a,b,c,d,e\}$,
$\tau_1=\{X,\phi,\{a\},\{b\},\{a,b\},\{a,b,c\}\}$,
$\tau_2=\{X,\phi,\{a,e\},\{a,e,d\}\}$,
$\{a,b,c\}\in 21-O^{gsr}(X)$, but $\{a,b,c\}\notin 21-C^{gs}(X)$.
(Arrow 7) In Example 1.3. $\{b,e\}\in 21-O^{gsr}(X)$, but . $\{b,e\}\notin 21-C^{rg}(X)$.

Remark 1.1. For any bitopological space (X,τ_1,τ_2) we note that:
(1) The classes $ij-C^{g}(X)$ and $ji-C^{s}(X)$ are independent
(2) The classes $ij-C^{rg}(X)$ and $ij-C^{gs}(X)$. are independent

The following example investigate the previous remark.

Example 1.4. Let (X,τ_1,τ_2) as in Example 1.3.
(1) $\{b,c,d\}\in 21-C^{rg}(X)$ but $\{b,c,d\}\notin 21-C^{gs}(X)$, also , $\{a,e\}\in 21-C^{gs}(X)$ but $\{a,e\}\notin 21-C^{rg}(X)$.
(2) $\{c,e\}\in 12-C^{g}(X)$ but $\{c,e\}\notin 21-C^{s}(X)$ also , $\{b\}\in 21-C^{s}(X)$ but $\{b\}\notin 12-C^{g}(X)$.

Theorem 1.1. For any bitopological space (X,τ_1,τ_2), $A\subseteq X$, the following are holds:

(1) If $A\in ij-C^{g}(X)\cap\tau_i$ then $A\in j-C(X)$.
(2) If $A\in ij-C^{gs}(X)\cap\tau_i$ then $A\in ji-C^{s}(X)$.
(3) If $A\in ij-C^{rg}(X)$ and $\tau_i =ij-O^{R}(X)$ then $A\in ij-C^{g}(X)$.
(4) If $A\in ij-C^{gsr}(X)$ and $\tau_i =ij-O^{R}(X)$ then $A\in ij-C^{gs}(X)$
(5) If $A\in ij-C^{gsr}(X)$ and $j-C(X)=ji-C^{s}(X)$ then $A\in ij-C^{rg}(X)$
(6) If $A\in ij-C^{gs}(X)$ and $j-C(X)=ji-C^{s}(X)$ then $A\in ij-C^{g}(X)$.

Proof: obvious.

Theorem 1.2. For any bitopological space (X,τ_1,τ_2). If $A\in ij-C^{gsr}(X)$ and $A\subseteq B\subseteq ji-cl^{s}(A)$, then $B\in ij-C^{gsr}(X)$.
Proof: Let $B\subseteq U$, $U\in ij-O^{R}(X)$. Since $A\subseteq B$ and $ij-C^{gsr}(X)$, then $ji-cl^{s}(A)\subseteq U$. Since $B\subseteq ji-cl^{s}(A)$, then we have
$ji-cl^{s}(B)\subseteq ji-cl^{s}(A)\subseteq U$. Consequently $B\in ij-C^{gsr}(X)$.

Theorem 1.3. Let (X_1,τ_1,τ_2) and (X_2,τ_1^*,τ_2^*) be two bitopological spaces. If $A\in ij-O^{gsr}(X_1)$ and $B\in i^*j^*-O^{gsr}(X_2)$, then $A\times B\in i\times i^*, j\times j^*-O^{gsr}(X_1\times X_2)$.
Proof: Let $A\in ij-O^{gsr}(X_1)$, and $B\in i^*j^*-O^{gsr}(X_2), W=A\times B\subseteq X_1\times X_2$ Let $F=F_1\times F_2\subseteq W, F\in i\times i^*$, $j\times j^*-C^{R}(X_1\times X_2)$
Then, there are $F_1\in ij-C^{R}(X_1)$ and $F_2\in i^*j^*-C^{R}(X_2), F_1\subseteq A, F_2\subseteq B$ and so $F_1\subseteq ji-\text{int}^{s}(A)$ and $F_2\subseteq i^*j^*-\text{int}^{s}(B)$.
Hence $F_1\times F_2\subseteq ij-\text{int}^{s}(A)\times i^*j^*-\text{int}^{s}(B)=j^*\times j, i^*\times i-\text{int}^{s}(A\times B)$
Therefore $A\times B\in i\times i^*, j\times j^*-O^{gsr}(X_1\times X_2)$

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Some types of ij -near continuous functions

In this section we introduce two types of continuous functions between bitopological spaces and study their properties.

Definition 2.1 [14]. A function $f:(X,\tau_1,\tau_2)\to(Y,\mu_1,\mu_2)$ is called:

(1) ij -semi-continuous if $\forall V\in i-C(Y), f^{-1}(V)\in ij-C^s(X)$,

(2) $ij-g$ -continuous if $\forall V\in j-C(Y), f^{-1}(V)\in ij-C^g(X)$,

(3) $ij-gs$ -continuous if $\forall V\in j-C(Y), f^{-1}(V)\in ij-C^{gs}(X)$,

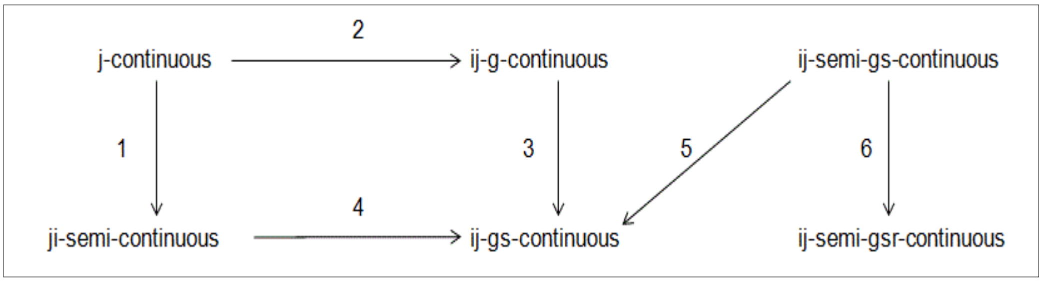
(4) i -continuous if $\forall V\in i-C(Y), f^{-1}(V)\in i-C(X)$.

Definition 2.2. A function $f:(X,\tau_1,\tau_2)\to(Y,\mu_1,\mu_2)$ is called:

(1) $ij-semi-gsr-$ continuous if $\forall V\in ji-C^s(Y)$, $f^{-1}(V)\in ij-C^{gsr}(X)$

(2) $ij-semi-gs-$ continuous if $\forall V\in ji-C^s(Y), f^{-1}(V)\in ij-C^{gs}(X)$

Theorem 2.1. The relationship between the previous concepts of continuity of functions between bitopological spaces are stated in the following diagram



$(Diagram\, 2.1)$

Proof: Straightforward.

In Diagram 2.1, the arrows are not reversible as one may see the following examples:

Example 2.1 Let $X=\{a,b,c,d\}$, $Y=\{u,v,w\}$, $\tau_1=\{x,\phi,\{a\},\{a,d\}\}$ and $\tau_2=\{x,\phi,\{a,b\},\{c,d\}\}$, $\mu_1=\{y,\phi,\{v\},\{v,w\}\}$ and $\mu_2=\{y,\phi,\{v\},\{v,u\}\}$ let . $f:(X,\tau_1,\tau_2)\to(Y,\mu_1,\mu_2)$

(arrows 1, 2) If f is defined by $f(a)=f(b)=w$ and $f(c)=u$, $f(d)=v$ We have f is 12 -semi-continuous, but it is not 1- continuous. Since there exist $\{u\}\in 1-C(Y)$ but $f^{-1}(\{u\})=\{c\}\notin 1-C(X)$ Also, f is $12-g$ -continuous, but it is not $2-$ continuous. Since there exist $\{u,w\}\in 2-C(Y)$ such that $f^{-1}(\{u,w\})=\{a,b,c\}\notin 2-C(X)$.

(arrows 3, 4) If f is defined by $f(a)=f(b)=u$, $f(c)=v$ and $f(d)=w$. We have f is $12-gs$ -continuous, but it is not $12-g-$continuous. Since there exist $\{w\}\in 2-C(Y)$ but $f^{-1}(\{w\})=\{d\}\notin 12-C^g(X)$ Also, f is not $21-$semi-continuous. Since there exist $\{u\}\in 2-C(Y)$ such that $f^{-1}(\{u\})=\{a,b\}\notin 12-C^s(X)$.

(arrows 5, 6) Example 2.2. Let $X=\{a,b,c\}, Y=\{u,v,w\}, \tau_1=\{X,\phi,\{a\},\{a,b\}\}$, $\tau_2=\{X,\phi,\{c\},\{a,c\}\}, \mu_1=\{Y,\phi,\{u\},\{v,w\}\}, \mu_1=\{Y,\phi,\{v\},\{v,u\}\}$. We have f is $12-gs$ -continuous and $12-$semi- $GSR-$ continuous but it is not $12-$semi $gs-$ continuous.

Since there exist $\{u\}\in 21-C^s(Y)$, such that $f^{-1}\{u\}=\{a\}\notin 12-C^{gs}(X)$.

Remark 2.1 For any function $f:(X,\tau_1,\tau_2)\to(Y,\mu_1,\mu_2)$ we note that:
(1) $ij-g$ -continuous and $ij-$semi- gs -continuous are independent.
(2) $ij-$semi-continuous and $ij-g$ -continuous are independent.
(3) $ij-gs$ -continuous and $ij-semi-gsr$ -continuous are independent.

The following example justifies the previous remark. Example 2.3 (i,ii) [resp. (iii,iv) and 2.4 (v,vi)] investigate Remark 2.1 (1) [resp. (2) and (3)].

Example 2.3. Let $f:(X,\tau_1,\tau_2)\to(Y,\mu_1,\mu_2)$ as in Example 2.1.

(i) If f is defined by $f(a)=f(b)=u, f(c)=v$ and $f(d)=w$. We have f is $12-semi-gs$ -continuous, but it is not $12-g-$continuous. Since there exist $\{w\}\in 2-C(Y)$ such that $f^{-1}(\{w\})=\{c,d\}\notin 12-C^g(X)$.

(ii) If f is defined by $f(a)=f(c)=v$, $f(b)=u$ and $f(d)=w$. We have f is $12-semi-gs$ -continuous, but it is not $12-g-$continuous. Since there exist $\{w\}\in 2-C(Y)$ such that $f^{-1}(\{w\})=\{d\}\notin 12-C^g(X)$

(iii) If f is defined by $f(a)=f(b)=u, f(c)=v$ and $f(d)=w$. We have f is $21-semi-$continuous, but it is not $12-g-$continuous. Since there exist $\{w\}\in 2-C(Y)$ such that $f^{-1}(\{w\})=\{d\}\notin 12-C^g(X)$

(iv) If f is defined by $f(a)=f(c)=v$, $f(b)=w$, and $f(d)=u$. We have f is $12-g$ -continuous, but it is not $21-semi-$continuous.Since there exist $\{v\}\in 2-C(Y)$ such that $f^{-1}(\{v\})=\{a,c\}\notin 21-C^s(X)$

(v) If f is defined by $f(a)=f(c)=w, f(b)=v$ and $f(d)=u$. We have f is $12-semi-gsr$ -continuous, but it is not $12-g-$continuous Since there exist. $\{w\}\in 2-C(Y)$ such that $f^{-1}(\{w\})=\{a,c\}\notin 12-C^{gs}(X)$

(vi) If f is defined by $f(a)=f(d)=u$, $f(b)=w$ and $f(c)=v$ We have f is $12-gs$ -continuous, but it is not $12-semi-gsr-$continuous. Since there exist $\{u\}\in 21-SC(Y)$ such that $f^{-1}(\{u\})=\{a,d\}\notin 12-C^{gsr}(X)$

Definition 2.3. A function $f:(X,\tau_1,\tau_2)\to(Y,\mu_1,\mu_2)$ is called:
(1) $ij-R$-map if $\forall V\in ij-O^R(Y), f^{-1}\in ij-O^R(X),$
(2) $ij-semi$ irresolute if $\forall V\in ij-C^s(Y), f^{-1}(V)\in ij-C^s(X),$
(3) $ij-r-closed$ if $\forall G\in ij-C^R(X), f(G)\in ij-C^R(Y),$
(4) $ij-semi-gs\text{-}closed$ if $\forall G\in ij-C^s(X), f(G)\in ji-C^{gs}(Y),$
(5) $ij-semi-rgs-closed$ if $\forall G\in ij-C^s(X), f(G)\in ji-C^{gsr}(Y).$

Lemma 2.1. For any surjection function $f:(X,\tau_1,\tau_2)\to(Y,\mu_1,\mu_2)$ the following are equivalent.
(a) f is $ij-semi-gs\text{-}closed$ function.
(b) For any $B\subseteq Y$, $U\in ij-O^s(X)$ such that $f^{-1}(B)\subseteq U$, there exist $V\in ji-O^{gs}(Y)$ such that $B\subseteq V$ and $f^{-1}(V)\subseteq U.$

Proof: Necessity, Let such that $B\subseteq Y, U\in ij-O^s(X)$ such that $f^{-1}(B)\subseteq U.$ Since f is $ij-semi-gs\text{-}closed$ function, then $f(U)\in ji-O^{gs}(Y)$ Put $V=f(U)$. Since $f^{-1}(B)\subseteq U$, then $B=f(f^{-1}(B))\subseteq f(U)=V$ and $f^{-1}(V)=f(f^{-1}(U))\subseteq U.$ Sufficiency, Let $G\in ij-O^s(X)$ such that $F\in i-C(Y)$, then $G\supseteq f^{-1}(F), F\subseteq Y.$ This implies that there exist $V\in ji-O^{gs}(Y)$ such that $F\subseteq V$ and $f^{-1}(V)\subseteq G$. Since $V\in ji-O^{gs}(Y)$, $F\in j-C(Y)$ and $F\subseteq V$. Consequently, $ij-\operatorname{int}^s(V)\supseteq F$ Since $V\subseteq f(G)$ then $F\subseteq ij-\operatorname{int}^s(V)\subseteq ij-\operatorname{int}^s(f(G))$. This implies that $f(G)\in ji-O^{gs}(Y)$. Therefore, f is $ij-semi-gs-closed$ function.

Lemma 2.2. For any surjection function $f:(X,\tau_1,\tau_2)\to(Y,\mu_1,\mu_2)$ the following are equivalent.

(a) f is $ij-semi-rgs-closed$ function.
(b) For any $B \subseteq Y, U \in ij-O^s(X)$ such that $f^{-1}(B) \subseteq U$, there exist $V \in ji-O^{gsr}(Y)$ such that and $B \subseteq V$ and $f^{-1}(V) \subseteq U.$

Proof: Similar to Lemma 2.1

Theorem 2.2. Let $f : (X,\tau_1,\tau_2) \to (Y,\mu_1,\mu_2)$ is $ij-semi-gs-$ continuous (resp. $ij-semi-gsr-$ continuous) function and $g : (Y,\mu_1,\mu_2) \to (Z,\eta_1,\eta_2)$ is $ij-semi$ -irresolute function, then $gof : (X,\tau_1,\tau_2) \to (Z,\eta_1,\eta_2)$, is $ij-semi-gs-$ continuous (resp. $ij-semi-gsr-$ continuous).

Proof: Let $V \in ij-C^s(Z)$, since g is $ij-semi-$irresolute, then $g^{-1}(V) \in ij-C^s(Y)$. Since f is $ij-semi-gs-$continuous, then $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$. Consequently, gof $ij-semi-gs-$continuous.

Some types of normality in bitopological spaces
In this section, we introduced three concepts of normality in bitopological spaces namely semibinormal, mild semibinormal, and almost semibinormal. We give a new characterization of these types of binormality by $ij-gsr$-open sets.

Definition 3.1 [12]. A bitopological space (X,τ_1,τ_2) is said to be semibinormal if given disjoint subsets $A, B, A \in i-C(X)$ and $B \in j-C(X)$, there are disjoint subsets U, V such that $U \in \tau_j$, $V \in \tau_j, A \subseteq U$ and $B \subseteq V$.

Definition 3.2. A bitopological space (X,τ_1,τ_2) is said to be semibinormal if given disjoint subsets $A, B, A \in i-C(X)$ and $B \in j-C(X)$, there are disjoint subsets U, V such that $U \in ji-O^s(X),\ V \in ij-O^s(X), A \subseteq U$ and $B \subseteq V$.

Theorem 3.1. For any bitopological space (X,τ_1,τ_2), the following statements are equivalent:

(a) X is semibinormal;
(b) for any disjoint sets $A \in i-C(X)$ and $B \in j-C(X)$, there exist $U \in ij-O^{gsr}(X),\ V \in ji-O^{gsr}(X)$ and $U \cap V = \phi$ such that $A \subseteq U$ and $B \subseteq V$.
(c) for any $A \in i-C(X),\ G \in \tau_j$ and $G \supseteq A$, there exists $U \in ij-O^{gsr}(X)$ such that $A \subseteq U \subseteq ji-cl^s(U) \subseteq G$.

Proof: $(a) \Rightarrow (b)$. Let $A \in i-C(X)$ and $B \in j-C(X)$, and $A \cap B = \phi$.
Since X is semibinormal, then there exist $U \in ji-O^s(X), V \in ij-O^s(X)$ and $U \cap V = \phi$ such that $A \subseteq U$ and $B \subseteq V$, this follows that, there exist $U \in ij-O^{gsr}(X),\ V \in ji-O^{gsr}(X)$ and $U \cap V = \phi$ such that $A \subseteq U$ and $B \subseteq V$.
$(b) \Rightarrow (c)$. Let $A \in i-C(X), G \in \tau_j$, and $G \supseteq A$. Then, $A \in i-C(X)$, $X \setminus G \in j-C(X)$, $(X \setminus G) \cap A = \phi$. Then, there exist $U \in ij-O^{gsr}(X)$, $V \in ji-O^{gsr}(X)$ and $U \cap V = \phi$ such that $A \subseteq U$ and $X \setminus G \subseteq V.$ Since $V \in ji-O^{gsr}(X),\ X \setminus G \in ji-RC(X)$ and $X \setminus G \subseteq V,$ then by using Lemma 1.1 (4) we have $ij-cl^s(V) \supseteq X \setminus G. U \cap V = \phi$ implies $U \cap ij-cl^s(V) = \phi$. Consequently, $A \subseteq U \subseteq X \setminus ij-cl^s(V) \subseteq G$ this follows that $A \subseteq U \subseteq ij-cl^s(U) \subseteq X \setminus ij-cl^s(V) \subseteq G$. Consequently, $A \subseteq U \subseteq ij-cl^s(U)$.
$(c) \Rightarrow (a)$. Let $A \in i-C(X),\ B \in j-C(X)$ and $A \cap B = \phi$. Then, $A \in i-C(X), X \setminus B \in \tau_j$ and $A \subseteq X \setminus B$. Consequently, there exist $G \in ij-O^{gsr}(X)$ such that $A \subseteq G \subseteq ij-cl^s(G) \subseteq X \setminus B$. Since $A \subseteq G, A \in ij-C^R(X)$ and $G \in ij-O^{gsr}(X)$ then, by using Lemma $1.1\,(4)$ we have $ji-\text{int}^s(G)$. This follows that $B \subseteq X \setminus ij-cl^s(G) = ij-cl^s(G^c), ji-\text{int}^s(G) \in ji-O^s(X), ij-cl^s(G^c) \in ij-O^s(X)$ and

$ji-\operatorname{int}^s(G)\cap ij-\operatorname{int}^s(G^c)=\phi$. Put $U=\operatorname{int}^j\left(cl^i\left(ji-\operatorname{int}^s(G)\right)\right)$ and $V=\operatorname{int}^i\left(cl^j\left(ij-\operatorname{int}^s(G^c)\right)\right)$. Then U,V are disjoint, $U\in ji-O^s(X)$ and $V\in ij-O^s(X)$ Such that $U\supseteq A$ and $V\supseteq B.$

Definition 3.3. A space (X,τ_1,τ_2) is said to be almost semibinormal if given disjoint subsets A and B, $A\in i-C(X), B\in ji-C^R(X),$ there are disjoint subsets U and V such that $U\in ji-O^s(X), V\in ij-O^s(X), A\subseteq U$ and $B\subseteq V$.

Theorem 3.2. For any bitopological space (X,τ_1,τ_2), the following statements are equivalent:

(a) X is almost semibinormal;

(b) for each disjoint sets $A\in i-C(X)$ and $B\in ji-C^R(X)$ there are disjoint subsets $U\in ij-O^{gs}(X)$ and $V\in ji-O^{gs}(X)$ such that $A\subseteq U$ and $V\subseteq B$.

(c) for each disjoint sets $A\in i-C(X)$ and $B\in ji-C^R(X)$ there are disjoint subsets $U\in ij-O^{gsr}(X)$ and $V\in ji-O^{gsr}(X)$ such that $A\subseteq U$ and $V\subseteq B$.

(d) for each $A\in i-C(X)$ and $K\in ji-O^R(X)$, and $K\supseteq A$ there exists $U\in ij-O^{gsr}(X)$ such that $A\subseteq U\subseteq ij-cl^s(U)\subseteq K.$

Proof: It is obvious that $(a)\Rightarrow(b)\Rightarrow(c)$, $(c)\Rightarrow(d)$. Let $A\in\tau_i^c$ and $K\in ji-O^R(X)$ and $K\supseteq A$. This implies that $A\in\tau_i^c$ and $X\setminus K\in ji-C^R(X)$ and $(X\setminus A)\cap A=\phi$
Then,there exists $U\in ij-O^{GSR}(X)$ and $V\in ji-O^{GSR}(X)$ such that $A\subseteq U, X\setminus K\subseteq V$, then by Lemma $1.1\,(4)$, we have $ij-\operatorname{int}^s(V)\supseteq X\setminus K$ Since $U\cap V=\phi$ implies $U\cap ij-\operatorname{int}^s(V)=\phi$.
Consequently, $A\subseteq U,\subseteq X\setminus ij-\operatorname{int}^s(V)\subseteq K$, this follows that $A\subseteq U\subseteq ij-\operatorname{int}^s(U)\subseteq X\,?\,ij-\operatorname{int}^s(V)\subseteq K$
Therefore, $A\subseteq U\subseteq ij-\operatorname{int}^s(U)\subseteq K.$
$(d)\Rightarrow(c)$ Let $A\in i-C(X)$, $B\in ji-C^R(X)$ and $A\cap B=\phi$. This implies that $A\in i-C(X)$, $X\setminus B\in ji-O^R(X)$ and $A\subseteq X\setminus B.$ Consequently, there exists $U\in ij-O^{gsr}(X)$ such that $A\subseteq U\subseteq ij-\operatorname{int}^s(U)\subseteq X\setminus B$. Since $A\subseteq U, A\in ij-C^R(X)$ and $U\in ij-O^{gsr}(X)$ then, by using Lemma $1.1\,(4)$ we have $A\subseteq ij-\operatorname{int}^s(U)$. This follows that
$B\subseteq X\setminus ij-cl^s(U)=ij-\operatorname{int}^s(U^c), ij-\operatorname{int}^s(U)\in ji-O^s(X), ij-\operatorname{int}^s(U^c)\in ij-O^s(X)$ and $ji-\operatorname{int}^s(U)\cap ij-\operatorname{int}^s(U^c)=\phi$. put $G=\operatorname{int}^j\left(cl^i\left(ji-\operatorname{int}^s(U)\right)\right)$ and $H=\operatorname{int}^i\left(cl^j\left(ij-\operatorname{int}^s(U^c)\right)\right)$. Then G,H are disjoint, $G\in ji-O^s(X)$ and $H\in ij-O^s(X)$ Such that $G\supseteq A$ and $H\supseteq B$.

Definition 3.4. A bitopological space (X,τ_1,τ_2) is said to be mildly semibinormal if given disjoint subsets $A\in ij-C^R(X)$ and $B\in ji-C^R(X)$, there are disjoint subsets $U\in ji-O^s(X)$ and $V\in ij-O^s(X)$ Such that $A\subseteq U$ and $V\subseteq B$.

Theorem 3.3. For any bitopological space (X,τ_1,τ_2), the following statements are equivalent:

(a) X is mildly semibinormal;

(b) for any $A\in ij-C^R(X)$ and $B\in ji-C^R(X)$ and $A\cap B=\phi$ there are $U\in ij-O^{gs}(X)$, $V\in ji-O^{gs}(X)$ and $U\cap V=\phi$ such that $A\subseteq U$ and $V\subseteq B$;

(c) for any $A\in ij-C^R(X)$ and $B\in ji-C^R(X)$ and $A\cap B=\phi$ there are $U\in ij-O^{grs}(X), V\in ji-O^{grs}(X)$, and $U\cap V=\phi$ such that $A\subseteq U$ and $V\subseteq B$.

(d) for any $A\in ij-C^R(X)$, $K\in ji-O^R(X)$ and $A\subseteq K$ there exists $U\in ij-O^{gs}(X)$, such that $A\subseteq U\subseteq ij-cl^s(U)\subseteq K$,

(e) for any $A \in ij - C^R(X)$, $K \in ji - O^R(X)$ and $A \subseteq K$ there exists $U \in ij - O^{gsr}(X)$, such that $A \subseteq U \subseteq ij - cl^s(U) \subseteq K$

Proof: Similar to that of Theorem 3.2

Preservation theorems

In this section, we prove that the three types of binormality properties are preserved under some types of function between bitopological spaces

Theorem 4.1. If $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ is $ij - semi - gs$-closed, $i-$ continuous, surjection and X is semibinormal then Y is also semibinormal.

Proof: Let A $A \in i - C(Y), B \in j - C(Y)$ and $A \cap B = \phi$, Since f is surjection i-continuous, then $f^{-1}(A) \in i - C(X)$, $f^{-1}(B) \in j - C(X)$ and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \phi$. Since X is semibinormal, there exist $U \in ji - O^s(X) V \in ij - O^s(X)$, and $U \cap V = \phi$, such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since f is $ij - semi - gs$-closed, by Lemma 2.1, there exist $G \in ij - O^{gs}(Y)$ and $H \in ji - O^{gs}(Y)$ such that $A \subseteq G, B \subseteq H$, $f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Since U *and* V are disjoint, G and H are disjoint. Since $G \in ij - O^{gs}(Y)$ and $H \in ji - O^{gs}(Y)$, by Lemma $1.1\,(3)$, then we have $A \subseteq ji - \text{int}^s(G), B \subseteq ji - \text{int}^s(H)$ and so $ji - \text{int}^s(G) \cap ji - \text{int}^s(H) = \phi$. Consequently, Y is also semibinormal.

Theorem 4.2. If $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ is $ij - semi - rgs$ -closed, $ij - R - map,$ surjection and X is mildly semibinormal then Y is also mildly semibinormal.

Proof: Let $A \in ij - C^R(Y), B \in ji - C^R(Y)$ and $A \cap B = \phi$, Since f is surjection $ij - R - map$, then $f^{-1}(A) \in ij - C^R(X), f^{-1}(B) \in ji - C^R(X)$, and $f^{-1}(A) \cap f^{-1}(B) = \phi$. Since X is mildly semibinormal, then there exist $U \in ji - O^s(X), V \in ij - O^s(X)$ and $U \cap V = \phi$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since f is $ij - semi - rgs -$ closed, by Lemma 2.1, there exist $G \in ij - O^{gsr}(Y)$ and $H \in ji - O^{gsr}(Y)$ such that $A \subseteq G, B \subseteq H, f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Since U *and* V are disjoint, G and H are disjoint. Since $G \in ij - O^{gsr}(Y)$ and $H \in ji - O^{gsr}(Y)$, by Lemma $1.1\,(4)$ then we have $A \subseteq ji - \text{int}^s(G)$, $B \subseteq ij - \text{int}^s(H)$ and so $ji - \text{int}^s(G) \cap ij - \text{int}^s(H) = \phi$. Consequently, Y is also mildly semibinormal.

Theorem 4.3. If $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ is $ij - semi - rgs$-closed, $ij - R - map$, i-continuous, surjection and X is almost semibinormal, then Y is also almost semibinormal.

Proof: Let $A \in i - C(Y), B \in ji - C^R(Y)$ and. Since f is $A \cap B = \phi.$ Since f is $ij - R - map$, then, $f^{-1}(B) \in ji - C^R(X)$. Since f is i-continuous, then $f^{-1}(A) \in i - C(X)$ and we have $f^{-1}(A) \cap f^{-1}(B) = \phi$. Since X is almost semibinormal, then there exist $U \in ji - O^s(X), V \in ij - O^s(X)$ and $U \cap V = \phi$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since f is $ij - semi - rgs$-closed, by Lemma 2.1, there exist $G \in ij - O^{gsr}(Y)$ and $H \in ji - O^{gsr}(Y)$ such that $A \subseteq G, B \subseteq H, f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Since U and V are disjoint, G and H are disjoint. Since $G \in ij - O^{gsr}(Y)$ and $H \in ji - O^{gsr}(Y)$, by Lemma $1.1\,(4)$, then we have $A \subseteq ji - \text{int}^s(G)$, $B \subseteq ij - \text{int}^s(H)$ and so $ji - \text{int}^s(G) \cap ij - \text{int}^s(H) = \phi$. Consequently, Y is also almost semibinormal.

Theorem 4.4. If $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ is $ij - semi - rgs$-continuous i-closed, injection and Y is semibinormal, then X is also semibinormal.

Proof: Let $A \in i-C(X), B \in j-C(X)$ and $A \cap B = \phi$. Since f is iclosed injection, then $f(A) \in i-C(Y)$, $f(B) \in j-C(Y)$ and $f(A) \cap f(B) = \phi$. By semibinormality of Y, there exist $U \in ji-O^s(Y), V \in ij-O^s(Y)$ and $U \cap V = \phi$ such that $f(A) \subseteq U$ and $f(B) \subseteq V$. Since f is $ij-semi-rgs$-continuous, $f^{-1}(U) \in ij-O^{gsr}(X)$ and $f^{-1}(V) \in ji-O^{gsr}(X)$ such that $A \subseteq f^{-1}(U), B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$. By Theorem 2.1 (b), therefore, X is semibinormal.

Theorem 4.5. If $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ is $ij-semi-grs$-continuous $ij-rc$ -preserving, injection and Y is mildly semibinormal then X is also mildly semibinormal

Proof: Let $A \in ij-C^R(X)$, $B \in ij-C^R(X)$, and $A \cap B = \phi$ Since f is $ij-rc$-preserving injection, then $f(A) \in ij-C^R(Y)$, $f(B) \in ji-C^R(Y)$ and $f(A) \cap f(B) = \phi$. By mild semibinormality of Y, there exist $U \in ji-O^s(Y), V \in ij-O^s(Y)$ and $U \cap V = \phi$ such that $f(A) \subseteq U$ and $f(B) \subseteq V$. Since f is $ij-semi-grs$-continuous, $f^{-1}(U) \in ij-GSRO(X)$ and $f^{-1}(V) \in ji-O^{gsr}(X)$ such that $A \subseteq f^{-1}(U)$, $B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$. By Theorem 2.3 (c), therefore, X is mildlysemibinormal.

Theorem 4.6. If $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ is $ij-semi-gsr$ -continuous $ij-rc$-preserving, i -closed injection and Y is almostsemibinormal then X is also almost semibinormal

Proof: Let $A \in i-C(Y), B \in ij-C^R(X)$ and $A \cap B = \phi$. Since f is $ij-rc$-preserving and i -closed injection, then $f(A) \in i-C(X)$, $f(B) \in ij-C^R(Y)$ and $f(A) \cap f(B) = \phi$. By almost semibinormality of Y, there exist $U \in ji-O^s(Y), V \in ij-O^s(Y)$ and $U \cap V = \phi$ such that $f(A) \subseteq U$ and $f(B) \subseteq V$. Since f is $ij-semi-gsr$ -continuous, $f^{-1}(U) \in ij-O^{gsr}(X)$ and $f^{-1}(V) \in ji-O^{gsr}(X)$ such that $A \subseteq f^{-1}(U)$, $B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$. By Theorem 2.2 (c) , therefore, X is almost semibinormal.

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