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On Existence of Mild and Positive Solutions of Impulsive Integro-Differential Equations

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Abstract: In the present paper, we prove the existence of mild and positive solutions of second order nonlocal impulsive initial value problem using Leary Schauder alternative and Leggett-Willaims fixed point theorem.

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INTRODUCTION

In many real life phenomena, the state of system changes abruptly and hence it is not possible to model such evolution processes with usual initial value problems. The discontinuities in the state of such evolution problems can be modeled with impulsive differential equations. Due to which the study of impulsive differential equations has became an area of attraction for young researchers. For more details, refer the monographs of Bainov and Simeonov [11], V. Lakashmikantham et al. [10].

As nonlocal condition takes more information at a time and decreases the negative effetcs than classical condition therefore modeling of evolution processes become more realistic and accurate.

In this paper, we study the second order integro-differential system of the type:

$$u''(t) = A_1 u(t) + f\Big(t, u_t, \int_0^t k(t,s)h(s,u_s)ds\Big),$$
$$t \in (0,T], t \neq \tau_k, k = 1, \dots, m \quad (1)$$
$$u(t) + (g(u_{t_1}, \dots, u_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0, \quad (2)$$
$$u'(0) = \eta, \quad \eta \in X \quad (3)$$
$$\Delta u(\tau_k) = I_k u(\tau_k), \quad k = 1, \dots, m, \quad (4)$$
$$\Delta u'(\tau_k) = \bar{I}_k u(\tau_k), \quad k = 1, \dots, m, \quad (5)$$

where $0 < t_1 < t_2 < \cdots < t_p \leq T$, $p \in \mathbb{N}$, A_1 is the infinitesimal generator of strongly continuous cosine family of bounded linear operators $\{C(t)\}_{t\in\mathbb{R}}$ on X, $k : [0,T] \times [0,T] \to \mathbb{R}$ is continuous function. f, g, h and ϕ are given functions satisfying some assumptions and $u_t(\theta) = u(t+\theta)$, for $\theta \in [-r, 0]$ and $t \in [0,T]$.

Motivated by the work given in [1]–[6], we study the existence of mild and positive solutions of second order impulsive functional integro-differential equations with nonlocal condition. The paper is organised as follows: section 2 contains preliminaries and hypotheses, section 3 deals with existence of mild solutions of impulsive second order problem. In section 4, we prove existence of triple positive solutions.

PRELIMINARIES AND HYPOTHESES

Let X be a Banach space with the norm $\|\cdot\|$. Let $C = \mathcal{C}([-r,0],X), 0 < r < \infty$, be the Banach space of all continuous functions $\psi : [-r,0] \to X$ endowed with supremum norm $\|\psi\|_C = \sup\{\|\psi(t)\| : -r \le t \le 0\}$ and B denote the set $\{u : [-r,T] \to X \mid u(t)$ is continuous at $t \neq \tau_k$, left continuous at $t = \tau_k$, and the right limit $u(\tau_k+0)$ exists for $k = 1,2,\dots,m\}$. Clearly, $B = PC([-r,T],X)$ is a Banach space with the supremum norm $\|u\|_B = \sup\{\|u(t)\| : t \in [-r,T]\backslash\{\tau_1,\tau_2,\dots,\tau_m\}\}$. For any $u \in B$ and $t \in [0,T] \setminus \{\tau_1,\tau_2,\dots,\tau_m\}$, we denote u_t the element of C given by $u_t(\theta) = u(t+\theta)$, for $\theta \in [-r,0]$ and ϕ is a given element of C. Let $AC^i((\tau_k,\tau_{k+1}),X)$ be the space of i-times differentiable functions $u : (\tau_k,\tau_{k+1}) \to X$, whose ith derivative, u^i, is absolutely continuous.

Definition 1. A one parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators in the Banach space X is called strongly continuous cosine family if and only if

1. $C(0) = I$ is the identity operator
2. $C(t+s) + C(t-s) = 2C(t)C(s) \quad \forall \quad t,s \in \mathbb{R}$
3. The map $t \mapsto C(t)(u)$ is strongly continuous for each $u \in X$.

The associated sine function is the family $\{S(t)\}_{t\in\mathbb{R}}$ of operators defined by $S(t)u = \int_0^t C(s)u\,ds$, for $u \in X, t \in \mathbb{R}$. The infinitesimal generator $A_1 : X \to X$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by $A_1u = \frac{d^2}{dt^2}C(t)u|_{t=0}$, $u \in D(A_1)$, where $D(A_1) = \{u \in X : C(.)u \in C^2(\mathbb{R},X)\}$. For more information on strongly continuous cosine and sine families, we refer the reader to [8], [9]. In this paper, we assume that, there exist positive constants M and N such that $\|C(t)\| \le M$ and $\|S(t)\| \le N$, for every $t \in [0,T]$.

Definition 2. A function $u \in B$ satisfying the equations:

$$
\begin{aligned}
u(t) &= C(t)[\phi(0) - (g(u_{t_1},\dots,u_{t_p}))(0)] + S(t)\eta \\
&+ \int_0^t S(t-s) f\big(s, u_s, \int_0^s k(s,\tau)h(\tau,u_\tau)d\tau\big)ds \\
&+ \sum_{0<\tau_k<t} \big[C(t-\tau_k)I_k u(\tau_k) - S(t-\tau_k)\bar{I}_k u(\tau_k)\big], \\
&\quad t \in (0,T], \\
u(t) &+ (g(u_{t_1},\dots,u_{t_p}))(t) = \phi(t), \quad -r \le t \le 0 \\
u'(0) &= \eta, \quad \eta \in X
\end{aligned}
$$

is said to be the mild solution of the initial value problem (1)–(5).

Definition 3. [Positive Solution] A function $u \in PC([-r,T],X) \cap AC^1((\tau_k,\tau_{k+1}),X)$ is said to be positive solution of equation (1)–(5) if u satisfies equation (1) a.e on $[0,T]/\{\tau_1,\tau_2,\dots,\tau_m\}$ along with the condition (2)–(5).

Theorem 4. [Leggett-Willaims fixed point Theorem] [7] Let E be a Banach Space, $C \supset E$ be a cone of E and $R > 0$ a constant. Let $C_R = \{y \in C : \|y\| < R\}$. Suppose that a concave non-negative continuous functional ψ exists on cone C with $\psi(y) \le \|y\|$ for $y \in \bar{C}_R$ and let $N : \bar{C}_R \to \bar{C}_R$ be completely continuous operator. Assume that they are numbers ρ, L and K with $0 < \rho < L < K \le R$ such that

1) $\{y \in C(\psi,L,K) | \psi(y) > L\} \neq 0$ and $\psi(N(y)) < L \quad \forall y \in C(\psi,L,K)$
2) $\|N(y)\| < \rho \quad y \in \bar{C}_\rho$
3) $\psi(N(y)) > L \quad \forall y \in C(\psi,L,R)$ with $\|N(y)\| > K$, where $C(\psi,L,K) = \{y \in C, \psi(y) \ge L$ and $\|y\| \le K\}$. Then N has at least three fixed points y_1,y_2,y_3 in $\bar{C}_R$. Furthermore, $y_1 \in C_\rho, y_2 \in \{y \in C(\psi,L,\mathbb{R}) | \psi(y) > L\}$, $y_3 \in \bar{C}_R - \{C(\psi,L,\mathbb{R}) \cup \bar{C}_\rho\}$.

Lemma 5. [12] Let for $t \ge t_0$, the following inequality hold

$$
u(t) \le a(t) + \int_{t_0}^{t} b(t,s)u(s)ds + \int_{t_0}^{t}\left(\int_{t_0}^{s} k(t,s,\tau)u(\tau)d\tau\right)ds + \sum_{t_0<\tau_k<t} \beta_k(t)u(t_k)
$$

where, $u,a \in PC([t_0,\infty),\mathbb{R}_+)$ is a nondecreasing, $b(t,s)$ and $k(t,s,\tau)$ are continuous and non negative functions for $t,s,\tau \ge t_0$ and are nondecreasing with respect to t, $\beta_k(t)(k \in \mathbb{N})$ are nondecreasing for $t \ge t_0$, then for $t \ge t_0$ the following inequality hold:

$$
u(t) \le a(t) \prod_{t_0<\tau_k<t} (1+\beta_k(t)) exp\left(\int_{t_0}^{t} b(t,s)ds\right) + \int_{t_0}^{t}\int_{t_0}^{s} k(t,s,\tau)d\tau ds
$$

Theorem 6. [Leary-Schauder Alternative] Let S be a convex subset of normed linear space E and assume $0 \in S$. Let $F : S \to S$ be a completely continuous operator and

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let $\varepsilon(F) = \{u \in S : u = \lambda F u, \text{for some} \quad 0 < \lambda < 1\}$. Then either $\varepsilon(F)$ is unbounded or F has fixed point.

Let us introduce the following hypotheses which are assumed thereafter for our convenience.

(H_1) Let $f : [0, T] \times C \times X \to X$ and $h : [0, T] \times C \to X$ be continuous functions such that, there exists continuous nondecreasing functions p and $q : [0, T] \to \mathbb{R}_+ = [0, \infty)$ such that

$$\|f(t, \psi, u)\| \leq p(t)(\|\psi\|_C + \|u\|),$$
$$\|h(t, \psi)\| \leq q(t)(\|\psi\|_C),$$

for every $t \in [0, T]$, $\psi \in C$ and $u \in X$.

(H_2) Let $g : C^p \to C$ such that there exists a constant $G \geq 0$ such that

$$\max \| g(u_{t_1}, u_{t_2}, \ldots, u_{t_p}) \| \leq G$$

(H_3) Let $I_k, \bar{I}_k : X \to X$ are functions such that there exists constants L_k and $\bar{L_k}$ satisfying

$$\|I_k(v)\| \leq L_k\|v\|, \quad v \in X, \quad k = 1, 2, \ldots, m.$$
$$\|\bar{I}_k(v)\| \leq \bar{L_k}\|v\|, \quad v \in X, \quad k = 1, 2, \ldots, m.$$
$$L_k^* = \max(L_k, \bar{L_k})$$

(H_4) For each $t \in [0, T]$, the function $f(t, ., .) : C \times X \to X$, $h(t, .) : C \to X$ are continuous and for each $(\psi, u) \in C \times X$, $\psi \in C$, the functions $f(., \psi, u) : [0, T] \to X$ and $h(., \psi) : [0, T] \to X$ are strongly measurable.

With these preparations we state our result to be proved in the present paper.

EXISTENCE OF MILD SOLUTIONS

Theorem 7. Suppose that the hypotheses (H_1)-(H_4) hold. Then the initial-value problem (1)–(5) has a mild solution u on $[-r, T]$.

Proof. To prove the existence of mild solution of the initial-value problem (1)–(5), first we establish the priori bounds on the solutions to the initial value problem $(1)_\lambda - (5)$ for $\lambda \in (0, 1)$,

$$u''(t) = A_1 u(t) + \lambda f\Big(t, u_t, \int_0^t k(t, s)h(s, u_s)ds\Big),$$
$$t \in (0, T], t \neq \tau_k, k = 1, \ldots, m \quad (6)$$
$$u(t) + (g(u_{t_1}, \ldots, u_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0, \quad (7)$$
$$u'(0) = \eta, \quad \eta \in X \quad (8)$$
$$\Delta u(\tau_k) = I_k u(\tau_k), \quad k = 1, \ldots, m, \quad (9)$$
$$\Delta u'(\tau_k) = \bar{I}_k u(\tau_k), \quad k = 1, \ldots, m. \quad (10)$$

Let $u(t)$ be a solution of given IVP (6)–(10) then it satisfies equivalent integral equation

$$u(t) = C(t)[\phi(0) - (g(u_{t_1}, \ldots, u_{t_p}))(0)] + S(t)\eta + \int_0^t \lambda S(t-s) f\big(s, u_s, \int_0^s k(s, \tau)h(\tau, u_\tau)d\tau\big)ds + \sum_{0<\tau_k<t} \big[C(t-\tau_k)I_k u(\tau_k) - S(t-\tau_k)\bar{I}_k u(\tau_k)\big], \quad t \in (0, T]. \quad (11)$$

Since k is continuous on a compact set $[0, T] \times [0, T]$, there exists a constant $L > 0$ such that $|k(t, s)| \leq L$ for $0 \leq s \leq t \leq T$. Also let $\|\phi\|_C = D$. Using hypotheses $(H_1) - (H_4)$ and the fact that $\lambda \in (0, 1)$, we have for $t \in [0, T]$

$$\begin{aligned}\|u(t)\| &\leq \|C(t)[\phi(0) - (g(u_{t_1}, \ldots, u_{t_p}))(0)]\| + \|S(t)\eta\| \\ &\quad + \|\lambda \int_0^t S(t-s) f(s, u_s, \int_0^s k(s, \tau)h(\tau, u_\tau)d\tau)ds\| \\ &\quad + \|\sum_{0<\tau_k<t} \big[C(t-\tau_k)I_k u(\tau_k) - S(t-\tau_k)\bar{I}_k u(\tau_k)\big]\| \\ &\leq M(D+G) + N\|\eta\| + \int_0^t Np(s)[\|u_s\|_C \\ &\quad + \int_0^s Lq(\tau)(\|u_\tau\|_C)\|d\tau]ds \\ &\quad + \sum_{0<\tau_k<t} (M+N)L_k^*\|u(\tau_k)\|. \end{aligned} \quad (12)$$

Let $R(t) = \sup\{p(t), Lq(t)\}$ and $R^* = \sup\{R(t) : t \in [-r, T]\}$. Define the function $z : [-r, T] \to \mathbb{R}$ by

$$z(t) = \sup\{\|u(s)\| : -r \leq s \leq t\}, \quad t \in [0, T]$$

Let $t^* \in [-r, t]$ be such that $z(t) = \|u(t^*)\|$. If $t^* \in [0, t]$ then from (12) we have

$$\begin{aligned}
z(t) &= \|u(t^*)\| \\
&\leq M(D+G) + N\|\eta\| + \int_0^{t^*} Np(s)[\|u_s\|_C + \int_0^s Lq(\tau)(\|u_\tau\|_C)\|d\tau]ds + \sum_{0<\tau_k<t}(M+N)L_k^*\|u(\tau_k)\| \\
&\leq M[D+G] + N\|\eta\| + \int_0^t NR(s)z(s)ds + \int_0^t \left(\int_0^s [NLR(s)R(\tau)](z(\tau))d\tau\right)ds \\
&+ \sum_{0<\tau_k<t}(M+N)L_k^*\|z(\tau_k)\| \qquad (13)
\end{aligned}$$

If $t^* \in [-r, 0]$ then

$$z(t) \leq \|\phi\|_C + G \leq D + G. \qquad (14)$$

$t^* \in [-r, T]$. In view of inequality (13) and (14), we can say that for $t^* \in [-r, T]$, the inequality (13) holds good. Applying the impulsive inequality given in lemma to equation (13), We get,

$$\begin{aligned}
z(t) &\leq [M(D+G) + N\|\eta\|] + \prod_{0<\tau_k<t}(1+(M+N)L_k^*)exp\left\{\int_0^t K_0R(s)ds + \int_0^t\int_0^s [K_0R(s)R(\tau)d\tau]ds\right\} \\
&\leq [M(D+G) + N\|\eta\|]\prod_{0<\tau_k<t}(1+(M+N)L_k^*)exp\left\{K_0R^*T + K_0(R^*)^2\frac{T^2}{2}\right\} \\
&= Q',
\end{aligned}$$

where Q' is some constant. Therefore, we have $\|u\|_B = \sup\{\|u(t)\| : t \in [-r, T]\} \leq Q'$. Now, we rewrite solution of initial value problem (1)-(5) as follows: For $\phi \in C$, define $\widehat{\phi} \in B$ by

$$\widehat{\phi}(t) = \begin{cases} \phi(t) - (g(u_{t_1}, \dots, u_{t_p}))(t) & \text{if } -r \leq t \leq 0 \\ C(t)[\phi(0) - (g(u_{t_1}, \dots, u_{t_p}))(0)] & \text{if } 0 < t \leq T \end{cases} \qquad (15)$$

If $v \in B$ and $u(t) = v(t) + \widehat{\phi}(t)$, $t \in [-r, T]$, then it is easy to see that v satisfies

$$v(t) = v_0 = 0; \quad -r \leq t \leq 0 \quad \text{and} \qquad (16)$$

$$\begin{aligned}
v(t) &= S(t)\eta + \int_0^t S(t-s)f\left(s, v_s + \widehat{\phi}_s, \int_0^s k(s,\tau)h(\tau, v_\tau + \widehat{\phi}_\tau)d\tau\right)ds \\
&+ \sum_{0<\tau_k<t}\big[C(t-\tau_k)I_k(v(\tau_k) + \widehat{\phi}(\tau_k)) - S(t-\tau_k)\bar{I}_k(v(\tau_k) + \widehat{\phi}(\tau_k))\big], \quad t \in (0, T] \qquad (17)
\end{aligned}$$

if and only if $u(t)$ satisfies the equivalent integro- differential equation to (1)-(5)

$$\begin{aligned}
u(t) &= C(t)[\phi(0) - (g(u_{t_1}, \dots, u_{t_p}))(0)] + S(t)\eta \\
&+ \int_0^t S(t-s)f\big(s, u_s, \int_0^s k(s,\tau)h(\tau, u_\tau)d\tau\big)ds \\
&+ \sum_{0<\tau_k<t}\big[C(t-\tau_k)I_k u(\tau_k) - S(t-\tau_k)\bar{I}_k u(\tau_k)\big], \quad t \in (0, T]. \qquad (18)
\end{aligned}$$

$$u(t) = \phi(t) - (g(u_{t_1}, \dots, u_{t_p}))(t), \quad -r \leq t \leq 0, \qquad (19)$$

$$u'(0) = \eta, \quad \eta \in X \qquad (20)$$

$$\Delta u(\tau_k) = I_k u(\tau_k), \quad k = 1, \dots, m, \qquad (21)$$

$$\Delta u'(\tau_k) = \bar{I}_k u(\tau_k), \quad k = 1, \dots, m. \qquad (22)$$

We define an operator $F : B_0 \to B_0$, $B_0 = \{v \in B : v_0 = 0\}$ by

$$(Fv)(t) = \begin{cases} 0 \quad \text{if} \qquad -r \le t \le 0 \\ S(t)\eta + \int_0^t S(t-s) f\left(s, v_s + \widehat{\phi}_s, \int_0^s k(s,\tau)h(\tau, v_\tau + \widehat{\phi}_\tau)d\tau\right) ds \\ + \sum\limits_{0<\tau_k<t} \left[C(t-\tau_k)I_k(v(\tau_k) + \widehat{\phi}(\tau_k)) - S(t-\tau_k)\bar{I}_k(v(\tau_k) + \widehat{\phi}(\tau_k))\right] \text{ if } t \in (0,T]. \end{cases} \tag{23}$$

From the definition of an operator F defined by the equation (23), it is to be noted that the equations (18)–(22) can be written as

$$v = Fv$$

and the integral equations (6)–(10) can be written as

$$v = \lambda F v.$$

First, we show that $F : B_0 \to B_0$ is continuous. Let $\{x_n\}$ be a sequence of elements of B_0 converging to x in B_0. Then there exists an integer J such that $\|x_n(t)\| \le J$ for all n and $t \in [0,T]$. So $x_n \in B_0$ and $x \in B_0$. Then by using hypothesis (H_4), we have

$$f(t, x_{n_t} + \widehat{\phi}_t, \int_0^t k(t,s)h(s, x_{n_s} + \widehat{\phi}_s)ds) \to f(t, x_t + \widehat{\phi}_t, \int_0^t k(t,s)h(s, x_s + \widehat{\phi}_s)ds)$$

and since I_k and $\bar{I}_k$ are continuous, we get, for each $t \in [0,T]$. Since $\|f(t, x_{n_t} + \widehat{\phi}_t, \int_0^t k(t,s)h(s, x_{n_s} + \widehat{\phi}_s)ds)\| - \|f(t, x_t + \widehat{\phi}_t, \int_0^t k(t,s)h(s, x_s + \widehat{\phi}_s)ds)\| \le 2h_{J'(t)}$ where $J' = \max\{J + \|\widehat{\phi}\|, (M+N)R^*[J + \|\widehat{\phi}\|]\}$. Then by dominated convergence theorem, we have

$$\begin{aligned} \|(Fx_n)(t) - (Fx)(t)\| \le \|S(t)\eta - S(t)\eta\| &+ \int_0^t \|S(t-s)[f(s, x_{n_s} + \widehat{\phi}_s, \int_0^s k(s,\tau)h(\tau, x_{n_\tau} + \widehat{\phi}_\tau)d\tau) \\ &- f(t, x_s + \widehat{\phi}_s, \int_0^s k(s,\tau)h(\tau, x_\tau + \widehat{\phi}_\tau)d\tau)]\|ds \\ &+ \|\sum_{0<\tau_k<t} \left[C(t-\tau_k)I_k(x_n(\tau_k) + \widehat{\phi}(\tau_k) - S(t-\tau_k)\bar{I}_k(x_n(\tau_k) + \widehat{\phi}(\tau_k)\right] \\ &- \sum_{0<\tau_k<t} \left[C(t-\tau_k)I_k(x(\tau_k) + \widehat{\phi}(\tau_k) - S(t-\tau_k)\bar{I}_k(x(\tau_k) + \widehat{\phi}(\tau_k)\right]\| \\ &\to 0 \quad \text{as} \quad n \to \infty, \forall \quad t \in [0,T]. \end{aligned}$$

Since, $\|Fx_n - Fx\|_B = \sup\limits_{t\in[-r,T]} \|(Fx_n)(t) - (Fx)(t)\|$, it follows that $\|Fu_n - Fu\|_B \to 0$ as $n \to \infty$ which implies $Fx_n \to Fx$ in B_0 as $x_n \to x$ in B_0. Therefore, F is continuous. Now, we prove that F is completely continuous i.e. F maps a bounded set of B_0 into a precompact set of B_0. Let $B_m = \{v \in B_0 : \|v\|_B \le m\}$ for $m \ge 1$. We show that F_{B_m} is uniformly bounded. Let $R^* = \sup\{R(t) : t \in [0,T]\}$ and $\|\phi\|_C \le D + G$. Then from the equation (23) and using hypotheses $(H_1) - (H_5)$ and the fact that $\|v\|_B \le m, v \in B_m$ implies $\|v_t\|_C \le m, t \in [0,T]$. We obtain,

$$\begin{aligned} \|(Fv)(t)\| \le \|S(t)\eta\| &+ \int_0^t \|S(t-s)f(s, v_s + \widehat{\phi}_s, \int_0^s k(s,\tau)h(\tau, v_\tau + \widehat{\phi}_\tau)d\tau)\|ds \\ &+ \sum_{0<\tau_k<t} \|[C(t-\tau_k)I_k(v(\tau_k) + \widehat{\phi}(\tau_k)) - S(t-\tau_k)\bar{I}_k(v(\tau_k) + \widehat{\phi}(\tau_k)]\| \\ &\le N\|\eta\| + NTR^*[m + D + G + \frac{LTR^*}{2}(m + D + G)] \\ &+ \sum_{0<\tau_k<t} (M+N)L_k^*[m + D + G]. \end{aligned}$$

This implies that the set $\{(Fv)(t) : \|v\|_B \leq m, -r \leq t \leq T\}$ is uniformly bounded in X and hence F_{B_m} is uniformly bounded. Now we show that F maps B_m into an equicontinuous family of functions with values in X. Let $v \in B_m$ and $t_1, t_2 \in [-r, T]$. Then from the equation (23) and using the hypotheses $(H_1) - (H_5)$ and we have,

Case 1: Suppose $0 \leq t_1 \leq t_2 \leq T$

$$\begin{aligned}
&\|(Fv)(t_2) - (Fv)(t_1)\| \\
&\leq \|S(t_2) - S(t_1)\|\|\eta\| + [\int_0^{t_1} \|S(t_2 - s) - S(t_1 - s)\| R^*[m + D + G + R^*T(m + D + G)]ds \\
&+ \int_{t_1}^{t_2} \|S(t_2 - s)\| R^*[m + D + G + R^*T(m + D + G)]ds] \\
&+ \sum_{0<\tau_k<t_1} \big[\|C(t_2 - \tau_k) - C(t_1 - \tau_k)\| + \|S(t_2 - \tau_k) - S(t_1 - \tau_k)\|\big] L_k^*(m + D + G) \\
&+ \sum_{t_1<\tau_k<t_1} \big[\|C(t_2 - \tau_k) - S(t_2 - \tau_k)\|\big] L_k^*(m + D + G)
\end{aligned}$$

Case 2: Suppose $-r \leq t_1 \leq 0 \leq t_2 \leq T$.
Proceeding as in Case 1, we get

$$\begin{aligned}
&\|(Fv)(t_2) - (Fv)(t_1)\| \\
&\leq \|S(t_2)\|\|\eta\| + \|S(t_2 - s)\| R^*[m + D + G + TR^*(m + D + G)]ds \\
&+ \sum_{0<\tau_k<t_2} \big[\|C(t_2 - \tau_k)\| + \|S(t_2 - \tau_k)\|\big] L_k^*(m + c_1 + G)
\end{aligned}$$

Case 3: Suppose $-r \leq t_1 \leq t_2 \leq 0$. Then

$$\|(Fv)(t_2) - (Fv)(t_1)\| = 0$$

The right hand side in the cases 1-3 are independent of $v \in B_m$ and tends to zero as $t_2 - t_1 \to 0$, since the compactness of cosine operator families for $t > 0$ implies continuity in uniform operator topology. Thus F maps B_m into an equicontinuous family of functions with values in X. We have already shown that F_{B_m} is an equicontinuous and uniformly bounded collection. To prove the set F_{B_m} is precompact in B, it is sufficient, by Arzela-Ascoli's argument, to show that F maps B_m into a precompact set in X.

Let $0 < t \leq T$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. Moreover for $v \in B_m$, we define

$$\begin{aligned}
(F_\epsilon v)(t) &= S(t)\eta + \int_0^{t-\epsilon} S(t - s) f(s, v_s + \widehat{\phi}_s, \int_0^s k(s, \tau) h(\tau, v_\tau + \widehat{\phi}_\tau) d\tau ds \\
&+ \sum_{0<\tau_k<t-\epsilon} \big[C(t - \tau_k) I_k(v(\tau_k) + \widehat{\phi}(\tau_k)) - S(t - \tau_k) \bar{I}_k(v(\tau_k) + \widehat{\phi}(\tau_k))\big]
\end{aligned}$$

Since $T(t)$ is the compact operator ,the set $Y_\epsilon(t) = \{(F_\epsilon v)(t) : v \in B_m\}$ is precompact in X for every ϵ, $0 < \epsilon < t$. Moreover for every $v \in B_m$, we have

$$\begin{aligned}
&(Fv)(t) - (F_\epsilon v)(t) \\
&= \int_{t-\epsilon}^{t} S(t - s) f(s, v_s + \widehat{\phi}_s, \int_0^s k(s, \tau) h(\tau, v_\tau + \widehat{\phi}_\tau) d\tau) ds \\
&+ \sum_{t-\epsilon<\tau_k<t} \big[C(t - \tau_k) I_k(v(\tau_k) + \widehat{\phi}(\tau_k)) - S(t - \tau_k) \bar{I}_k(v(\tau_k) + \widehat{\phi}(\tau_k))\big]
\end{aligned}$$

By making use of hypotheses $(H_1) - (H_5)$ and the fact that $\|v\|_B \leq m, v \in B_m$ implies $\|v_t\|_C \leq m, t \in [0, T]$, we have

$$\begin{aligned}
&\|(Fv)(t) - (F_\epsilon v)(t))\| \\
&\leq \int_{t-\epsilon}^{t} \|S(t-s)f(s, v_s + \widehat{\phi}_s, \int_0^s k(s,\tau)h(\tau, v_\tau + \widehat{\phi}_\tau)d\tau\|ds \\
&+ \sum_{t-\epsilon<\tau_k<t} \|\big[C(t-\tau_k)I_k(v(\tau_k) + \widehat{\phi}(\tau_k)) - S(t-\tau_k)\bar{I}_k(v(\tau_k) + \widehat{\phi}(\tau_k))\big]\| \\
&\leq MR^*\{m + D + G + TR^*[(m + D + G)]\}\epsilon + \sum_{t-\epsilon<\tau_k<t} (M+N)L_k^*(m + D + G)
\end{aligned}$$

This shows that there exists precompact sets arbitrarily close to the set $\{(Fv)(t) : v \in B_m\}$. Hence the set $\{(Fv)(t) : v \in B_m\}$ is precompact in X. This complete the proof that F is completely continuous operator. Moreover, the set

$$\varepsilon(F) = \{v \in B_0 : v = \lambda F v, \quad 0 < \lambda < 1\},$$

is bounded in B, since for every v in $\varepsilon(F)$, the function $x(t) = v(t) + \widehat{\phi}(t)$ is a solution of initial value problem $(6)_\lambda$-(10) for which we have proved that $\|x\|_B \leq Q'$ and hence $\|v\|_B \leq Q' + D + G$. Now, by virtue of Theorem 2.4, the operator F has a fixed point $\tilde{v}$ in B_0. Then $\tilde{u} = \tilde{v} + \widehat{\phi}$ is a solution of the initial value problem (1)-(5). ■

EXISTENCE OF POSITIVE SOLUTIONS

To prove that positive solution by Legget-Williams fixed point theorem, we define a concave non-negative continuous functional $\psi : \mathcal{C} \to [0, \infty)$ with
$\psi(\lambda x + (1-\lambda)y) \geq \lambda\psi(x) + (1-\lambda)\psi(y), \quad \forall x, y \in \mathcal{C}, \lambda \in [0, 1]$ and the hypotheses defined below:

(H_1') Let f is L^1 Caratheodary, there exists L^1 Caratheodary function p' and contains $\rho" > 0$ with $0 < \widetilde{M} < 1$ such that $\|f(t, u, v)\| < \widetilde{M}p'(t)$, for a. e.$t \in [0, T]$
$\|C(t)(\phi - g)\| + \|S(t)\eta\| + N\widetilde{M}\int_0^T p'(t)dt + \sum(L_k + \bar{L}_k) < \rho"$

(H_2') There exists $L" > \rho"$, $\widetilde{M} \leq \widetilde{M}_1 < 1$ and the interval $[a, b] \subset (0, T)$ such that
$\min_{t\in[a,b]} \Big[C(t)(\phi(0) - T(t)(g(u_{t_1}, \ldots, u_{t_p}))(0)) + S(t)\eta$
$+ \int_0^t S(t-s)f(s, u_s, \int_0^s k(s,\tau)h(\tau, u_\tau)d\tau)ds + \sum_{0<\tau_k<t} \big[C(t-\tau_k)I_k u(\tau_k) - S(t-\tau_k)\bar{I}_k u(\tau_k)\big]\Big] \geq \widetilde{M}_1\{C(t)(\phi(0) - T(t)(g(u_{t_1}, \ldots, u_{t_p}))(0)) + S(t)\eta$
$+ \int_0^T S(t-s)f(s, u_s, \int_0^s k(s,\tau)h(\tau, u_\tau)d\tau)ds + \sum_{0<\tau_k<t} \big[C(t-\tau_k)I_k u(\tau_k) - S(t-\tau_k)\bar{I}_k u(\tau_k)\big]\} > L"$

(H_3') Let $I_k, \bar{I}_k : X \to X$ are functions such that there exists constants L_k and $\bar{L}_k$
$\|I_k(u)\| \leq L_k, \quad u \in X, \quad k = 1, 2, \ldots, m.$
$\|\bar{I}_k(u)\| \leq \bar{L}_k$

(H_4') There exists $R" > L"$ and $\widetilde{M}_2$ with $\widetilde{M} \leq \widetilde{M}_2 < 1$ such that
$\|C(t)(\phi - g)\| + \|S(t)\eta\| + N\widetilde{M}_2\int_0^T p'(t)dt + \sum(L_k + \bar{L}_k) < R"$

Theorem 8. If the hypotheses (H_1')-(H_4') hold. Then the nonlocal impulsive IVP(1)–(5) has three positive solutions u_1, u_2 and u_3 with $\|u_1\| < \rho"$, $\|u_2\| > L"$ and $\|u_1\| > \rho"$ with $\min_{t\in[a,b]} u_3(t) < L"$.

Proof. Consider the following well defined operator

$$(F_1u)(t) = \begin{cases} \phi(t) - (g(u_{t_1}, \ldots, u_{t_p}))(t) \quad \text{if} \quad -r \le t \le 0 \\ C(t)[\phi(0) - (g(u_{t_1}, \ldots, u_{t_p}))(0)] + S(t)\eta \\ + \int_0^t S(t-s) f\big(s, u_s, \int_0^s k(s,\tau)h(\tau, u_\tau)d\tau\big)ds \\ + \sum_{0<\tau_k<t} \big[C(t-\tau_k)I_k u(\tau_k) - S(t-\tau_k)\bar{I}_k u(\tau_k)\big] \\ \quad \text{if } t \in [0, T] \end{cases}$$

Now let $\mathcal{C} = \{u \in PC([-r, T], X) : u(t) \ge 0, t \in [-r, T]\}$ be a cone in PC([-r, T], X). Since f, h and I_k, $\bar{I}_k$ all are positive functions, $F_1(\mathcal{C}) \subset \mathcal{C}$. Clearly, the operator F_1 is completely continuous. Let $\psi : \mathcal{C} \to [0, \infty)$ be defined by

$$\psi(u) = \min_{t\in[a,b]} u(t), \quad [a, b] \subset (0, T]$$

ψ is nonnegative concave continuous functional and $\psi(y) \le \|x\|_{PC([-r,T],X)}$. Now, it remains to show that the hypotheses for Leggett-William fixed point theorem holds:

Claim 1:
$\{u \in \mathcal{C}(\psi, L", K") : \psi(u) > L"\} \ne \phi$ and $\psi(F(u)) > L" \quad \forall \mathcal{C}(\psi, L", K")$. Let $K"$ be such that $L"M"^{-1} \le K" \le R"$ and $u(t) = \frac{L"+K"}{2}, \forall t \in [-r, T]$. By the definition of $\mathcal{C}(\psi, L", K"), u \in \mathcal{C}(\psi, L", K")$ then u belongs to $\{u \in \mathcal{C}(\psi, L", K") : \psi(u) > L"\}$ hence it is non empty. Also if $u \in \mathcal{C}(\psi, L", K")$ then by using the hypotheses $[H_2']$ and $[H_3']$, we get

$$\begin{aligned} \psi(F_1(u)) &= \min_{t\in[a,b]}\Big\{C(t)[\phi(0) - (g(u_{t_1}, \ldots, u_{t_p}))(0)] + S(t)\eta \\ &\quad + \int_0^t S(t-s) f\big(s, u_s, \int_0^s k(s,\tau)h(\tau, u_\tau)d\tau\big)ds \\ &\quad + \sum_{0<\tau_k<t} \big[C(t-\tau_k)I_k u(\tau_k) - S(t-\tau_k)\bar{I}_k u(\tau_k)\big]\Big\} \\ &\ge \widetilde{M}_2\{C(t)(\phi(0) - T(t)(g(u_{t_1}, \ldots, u_{t_p}))(0)) \\ &\quad + S(t)\eta \\ &\quad + \int_0^t S(t-s) f(s, u_s, \int_0^s k(s,\tau)h(\tau, u_\tau)d\tau)ds \\ &\quad + \sum_{0<\tau_k<t} \big[C(t-\tau_k)I_k u(\tau_k) - S(t-\tau_k)\bar{I}_k u(\tau_k)\big]\} > L" \end{aligned}$$

Claim 2: $\|F_1(u)\|_{PC} < \rho", \quad \forall u \in C_{\rho"}$.
We use the hypotheses $[H_1']$ and $[H_3']$. Now consider,

$$\begin{aligned} \|F_1(u)\| &\le \|C(t)[\phi(0) - (g(u_{t_1}, \ldots, u_{t_p}))(0)] + S(t)\eta\| \\ &\quad + \|\int_0^t S(t-s) f\big(s, u_s, \int_0^s k(s,\tau)h(\tau, u_\tau)d\tau\big)ds\| \\ &\quad + \|\sum_{0<\tau_k<t} \big[C(t-\tau_k)I_k u(\tau_k) - S(t-\tau_k)\bar{I}_k u(\tau_k)\big]\| \\ &\le \|C(t)(\phi - g)\| + \|S(t)\eta\| + N\widetilde{M}\int_0^T p'(s)ds \\ &\quad + \sum (L_k + \bar{L}_k) \\ &< \rho" \end{aligned}$$

Claim 3: $\psi(F_1(u)) > L"$, for each $u \in \mathcal{C}(\psi, L", R")$ with $\|F_1(u)\| \ge K"$.
Let $u \in \mathcal{C}(\psi, L", R")$ with $\|F_1(u)\| \ge K"$

$$\begin{aligned} \psi(F_1(u)) &= \min_{t\in[a,b]}\Big\{C(t)[\phi(0) - (g(u_{t_1}, \ldots, u_{t_p}))(0)] + S(t)\eta \\ &\quad + \int_0^t S(t-s) f\big(s, u_s, \int_0^s k(s,\tau)h(\tau, u_\tau)d\tau\big)ds \\ &\quad + \sum_{0<\tau_k<t} \big[C(t-\tau_k)I_k u(\tau_k) - S(t-\tau_k)\bar{I}_k u(\tau_k)\big]\Big\} \\ &\ge \widetilde{M}_2\{C(t)(\phi(0) - T(t)(g(u_{t_1}, \ldots, u_{t_p}))(0)) + S(t)\eta \\ &\quad + \int_0^t S(t-s) f(s, u_s, \int_0^s k(s,\tau)h(\tau, u_\tau)d\tau)ds \\ &\quad + \sum_{0<\tau_k<t} \big[C(t-\tau_k)I_k u(\tau_k) - S(t-\tau_k)\bar{I}_k u(\tau_k)\big]\} \\ &= \widetilde{M}_2\|F_1(u)\| > \widetilde{M}_2 K" \ge L" \end{aligned}$$

The operator F_1 satisfies all the hypotheses of Leggett–Williams fixed point theorem. Therefore F has at least three fixed points u_1, u_2 and u_3 with $u_1 \in C_{\rho"}, u_2 \in \{u \in \mathcal{C}(\psi, L", R") : \psi(u) \ge L"\}$ and $u_3 \in C_{R"} - \{\mathcal{C}(\psi, L", R") \cup C_{\rho"}\}$. ∎

REFERENCES

[1] Rupali S. Jain, M. B. Dhakne; On mild solutions of second order semilinear functional integrodifferential equations, Fasciculi Mathematici, Nr 3, 2014, 65–75.

[2] S. D. Kadam, R. S. Jain and B. Surendranath Reddy; On the global existence of mild solutions of impulsive integro-differential equations with nonlocal condition, Open J. of Applied and theoretical mathematics, Vol. 2(4), Dec 2016, 1027–1040.

[3] Travis C. C. , Webb G. F.; An abstract second order semilinear voltera integrodifferential equations, SIAM J. Math Anal., 10 (1979), 412–424.

[4] R. P. Agarwal, D. O'Regan: Multiple positive solutions for second order differential equations, Applied Mathematics and Computation, 114(2000), no. 1, 51–59.

[5] M. Benchora, A. Ouahab, J. Henderson and S. Ntouyas; A note on multiple solutions of impulsive functional differential equations, Communications on Applied Nonlinear Ananlysis, 12(2005), no. 3, 61–70.

[6] E. Liz and J. J. Nieto, Positive solutions of linear impulsive differential equations, Communications in Applied Analysis, 2(1998), no. 4, 565–571.

[7] Williams Legett, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, Indiana University Mathematics Journal, 428, 1979, 673–688.

[8] Fattorini H. O.; Second order linear diff. equations in Banach spaces, North Holland Mathematics Studies, 1985.

[9] Travis C. C., Webb G. F.; Cosine families and abstract nonlinear second order differential equations, Acta Mathematica Academiae Scientiarum Hungaricae 32(1978), 75–96, https://doi.org/10.1007/BF01902205.

[10] V. Lakshmikanthamm, D. D. Bainov and P. S. Simeonov; Theory of impulsive differntial equations, World Scientific, Singapore 1889.

[11] Drumi Bainov and Pavel Simeonov; Impusive Differential Equations: Periodic Solutions and Applications, John Wiley and Sons, Inc. New York 1993.

[12] Drumi D. Bainov and Snezhana G. Haristova; Integral Inequalities of Gronwall type for piecewise continuous functions, J. of Applied Mathematics and Stochastic Analysis, 10:1(1997), 89–94.