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r-Generalization of Phi Functions For The Subsets Of {m,m+1,…,n}

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Abstract

A nonempty finite set A of positive integers is r-relatively prime if greatest r[th] power common divisor of elements of A is 1. In this case we write $\gcd_r(A)=1$. Let $f^{(r)}(m,n)$ be the number of r-relatively prime subsets of $\{m, m+1, ..., n\}$ and the number of sets in $f^{(r)}(m,n)$ of cardinality k is $f_k^{(r)}(m,n)$. The number of nonempty subsets which are r-relatively prime to n is $\Phi^{(r)}(m,n)$ and the number of sets in $\Phi^{(r)}(m,n)$ of cardinality k is $\Phi_k^{(r)}(m,n)$. We obtained exact formulae and asymptotic estimates for these functions $f^{(r)}(m,n)$, $f_k^{(r)}(m,n)$, $\Phi^{(r)}(m,n)$ and $\Phi_k^{(r)}(m,n)$ in [4]. In this paper we find simple explicit formulae for these four functions which simplify the results in [4] and also find the asymptotic estimates for these functions.

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INTRODUCTION

Let A be a nonempty subset of {1,2,…,n}. The greatest common divisor of elements of A is denoted as $\gcd(A)$. We say that A is relatively prime if $\gcd(A)=1$, and that A is relatively prime to n if $\gcd(A\cup n)=1$. Nathanson [1] defined $f(n)$ is the number of relatively prime subsets of {1,2,…,n} and for $k\geq 1$, $f_k(n)$ is the number of sets in $f(n)$ of cardinality k. The number of nonempty subsets which are relatively prime to n is $\Phi(n)$ and the number of sets in $\Phi(n)$ of cardinality k is $\Phi_k(n)$. M.El.Bachraoui[3] generalized these four functions for the set $\{m, m+1, ..., n\}$. The set A is r-relatively prime if the greatest r[th] power common divisor of elements of A is 1. In this case we write $\gcd_r(A)=1$. The set A is r-relatively prime to n if the greatest r[th] power common divisor of elements of A and n is 1. In this case we write $(\gcd_r(A),n)_r=1$. In [5] we defined the following functions:

$$f^{(r)}(n)=\#\{A\subseteq\{1,2,...,n\}: A\neq\phi, \gcd_r(A)=1\}$$

$$f_k^{(r)}(n)=\#\{A\subseteq\{1,2,...,n\}: \#A=k, \gcd_r(A)=1\}$$

$$\Phi^{(r)}(n)=\#\{A\subseteq\{1,2,...,n\}: A\neq\phi, (\gcd_r(A),n)_r=1\}$$

$$\Phi_k^{(r)}(n)=\#\{A\subseteq\{1,2,...,n\}: \#A=k, (\gcd_r(A),n)_r=1\}$$

and obtained the exact formulae and asymptotic estimates for these functions in [5]. We generalized these four functions for the set $\{m, m+1, ..., n\}$ where $n\geq m$, and obtained exact formulae for the functions $f^{(r)}(m,n)$, $f_k^{(r)}(m,n)$, $\Phi^{(r)}(m,n)$ and $\Phi_k^{(r)}(m,n)$ in [4]. In the present paper we further simplify the exact formulae which are obtained in [4] and find the asymptotic estimates for these four functions.

DEFINITIONS

$$f^{(r)}(m,n)=\#\{A\subseteq\{m,m+1,...,n\}: A\neq\phi, \gcd_r(A)=1\}$$

$$f_k^{(r)}(m,n)=\#\{A\subseteq\{m,m+1,...,n\}: \#A=k, \gcd_r(A)=1\}$$

$$\Phi^{(r)}(m,n)=\#\{A\subseteq\{m,m+1,...,n\}: A\neq\phi, (\gcd_r(A),n)_r=1\}$$

$$\Phi_k^{(r)}(m,n)=\#\{A\subseteq\{m,m+1,...,n\}: \#A=k, (\gcd_r(A),n)_r=1\}$$

We obtain the explicit formulae and asymptotic estimates for these four functions. The following inequality is used.

$$[x]-[y]\leq[x-y]+1$$

Theorem 1 : Let m,n be non-negative integers. Then for $m < n$,

(i) $$f^{(r)}(m,\ n) = \sum_{1 \le d^r \le n} \mu_r\left(d^r\right)\left(2^{\left\lfloor \frac{n}{d^r} \right\rfloor - \left\lfloor \frac{m-1}{d^r} \right\rfloor} - 1\right)$$

(ii) $$0 \le 2^{n-m+1} - 2^{\left\lfloor \frac{n}{2^r} \right\rfloor - \left\lfloor \frac{m-1}{2^r} \right\rfloor} - f^{(r)}(m,\ n) \le 2n \cdot 2^{\left\lfloor \frac{n-m+1}{3^r} \right\rfloor}$$

if $\{m,\ m+1,\ ...,\ n\}$ contains multiples of 2^r, and

$$0 \le -2^{n-m+1} - f^{(r)}(m,\ n) \le 2n \cdot 2^{\left\lfloor \frac{n-m+1}{3^r} \right\rfloor} + 2^{\left\lfloor \frac{n}{2^r} \right\rfloor - \left\lfloor \frac{m-1}{2^r} \right\rfloor}$$

if $\{m,\ m+1,\ ...,\ n\}$ has no multiples of 2^r.

Proof **: (i)** We have proved in [4], that

$$f^{(r)}(m,\ n) = \sum_{1 \le d^r \le n} \mu_r\left(d^r\right)\left(2^{\left\lfloor \frac{n}{d^r} \right\rfloor} - 1\right) - \sum_{i=1}^{m-1}\left(\sum_{d^r \mid i} \mu_r\left(d^r\right) 2^{\left\lfloor \frac{n}{d^r} \right\rfloor - \left\lfloor \frac{i}{d^r} \right\rfloor}\right)$$

Which can be written as

$$f^{(r)}(m,\ n) = \sum_{1 \le d^r \le n} \mu_r\left(d^r\right)\left(2^{\left\lfloor \frac{n}{d^r} \right\rfloor} - 1\right) - \sum_{1 \le d^r \le m-1} \mu_r\left(d^r\right) 2^{\left\lfloor \frac{n}{d^r} \right\rfloor} \sum_{i=1}^{m-1} 2^{-\frac{i}{d^r}}$$

$$= \sum_{1 \le d^r \le n} \mu_r\left(d^r\right)\left(2^{\left[\frac{n}{d^r} \right]} - 1\right) - \sum_{1 \le d^r \le m-1} \mu_r\left(d^r\right) 2^{\left[\frac{n}{d^r} \right]}\left(\sum_{j=1}^{\left\lfloor \frac{m-1}{d^r} \right\rfloor} 2^{-j}\right)$$

$$= \sum_{1 \le d^r \le n} \mu_r\left(d^r\right) 2^{\left[\frac{n}{d^r} \right]}\left(1 - \sum_{j=1}^{\left\lfloor \frac{m-1}{d^r} \right\rfloor} 2^{-j}\right) - \sum_{1 \le d^r \le n} \mu_r\left(d^r\right)$$

Note that $\left[\frac{m-1}{d^r} \right] = 0$ if $m \le d^r \le n$

$$= \sum_{1 \le d^r \le n} \mu_r\left(d^r\right) 2^{\left[\frac{n}{d^r} \right]}\left[1 - \left(1 - 2^{-\left[\frac{m-1}{d^r} \right]}\right)\right] - \sum_{1 \le d^r \le n} \mu_r\left(d^r\right)$$

$$= \sum_{1 \le d^r \le n} \mu_r\left(d^r\right)\left(2^{\left[\frac{n}{d^r}\right]-\left[\frac{m-1}{d^r}\right]}-1\right).$$

(ii) Let $1 \le d^r \le n.$ Then $m \le a \le n$ and $d^r \mid a$ if and only if

$$\left[\frac{m}{d^r}\right] \le \frac{a}{d^r} \le \left[\frac{n}{d^r}\right].$$

Which gives that $A \subseteq \{m,\ m+1,\ ...,\ n\}$ and $\gcd_r(A) = d^r$ if and only if

$A^1 = \frac{1}{d^r} * A \subseteq \left\{\left[\frac{m}{d^r}\right],\ \left[\frac{m}{d^r}\right]+1,\ ...,\ \left[\frac{n}{d^r}\right]\right\}$ and $\gcd_r\left(A^1\right) = 1.$ Therefore

$$2^{n-(m-1)}-1 = \sum_{1 \le d^r \le n} f^{(r)}\left(\left[\frac{m}{d^r}\right],\ \left[\frac{n}{d^r}\right]\right).$$

$$\Rightarrow 2^{n-(m-1)}-1 = f^{(r)}(m,\ n) + f^{(r)}\left(\left[\frac{m}{2^r}\right],\ \left[\frac{n}{2^r}\right]\right) + \sum_{3 \le d^r \le n} f^{(r)}\left(\left[\frac{m}{d^r}\right],\ \left[\frac{n}{d^r}\right]\right)$$

$$\Rightarrow 2^{n-(m-1)}-1 \le f^{(r)}(m,\ n) + \left(2^{\left[\frac{n}{2^r}\right]-\left[\frac{m-1}{2^r}\right]}-1\right) + \sum_{3 \le d^r \le n} 2^{\left(\left[\frac{n}{d^r}\right]-\left[\frac{m-1}{d^r}\right]\right)}$$

$$\Rightarrow 2^{n-(m-1)} \le f^{(r)}(m,\ n) + 2^{\left[\frac{n}{2^r}\right]-\left[\frac{m-1}{2^r}\right]} + n \cdot 2^{\left[\frac{n}{3^r}\right]-\left[\frac{m-1}{3^r}\right]}$$

$$\le f^{(r)}(m,\ n) + 2^{\left[\frac{n}{2^r}\right]-\left[\frac{m-1}{2^r}\right]} + n \cdot 2^{\left[\frac{n-m+1}{3^r}\right]+1}$$

Since $[x]-[y] \le [x-y]+1.$

$$2^{n-m+1} - 2^{\left[\frac{n}{2^r}\right]-\left[\frac{m-1}{2^r}\right]} - 2n \cdot 2^{\left[\frac{n-m+1}{3^r}\right]} \le f^{(r)}(m,\ n) \qquad \text{……….} \ (1)$$

and hence the lower bound for $f^{(r)}(m,\ n)$ is obtained.

The upper bound for $f^{(r)}(m,\ n)$ is obtained as follows **:**

If the set $\{m,\ m+1,\ ...,\ n\}$ contains multiples of $2^r,$ then

$$f^{(r)}(m,\ n) \le 2^{n-m+1} - 2^{\left\lfloor \frac{n}{2^r} \right\rfloor - \left\lfloor \frac{m-1}{2^r} \right\rfloor}$$

$$\Rightarrow 0 \le 2^{n-m+1} - 2^{\left\lfloor \frac{n}{2^r} \right\rfloor - \left\lfloor \frac{m-1}{2^r} \right\rfloor} - f^{(r)}(m,\ n). \qquad \text{………} (2)$$

From equations (1) and (2)

$$\Rightarrow 0 \le 2^{n-m+1} - 2^{\left\lfloor \frac{n}{2^r} \right\rfloor - \left\lfloor \frac{m-1}{2^r} \right\rfloor} - f^{(r)}(m,\ n) \le 2n \cdot 2^{\left\lfloor \frac{n-m+1}{3^r} \right\rfloor}$$

If the set $\{m,\ m+1,\ ...,\ n\}$ has no multiples of 2^r, then

$$f^{(r)}(m,\ n) \le 2^{n-m+1}.$$

Hence

$$0 \le 2^{n-m+1} - f^{(r)}(m,\ n) \le 2n \cdot 2^{\left\lfloor \frac{n-m+1}{3^r} \right\rfloor} + 2^{\left\lfloor \frac{n}{2^r} \right\rfloor - \left\lfloor \frac{m-1}{2^r} \right\rfloor}$$

$$\le 2n \cdot 2^{\left\lfloor \frac{n-m+1}{3^r} \right\rfloor} + 2 \cdot 2^{\left\lfloor \frac{n-m+1}{2^r} \right\rfloor}$$

$$= 2\left[n \cdot 2^{\left\lfloor \frac{n-m+1}{3^r} \right\rfloor} + 2^{\left\lfloor \frac{n-m+1}{2^r} \right\rfloor} \right].$$

Theorem 2 : Let m,n be non-negative integers. Then for $m < n$, $k \ge 1$,

(i) $$f_k^{(r)}(m,\ n) = \sum_{1 \le d^r \le n} \mu_r\left(d^r\right) \binom{\left\lfloor \frac{n}{d^r} \right\rfloor - \left\lfloor \frac{m-1}{d^r} \right\rfloor}{k}$$

(ii) $$0 \le \binom{n-m+1}{k} - \binom{\left\lfloor \frac{n}{2^r} \right\rfloor - \left\lfloor \frac{m-1}{2^r} \right\rfloor}{k} - f_k^{(r)}(m,\ n) \le n \binom{\left\lfloor \frac{n-m+1}{2^r} \right\rfloor + 1}{k}$$

if $\{m,\ m+1,\ ...,\ n\}$ contains multiplies of 2^r and

$$0 \le \binom{n-m+1}{k} - f_k^{(r)}(m,\ n) \le n \binom{\left\lfloor \frac{n-m+1}{2^r} \right\rfloor + 1}{k}$$

if $\{m,\ m+1,\ ...,\ n\}$ does not contain multiples of 2^r.

Proof : (i) In [4] we have proved that

$$f_k^{(r)}(m,\ n)=\sum_{1\le d^r\le n}\mu_r\left(d^r\right)\binom{\left\lfloor \frac{n}{d^r}\right\rfloor}{k}-\sum_{i=1}^{m-1}\sum_{d^r\mid i}\mu_r\left(d^r\right)\binom{\left\lfloor \frac{n}{d^r}\right\rfloor-\frac{i}{d^r}}{k-1}$$

For $K\ge 1$ and $0\le M\le N$,we have

$$\binom{N}{K}-\sum_{j=1}^{M}\binom{N-j}{K-1}=\binom{N-M}{\mathrm{K}}$$

$$f_k^{(r)}(m,\ n)=\sum_{1\le d^r\le n}\mu_r\left(d^r\right)\binom{\left\lfloor \frac{n}{d^r}\right\rfloor}{k}-\sum_{1\le d^r\le m-1}\mu_r\left(d^r\right)\sum_{\substack{i=1\\ d^r\mid i}}^{m-1}\binom{\left\lfloor \frac{n}{d^r}\right\rfloor-\frac{i}{d^r}}{k-1}$$

$$=\sum_{1\le d^r\le n}\mu_r\left(d^r\right)\binom{\left[\frac{n}{d^r}\right]}{k}-\sum_{1\le d^r\le m-1}\mu_r\left(d^r\right)\sum_{j=1}^{\left\lfloor\frac{m-1}{d^r}\right\rfloor}\binom{\left[\frac{n}{d^r}\right]-j}{k-1}$$

$$=\sum_{1\le d^r\le m-1}\mu_r\left(d^r\right)\left[\binom{\left[\frac{n}{d^r}\right]}{k}-\sum_{j=1}^{\left\lfloor\frac{m-1}{d^r}\right\rfloor}\binom{\left[\frac{n}{d^r}\right]-j}{k-1}\right]+\sum_{m\le d^r\le n}\mu_r\left(d^r\right)\binom{\left[\frac{n}{d^r}\right]}{k}$$

$$=\sum_{1\le d^r\le m-1}\mu_r\left(d^r\right)\binom{\left\lfloor\frac{n}{d^r}\right\rfloor-\left\lfloor\frac{m-1}{d^r}\right\rfloor}{k}+\sum_{m\le d^r\le n}\mu_r\left(d^r\right)\binom{\left\lfloor\frac{n}{d^r}\right\rfloor-\left\lfloor\frac{m-1}{d^r}\right\rfloor}{k}$$

Note that $\left[\frac{m-1}{d^r}\right]=0$ if $m\le d^r\le n$.

$$=\sum_{1\le d^r\le n}\mu_r\left(d^r\right)\binom{\left\lfloor\frac{n}{d^r}\right\rfloor-\left\lfloor\frac{m-1}{d^r}\right\rfloor}{k}$$. Which proves (i).

(ii) The upper bound for $f_k^{(r)}(m,\ n)$ is obtained by deleting k-element sets of multiples of 2^r if they belong to the set $\{m,\ m+1,\ \ldots,\ n\}$. If the set contains multiples of 2^r, then the upper bound for $f_k^{(r)}(m,\ n)$ is obtained by deleting sets of order k from the set

$$\left\{\left[\frac{m}{2^r}\right],\ \left[\frac{m+1}{2^r}\right],\ \ldots,\ \left[\frac{n}{2^r}\right]\right\}$$

Hence

$$f_k^{(r)}(m,\ n) \le \binom{n-m+1}{k} - \binom{\left\lfloor \frac{n}{2^r} \right\rfloor - \left\lfloor \frac{m-1}{2^r} \right\rfloor}{k}$$

The lower bound for $f_k^{(r)}(m,\ n)$ is obtained as follows:

$$\binom{n-m+1}{k} = \sum_{1 \le d^r \le n} f_k^{(r)}\left(\left[\frac{m}{d^r}\right],\ \left[\frac{n}{d^r}\right]\right)$$

$$\le f_k^{(r)}(m,\ n) + \binom{\left\lfloor \frac{n}{2^r} \right\rfloor - \left\lfloor \frac{m-1}{2^r} \right\rfloor}{k} + \sum_{3^r \le d^r \le n} \binom{\left\lfloor \frac{n}{d^r} \right\rfloor - \left\lfloor \frac{m-1}{d^r} \right\rfloor}{k}$$

$$\le f_k^{(r)}(m,\ n) + \binom{\left\lfloor \frac{n}{2^r} \right\rfloor - \left\lfloor \frac{m-1}{2^r} \right\rfloor}{k} + n\binom{\left\lfloor \frac{n-m+1}{3^r} \right\rfloor + 1}{k}$$

$$\therefore\ 0 \le \binom{n-m+1}{k} - \binom{\left\lfloor \frac{n}{2^r} \right\rfloor - \left\lfloor \frac{m-1}{2^r} \right\rfloor}{k} - f_k^{(r)}(m,\ n) \le n\binom{\left\lfloor \frac{n-m+1}{3^r} \right\rfloor + 1}{k}.$$

If the set $\{m,\ m+1,\ ...,\ n\}$ does not contain multiples of 2^r, then

$$f_k^{(r)}(m,\ n) \le \binom{n-m+1}{k} \Rightarrow\ 0 \le \binom{n-m+1}{k} - f_k^{(r)}(m,\ n).$$

Also

$$\binom{n-m+1}{k} = \sum_{1 \le d^r \le n} f_k^{(r)}\left(\left[\frac{m}{d^r}\right],\ \left[\frac{n}{d^r}\right]\right)$$

$$= f_k^{(r)}(m,\ n) + \sum_{2 \le d^r \le n} f_k^{(r)}\left(\left[\frac{m}{d^r}\right],\ \left[\frac{n}{d^r}\right]\right)$$

$$\le f_k^{(r)}(m,\ n) + n\binom{\left\lfloor \frac{n}{2^r} \right\rfloor - \left\lfloor \frac{m-1}{2^r} \right\rfloor}{k}$$

$$\le f_k^{(r)}(m,\ n) + n\binom{\left\lfloor \frac{n-m+1}{2^r} \right\rfloor + 1}{k}$$

$$\therefore\ 0 \le \binom{n-m+1}{k} - f_k^{(r)}(m,\ n) \le n \binom{\left\lfloor \frac{n-m+1}{2^r} \right\rfloor + 1}{k}.$$

Which proves (ii).

***Theorem* 3:** Let m,n be non-negative integers. Then for, $m < n$

(i) $$\Phi^{(r)}(m,\ n) = \sum_{d^r \mid n} \mu_r\left(d^r\right)\left(2^{\frac{n}{d^r} - \left\lfloor \frac{m-1}{d^r} \right\rfloor}\right)$$

(ii) If p is the smallest prime such that $p^r \mid n$, then

$$0 \le 2^{n-m+1} - 2^{\frac{n}{p^r} - \left\lfloor \frac{m-1}{p^r} \right\rfloor} - \Phi^{(r)}(m,\ n) \le 2n \cdot 2^{\left\lfloor \frac{n-m+1}{p^r} \right\rfloor}.$$

Proof **: (i)** In [4] we obtained

$$\Phi^{(r)}(m,\ n) = \sum_{d^r \mid n} \mu_r\left(d^r\right) 2^{\frac{n}{d^r}} - \sum_{i=1}^{m-1} \sum_{d^r \mid \mathrm{gcd}_r(i,\ n)} \mu_r\left(d^r\right) 2^{\frac{n-i}{d^r}}.$$

Which can be written as

$$\Phi^{(r)}(m,\ n) = \sum_{d^r \mid n} \mu_r\left(d^r\right) 2^{\frac{n}{d^r}} - \sum_{d^r \mid n} \mu_r\left(d^r\right) \sum_{i=1}^{m-1} 2^{\frac{n-i}{d^r}}$$

$$= \sum_{d^r \mid n} \mu_r\left(d^r\right) 2^{\frac{n}{d^r}} - \sum_{d^r \mid n} \mu_r\left(d^r\right) 2^{\frac{n}{d^r}} \sum_{i=1}^{m-1} 2^{\frac{-i}{d^r}}$$

$$= \sum_{d^r \mid n} \mu_r\left(d^r\right) 2^{\frac{n}{d^r}} - \sum_{d^r \mid n} \mu_r\left(d^r\right) 2^{\frac{n}{d^r}} \sum_{j=1}^{\left\lfloor \frac{m-1}{d^r} \right\rfloor} 2^{-j}$$

$$= \sum_{d^r \mid n} \mu_r\left(d^r\right) 2^{\frac{n}{d^r}} \left(1 - \sum_{j=1}^{\left\lfloor \frac{m-1}{d^r} \right\rfloor} 2^{-j}\right)$$

$$= \sum_{d^r \mid n} \mu_r\left(d^r\right) 2^{\frac{n}{d^r}} \left(1 - \left(1 - 2^{-\left\lfloor \frac{m-1}{d^r} \right\rfloor}\right)\right)$$

$$= \sum_{d^r \mid n} \mu_r\left(d^r\right)\left(2^{\frac{n}{d^r}} \cdot 2^{-\left\lfloor \frac{m-1}{d^r} \right\rfloor}\right)$$

$$= \sum_{d^r \mid n} \mu_r\left(d^r\right) 2^{\frac{n}{d^r} - \left\lfloor \frac{m-1}{d^r} \right\rfloor}$$

which proves (i).

(ii) For the smallest prime divisor p of n such that $p^r \mid n$, if we delete all subsets of $\{m, m+1, \ldots, n\}$ whose elements are multiples of p^r, we get

$$\Phi^{(r)}(m, n) \le 2^{n-(m-1)} - 2^{\frac{n}{p^r} - \left\lfloor \frac{m-1}{p^r} \right\rfloor}$$

$$\Rightarrow \ 0 \le 2^{n-(m-1)} - 2^{\frac{n}{p^r} - \left\lfloor \frac{m-1}{p^r} \right\rfloor} - \Phi^{(r)}(m, n).$$

The lower bound for $\Phi^{(r)}(m, n)$ can be obtained as follows:

$$\Phi^{(r)}(m, n) = \sum_{d^r \mid n} \mu_r\left(d^r\right)\left(2^{\frac{n}{d^r} - \left\lfloor \frac{m-1}{d^r} \right\rfloor}\right)$$

$$= \mu_r(1) 2^{n-(m-1)} + \mu_r\left(p^r\right) 2^{\frac{n}{p^r} - \left\lfloor \frac{m-1}{p^r} \right\rfloor} + \sum_{\substack{d^r \mid n \\ d > p}} \mu_r\left(d^r\right)\left(2^{\frac{n}{d^r} - \left\lfloor \frac{m-1}{d^r} \right\rfloor}\right)$$

$$\Phi^{(r)}(m, n) = 2^{n-m+1} - 2^{\frac{n}{p^r} - \left\lfloor \frac{m-1}{p^r} \right\rfloor} + \sum_{\substack{d^r \mid n \\ d > p}} \mu_r\left(d^r\right)\left(2^{\frac{n}{d^r} - \left\lfloor \frac{m-1}{d^r} \right\rfloor}\right)$$

$$\Phi^{(r)}(m, n) - 2^{n-m+1} + 2^{\frac{n}{p^r} - \left\lfloor \frac{m-1}{p^r} \right\rfloor} = \sum_{\substack{d^r \mid n \\ d > p}} \mu_r\left(d^r\right)\left(2^{\frac{n}{d^r} - \left\lfloor \frac{m-1}{d^r} \right\rfloor}\right)$$

$$\geq \sum_{\substack{d^r | n \\ d>p}} (-1) 2^{\frac{n}{d^r} - \left\lfloor \frac{m-1}{d^r} \right\rfloor}$$

$$\geq (-1) n \cdot 2^{\left\lfloor \frac{n-m+1}{p^r} \right\rfloor + 1}$$

$$= -2n \cdot 2^{\left\lfloor \frac{n-m+1}{p^r} \right\rfloor}$$

$$\Rightarrow \; 2^{n-m+1} - 2^{\frac{n}{p^r} - \left\lfloor \frac{m-1}{p^r} \right\rfloor} - \Phi^{(r)}(m,\, n) \leq 2n \cdot 2^{\left\lfloor \frac{n-m+1}{p^r} \right\rfloor}$$

$$\therefore \; 0 \leq 2^{n-m+1} - 2^{\frac{n}{p^r} - \left\lfloor \frac{m-1}{p^r} \right\rfloor} - \Phi^{(r)}(m,\, n) \leq 2n \cdot 2^{\left\lfloor \frac{n-m+1}{p^r} \right\rfloor}.$$

which proves (ii).

Theorem 4 : Let m,n be non-negative integers. Then for $m < n$,

(i) $$\Phi_k^{(r)}(m,\, n) = \sum_{d^r | n} \mu_r\left(d^r\right) \binom{\frac{n}{d^r} - \left\lfloor \frac{m-1}{d^r} \right\rfloor}{k}$$

and

(ii) $$0 \leq \binom{n-m+1}{k} - \binom{\frac{n}{p^r} - \left\lfloor \frac{m-1}{p^r} \right\rfloor}{k} - \Phi_k^{(r)}(m,\, n) \leq n \binom{\left\lfloor \frac{n-m+1}{p^r} \right\rfloor + 1}{k}$$

Proof **: (i)** Let p be the smallest prime such that $p^r | n$. In [4] , we obtained

$$\Phi_k^{(r)}(m,\, n) = \sum_{d^r | n} \mu_r\left(d^r\right) \binom{\frac{n}{d^r}}{k} - \sum_{d^r | n} \mu_r\left(d^r\right) \sum_{\substack{i=1 \\ d^r | i}}^{m-1} \binom{\frac{n-i}{d^r}}{k-1}$$

$$= \sum_{d^r | n} \mu_r\left(d^r\right) \left(\binom{\frac{n}{d^r}}{k} - \sum_{j=1}^{\left\lfloor \frac{m-1}{d^r} \right\rfloor} \binom{\frac{n}{d^r} - j}{k-1} \right)$$

$$= \sum_{d^r \mid n} \mu_r\left(d^r\right) \binom{\frac{n}{d^r} - \left\lfloor \frac{m-1}{d^r} \right\rfloor}{k}$$

Note that $\binom{N}{K} - \sum_{j=1}^{M} \binom{N-j}{K-1} = \binom{N-M}{\mathrm{K}}$

which proves (i).

(ii) Consider

$$\Phi_k^{(r)}(m,\, n) = \sum_{d^r \mid n} \mu_r\left(d^r\right) \binom{\frac{n}{d^r} - \left\lfloor \frac{m-1}{d^r} \right\rfloor}{k}$$

$$\geq \mu_r(1) \binom{n-(m-1)}{k} - \binom{\frac{n}{p^r} - \left\lfloor \frac{m-1}{p^r} \right\rfloor}{k} - \sum_{\substack{d^r \mid n \\ d > p}} \binom{\frac{n}{d^r} - \left[\frac{m-1}{d^r} \right]}{k}$$

$$\geq \binom{n-(m-1)}{k} - \binom{\frac{n}{p^r} - \left\lfloor \frac{m-1}{p^r} \right\rfloor}{k} - \sum_{\substack{d^r \mid n \\ d > p}} \binom{\left[\frac{n-m+1}{d^r} \right] + 1}{k}$$

$$\geq \binom{n-(m-1)}{k} - \binom{\frac{n}{p^r} - \left\lfloor \frac{m-1}{p^r} \right\rfloor}{k} - n \cdot \binom{\left\lfloor \frac{n-m+1}{p^r} \right\rfloor + 1}{k}.$$

The upper bound is obtained by deleting k -element sets of $\{m,\, m+1,\, \ldots,\, n\}$ whose elements are multiples of p^r, we get

$$\Phi_k^{(r)}(m,\, n) \leq \binom{n-(m-1)}{k} - \binom{\frac{n}{p^r} - \left\lfloor \frac{m-1}{p^r} \right\rfloor}{k}.$$

$$\Rightarrow\ 0 \leq \binom{n-(m-1)}{k} - \binom{\frac{n}{p^r} - \left\lfloor \frac{m-1}{p^r} \right\rfloor}{k} - \Phi_k^{(r)}(m,\, n) \leq n \binom{\left\lfloor \frac{n-m+1}{p^r} \right\rfloor + 1}{k}$$

which proves (ii).

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