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Degree Square Sum Polynomial of some Special Graphs

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Abstract: *Degree square sum matrix* $DSS(G)$ *of a graph* G is a square matrix of order equal to the order of G with its $(i, j)^{th}$ entry as ${d_i}^2 + {d_j}^2$ if $i \neq j$ and zero otherwise, where d_i and d_j are the degrees of i^{th} and j^{th} vertices of G respectively. In this paper, we obtain the degree square sum polynomial of some *special graphs*.

AMS subject classification: 05C50.
Keywords: Degree square sum matrix, degree square sum polynomial, transformation graph, graph operation.

INTRODUCTION

We consider in this paper, a graph G as a simple, undirected graph with n vertices and m edges. Let $V(G)$ be a vertex set and $E(G)$ be an edge set of G. The *degree* $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident to it in G. The graph G is *r-regular* if the degree of each vertex in G is r. Let $v_1, v_2, \dots, v_n$ be the vertices of G and let $d_i = d_G(v_i)$. For undefined graph theoretic terminologies and notations, we refer [11] or [15].

The literature of Graph Theory has several graph polynomials based on the matrices associated with a graph. In [4], We have introduced *degree square sum matrix* wherein we have given bounds for degree square sum energy of a graph and we have computed the degree square sum polynomial of graphs obtained by some graph operations. In this paper, we obtain the degree square sum polynomial of some special graphs.

The degree square sum matrix of a graph G is an $n \times n$ matrix denoted by $DSS(G) = [dss_{ij}]$ and whose elements are defined as

$$dss_{ij} = \begin{cases} {d_i}^2 + {d_j}^2 & if\ i \neq j, \\ 0 & otherwise. \end{cases}$$

Let I_n be an identity matrix (order n) and J_n be a matrix (order n) whose all entries are equal to 1. The *degree square sum polynomial* of a graph G is defined as

$$P_{DSS(G)}(\mu) = det(\mu I_n - DSS(G)).$$

Let $\mu_1, \mu_2, \dots, \mu_n$ are called the *degree square sum eigenvalues* of G. For an *r-regular* graph of order n,

$$DSS(G) = 2r^2 J_n - 2r^2 I_n$$

and

$$P_{DSS(G)}(\mu) = [\mu - 2r^2(n-1)][\mu + 2r^2]^{n-1}. \quad (1)$$

For more details on degree square sum polynomial of a graph, refer [4].

Lemma 1. [20] If a, b, c and d are real numbers, then the determinant of the form

$$\begin{vmatrix} (\mu + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\mu + b)I_{n_2} - bJ_{n_2} \end{vmatrix} \quad (2)$$

of order $n_1 + n_2$ can be expressed in the simplified form as

$$(\mu+a)^{n_1-1}(\mu+b)^{n_2-1}\{[\mu-(n_1-1)a][\mu-(n_2-1)b]-n_1n_2cd\}.$$

The above lemma is useful for proving the forthcoming theorems.

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DEGREE SQUARE SUM POLYNOMIAL OF SOME CYCLE RELATED AND PRODUCT RELATED GRAPHS

The details of the following graphs can be found in [10].

Let $C_n^{(t)}$ denote the one-point union of t cycles of length n. The graph $C_3^{(t)}$ is called a *Friendship graph.*

Theorem 2. If $C_3^{(t)}$ is a friendship graph of order $2t+1$, then the degree square sum polynomial of $C_3^{(t)}$ is

$$P_{DSS(C_3^{(t)})}(\mu) = (\mu+8)^{n-2}\{[\mu-8(n-2)]-(n-1)[(n-1)^2+4]^2\}.$$

Proof. The graph $C_3^{(t)}$ of order $2t+1$ has two types of verices namely $n-1$ vertices of degree 2 and 1 vertices of degree $n-1$. Hence,

$$DSS(C_3^{(t)}) = \begin{bmatrix} 8(J_{n-1}-I_{n-1}) & (2^2+(n-1)^2)J_{(n-1)\times 1} \\ (2^2+(n-1)^2)J_{1\times(n-1)} & 2(n-1)^2(J_1-I_1) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(C_3^{(t)})}(\mu) &= |\mu I - DSS(C_3^{(t)})| \\ &= \begin{vmatrix} (\mu+8)I_{n-1}-8J_{n-1} & -((n-1)^2+4)J_{(n-1)\times 1} \\ -((n-1)^2+4)J_{1\times(n-1)} & (\mu+2(n-1)^2)I_1-2(n-1)^2J_1 \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

The *helm* H_n is a graph obtained from a wheel W_n with cetral vertex c, by attaching a pendant edge to each rim vertex of W_n. A *closed helm* is the graph with central vertex c, obtained from a helm by joining each pendant vertex to form a cycle.

Theorem 3. If the graph $H_n - c$ of order $2n$ and size $2n$ is a helm without central vertex, then

$$P_{DSS(H_n-c)}(\mu) = (\mu+18)^{n-1}(\mu+2)^{n-1}\{[\mu-18(n-1)(\mu-2(n-1))-100n^2\}.$$

Proof. The helm $H_n - c$ without central vertex is a graph of order $2n$, which has two types of vertices. The n vertices have degree 3 and the remaining n vertices have degree 1. Hence,

$$DSS(H_n-c) = \begin{bmatrix} 18(J_n-I_n) & 10J_{n\times n} \\ 10J_{n\times n} & 2(J_n-I_n) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(H_n-c)}(\mu) &= |\mu I - DSS(H_n-c)| \\ &= \begin{vmatrix} (\mu+18)I_n-18J_n & -10J_{n\times n} \\ -10J_{n\times n} & (\mu+2)I_n-2J_n \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 4. If the graph $H_n' - c$ of order $2n$ and size $3n$ is a closed helm without central vertex, then

$$P_{DSS(H_n'-c)}(\mu) = (\mu-54(2n-1))(\mu+54)^{2n-1}.$$

Proof. The closed helm without central vertex $H'_n - c$ is 3-regular graph with $2n$ vertices. Hence, the result follows from Eq. (1). ■

Remark 5. The graph $W_c = C_n \circ K_1$ is called a *Crown* graph. The *crown* $W_c = C_n \circ K_1$ is a unicyclic graph of order $2n$ and size $2n$. That is the crown is obtained from the helm by removing the central vertex c.

The *sunflower graph* SF_n is a graph obtained from a wheel with central vertex c, n-cycle $v_0, v_1, \dots, v_{n-1}$ and additional n vertices $w_0, w_1, \dots, w_{n-1}$ where w_i is joined by edges to v_i, v_{i+1} for $i = 0, 1, \dots, n-1$ where $i+1$ is taken modulo n.

Theorem 6. If the graph $SF_n - c$ of order $2n$ and size $3n$ is a sunflower graph without central vertex, then

$$P_{DSS(SF_n-c)}(\mu) = (\mu+18)^{n-1}(\mu+8)^{n-1}\{[\mu-18(n-1)][\mu-8(n-1)]-169n^2\}.$$

Proof. The sunflower graph $SF_n - c$ without central vertex is a graph of order $2n$, which has two types of vertices. The n vertices have degree 3 and the remaining n vertices have degree 2. Hence,

$$DSS(SF_n - c) = \begin{bmatrix} 18(J_n - I_n) & 13J_{n\times n} \\ 13J_{n\times n} & 8(J_n - I_n) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(SF_n-c)}(\mu) &= |\mu I - DSS(SF_n - c)| \\ &= \begin{vmatrix} (\mu+18)I_n - 18J_n & -13J_{n\times n} \\ -13J_{n\times n} & (\mu+8)I_n - 8J_n \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

The *lotus inside a circle* LC_n is a graph obtained from the cycle $C_n : w_1w_2\dots w_nw_1$ and a star $K_{1,n}$ with central vertex u and the end vertices $u_i, (1 \le n)$ by joining each u_i to w_i and $w_{i+1(mod\ n)}$.

Remark 7. The lotus inside a circle without central vertex is isomorphic to sunflower without the central vertex.

The *double cone* $DC_n = C_n + 2K_1$ is a graph with $n+2$ vertices and $3n$ edges.

Theorem 8. If DC_n is a double cone of order $n+2$ and size $3n$, then

$$P_{DSS(DC_n)}(\mu) = (\mu+32)^{n-1}(\mu+2n^2)\{[\mu-32(n-1)][\mu-2n^2]-2n(n^2+16)^2\}.$$

Proof. The double cone is a graph of of order $n+2$ has two types of vertices. The n vertices are of degree 4 and the remaining 2 vertices with degree n. Hence,

$$DSS(DC_n) = \begin{bmatrix} 32(J_n - I_n) & (n^2+16)J_{n\times 2} \\ (n^2+16)J_{2\times n} & 2n^2(J_2 - I_2) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(DC_n)}(\mu) &= |\mu I - DSS(DC_n)| \\ &= \begin{vmatrix} (\mu+32)I_n - 32J_n & -(n^2+16)J_{n\times 2} \\ -(n^2+16)J_{2\times n} & (\mu+2n^2)I_2 - 2n^2J_2 \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the expected result. ■

The graph $S_m \times P_2$ (where S_m is a star with $m+1$ vertices) is called a *Book graph* B_m. B_m is a graph of order $2(m+1)$.

Theorem 9. If B_m is a book graph, then

$$P_{DSS(B_m)}(\mu) = (\mu+8)^{2m-1}(\mu+2(m+1)^2)\{[\mu-8(2m-1)][\mu-2(m+1)^2]-4m((m+1)^2+4)^2\}.$$

Proof. The Book graph B_m has two types of vertices. The $2m$ vertices with degree 2 and 2 vertices are with degree $m+1$. Hence,

$$DSS(B_m) = \begin{bmatrix} 8(J_{2m}-I_{2m}) & (2^2+(m+1)^2)J_{2m\times 2} \\ (2^2+(m+1)^2)J_{2\times 2m} & 2(m+1)^2(J_2-I_2) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(B_m)}(\mu) &= |\mu I - DSS(B_m)| \\ &= \begin{vmatrix} (\mu+8)I_{2m}-8J_{2m} & -((m+1)^2+4)J_{2m\times 2} \\ -((m+1)^2+4)J_{2\times 2m} & (\mu+2(m+1)^2)I_2-2(m+1)^2J_2 \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

The graph $P_n \times P_2$ is called a *Ladder graph* L_n.

Theorem 10. If L_n is a ladder graph, then

$$P_{DSS(L_n)}(\mu) = (\mu+18)^{2n-5}(\mu+8)^3\{[\mu-18(2n-5)](\mu-24)-676(2n-4)\}.$$

Proof. The Ladder graph Ln is a graph of order $2n$ and has two types of vertices. The 4 vertices have degree 2 and $2n-4$ vertices have degree 3. Hence,

$$DSS(L_n) = \begin{bmatrix} 18(J_{2n-4}-I_{2n-4}) & 13J_{(2n-4)\times 4} \\ 13J_{4\times(2n-4)} & 8(J_4-I_4) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(L_n)}(\mu) &= |\mu I - DSS(L_n)| \\ &= \begin{vmatrix} (\mu+18)I_{2n-4}-18J_{2n-4} & -13J_{(2n-4)\times 4} \\ -13J_{4\times(2n-4)} & (\mu+8)I_4-8J_4 \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the expected result. ■

The *prism* $\Pi_n = C_n \times P_2$ is a 3-regular graph of order $2n$ and size $3n$

Theorem 11. If the prism Π_n is a graph of order $2n$ and size $3n$, then

$$P_{DSS(\Pi_n)}(\mu) = (\mu-18(2n-1))(\mu+18)^{2n-1}.$$

Proof. The prism Π_n is 3-regular graph with $2n$ vertices. Hence, the result follows from Eq. (1). ■

The *point splitting graph* $S'(G)$ of a graph G is obtained from a graph G by adding for each vertex v of G a new vertex v' so that v' is adjacent to every vertex that is adjacent to v.

Theorem 12. If G is an r-regular graph of order n, then

$$P_{DSS(S'(G))}(\mu) = (\mu + 8r^2)^{n-1}(\mu + 2r^2)n - 1[\mu - 8(n-1)][\mu - 2(n-1)r^2] - 25r^4n^2\}.$$

Proof. The point splitting graph of an r-regular graph is a graph of order $2n$ and has two types of vertices. The n vertices have degree $2r$ and the remaining n vertices have degree r. Hence,

$$DSS(S'(G)) = \begin{bmatrix} 2(2r)^2(J_n - I_n) & ((2r)^2 + r^2)J_{n\times n} \\ ((2r)^2 + r^2)J_{n\times n} & 2r^2(J_n - I_n) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(S'(G))}(\mu) &= |\mu I - DSS(S'(G))| \\ &= \begin{vmatrix} (\mu + 8r^2)I_n - 8r^2J_n & -5r^2J_{n\times m} \\ -5r^2J_{m\times n} & 2r^2I_m - 2r^2J_m \end{vmatrix}. \end{aligned}$$

Now by using Lemma 1, we get the required result. ■

The line splitting graph[14] $L_s(G)$ of a graph G is a graph with vertex set $E(G) \cup E_1(G)$ with two vertices adjacent if they correspond to adjacent edges of G or one corresponds to an element e'_i of $E_1(G)$ and the other to an element e_j of $E(G)$ where e_j is in $N(e_i)$.

Theorem 13. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(L_s(G))}(\mu) &= [\mu + 2(2r-2)^2]^{m-1}[\mu + 2(4r-4)^2]^{m-1}\{[\mu - 2(m-1)(2r-2)^2][\mu \\ &\quad -2(m-1)(4r-4)^2] - m^2[(2r-2)^2 + (4r-4)^2]^2\}. \end{aligned}$$

Proof. The line splitting graph of an r-regular graph of size m is a graph of order $2m$. $L_s(G)$ has 2 types of vertices. The m vertices have degree $2r-2$ and the remaining m vertices have degree $4r-4$. Hence,

$$DSS(L_s(G)) = \begin{bmatrix} 2(2r-2)^2(J_m - I_m) & [(2r-2)^2 + (4r-4)^2]J_{m\times m} \\ [(2r-2)^2 + (4r-4)^2]J_{m\times m} & 2(4r-4)^2(J_m - I_m) \end{bmatrix}$$

Therefore,

$$P_{DSS(L_s(G))}(\mu) = |\mu I - DSS(L_s(G))|$$

$$= \begin{vmatrix} [\mu + 2(2r-2)^2]I_m - 2(2r-2)^2J_m & -[(2r-2)^2 + (4r-4)^2]J_{m\times m} \\ -[(2r-2)^2 + (4r-4)^2]J_{m\times m} & (\mu + 2(4r-4)^2)I_m - 2(4r-4)^2J_m \end{vmatrix}.$$

Next by using Lemma 1, we get the desired result. ■

Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup \cdots \cup S_t \cup T$ where each S_i is a set of vertices having at least two vertices and having the same degree and $T = V - \cup_{i=1}^{t} S_i$. The *degree splitting graph* [17] $DS(G)$ is obtained from G by adding vertices $w_1, w_2, ..., w_t$ and joining w_i to each vertex of $S_i (1 \le i \le t)$.

Theorem 14. If G is an r-regular graph of order n, then

$$P_{DSS(DS(G))}(\mu) = (\mu + 2(r+1)^2)^{n-1}\{(\mu - 2(n-1)(r+1)^2) - n[(r+1)^2 + n^2]^2\}.$$

Proof. The degree splitting graph $DS(G)$ of a r-regular graph G with n vertices is graph, which has two types of vertices. The n vertices with degree $(r+1)$ and the remaining 1 vertex is of degree n. Hence,

$$DSS(DS(G)) = \begin{bmatrix} 2(r+1)^2(J_n - I_n) & ((r+1)^2+n^2)J_{n\times 1} \\ ((r+1)^2+n^2)J_{1\times n} & 2n^2(J_1 - I_1) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(DS(G))}(\mu) &= |\mu I - DSS(DS(G))| \\ &= \begin{vmatrix} (\mu+2(r+1)^2)I_n - 2(r+1)^2 J_n & -((r+1)^2+n^2)J_{n\times 1} \\ -((r+1)^2+n^2)J_{1\times n} & 0 \end{vmatrix}. \end{aligned}$$

By using Lemma 1, we get the desired result. ■

The *book with triangular pages* $B_t = P_2 + tK_1$ is a graph with $(n+2)$ vertices and $2n+1$ edges.

Theorem 15. If B_t is a book with triangular pages of order $(t+2)$ and size $2t+1$, then

$$P_{DSS(B_t)}(\mu) = (\mu+8)^{t-1}(\mu+2(t+1)^2)\{(\mu-8(t-1))[\mu-2(t+1)^2] - 2t((t+1)^2+4)^2\}.$$

Proof. The book B_t with triangular pages has two types of vertices. The t vertices are of degree 2 and the remaining 2 vertices are of degree $t+1$. Hence,

$$DSS(B_t) = \begin{bmatrix} 8(J_t - I_t) & (2^2+(t+1)^2)J_{t\times 2} \\ (2^2+(t+1)^2)J_{2\times t} & 2(t+1)^2(J_2 - I_2) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(B_t)}(\mu) &= |\mu I - DSS(B_t)| \\ &= \begin{vmatrix} (\mu+8)I_t - 8J_t & -((t+1)^2+4)J_{t\times 2} \\ -((t+1)^2+4)J_{2\times t} & (\mu+2(t+1)^2)I_2 - 2(t+1)^2 J_2 \end{vmatrix}. \end{aligned}$$

Now by using Lemma 1, we get the required result. ■

The *shadow graph* $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G', G'' and joining each vertex v' in G' to the neighbors of the corresponding vertex v'' in G''.

Theorem 16. If G is an r-regular graph of order n, then

$$P_{DSS(D_2(G))}(\mu) = [\mu - 8r^2(2n-1)](\mu+8r^2)^{2n-1}.$$

Proof. The shadow graph $D_2(G)$ of an r-regular graph G is a $2r$-regular graph with $2n$ vertices. Hence the result follows from Eq. (1). ■

Theorem 17. If H is a graph having n vertices of degree x and m vertices of degree y, then

$$P_{DSS(D_2(H))}(\mu) = (\mu+8x^2)^{2n-1}(\mu+8y^2)^{2m-1}\{[\mu-8x^2(2n-1)][\mu-8y^2(2m-1) - 4mn(4x^2+4y^2)^2\}.$$

Proof. The shadow graph $D_2(H)$ is a graph, which has two types of vertices. The $2n$ vertices with degree $2x$ and the remaining $2m$ vertices are of degree $2y$ since H has n vertices with degree x and m vertices with degree y. Hence,

$$DSS(D_2(H)) = \begin{bmatrix} 2(2x)^2(J_{2n} - I_{2n}) & ((2x)^2+(2y)^2)J_{2n\times 2m} \\ ((2x)^2+(2y)^2)J_{2m\times 2n} & 2(2y)^2(J_{2m} - I_{2m}) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(D_2(H))}(\mu) &= |\mu I - DSS(D_2(H))| \\ &= \begin{vmatrix} (\mu + 2(2x)^2)I_{2n} - 2(2x)^2 J_{2n} & -((2x)^2 + (2y)^2)J_{2n\times 2m} \\ -((2x)^2 + (2y)^2)J_{2m\times 2n} & (\mu + 2(2y)^2)I_{2m} - 2(2y)^2 J_{2m} \end{vmatrix}. \end{aligned}$$

By using Lemma 1, we get the desired result. ∎

The *triangular snake* T_n is a graph obtained from the path P_n by replacing each edge of the path by a triangle C_3.

Theorem 18. If T_n is a triangular snake of order $2n-1$, then

$$P_{DSS(T_n)}(\mu) = (\mu+8)^n(\mu+32)^{n-3}\{(\mu-8n)(\mu-32(n-3)) - 400(n+1)(n-2)\}.$$

Proof. The graph triangular snake T_n has two types of vertices. The $n+1$ vertices with degree 2 and the remaining $n-2$ vertices are of degree 4. Hence,

$$DSS(T_n) = \begin{bmatrix} 8(J_{n+1} - I_{n+1}) & 20J_{(n+1)\times(n-2)} \\ 20J_{(n-2)\times(n+1)} & 32(J_{n-2} - I_{n-2}) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T_n)}(\mu) &= |\mu I - DSS(T_n)| \\ &= \begin{vmatrix} (\mu+8)I_{n+1} - 8J_{n+1} & -20J_{(n+1)\times(n-2)} \\ -20J_{(n-2)\times(n+1)} & (\mu+32)I_{n-2} - 32J_{n-2} \end{vmatrix}. \end{aligned}$$

By using Lemma 1, we get the desired result. ∎

The *alternate triangular snake* $A(T_n)$ is obtained from a path $v_1v_2...v_n$ by joining v_i and v_{i+1} (alternatively) to new vertex v_i. That is every alternate edge of a path is replaced by C_3.

Theorem 19. If $A(T_{2n}), (n \geq 2)$ is an alternate triangular snake of order $3n$, then

$$P_{DSS(A(T_{2n}))}(\mu) = (\mu+8)^{n+1}(\mu+18)^{2n-3}\{(\mu-8(n+1))(\mu-18(2n-3))169(n+2)(2n-2)\}.$$

Proof. The alternate triangular snake $A(T_{2n})$ is a graph, which has two types of vertices. The $n+2$ vertices have degree 2 and the remaining $2n-2$ vertices have degree 3. Hence,

$$DSS(A(T_{2n})) = \begin{bmatrix} 8(J_{n+2} - I_{n+2}) & 13J_{(n+2)\times(2n-2)} \\ 13J_{(2n-2)\times(n+2)} & 18(J_{2n-2} - I_{2n-2}) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(A(T_{2n}))}(\mu) &= |\mu I - DSS(T_{2n})| \\ &= \begin{vmatrix} (\mu+8)I_{n+2} - 8J_{n+2} & -13J_{(n+2)\times(2n-2)} \\ -13J_{(2n-2)\times(n+2)} & (\mu+18)I_{2n-2} - 18J_{2n-2} \end{vmatrix}. \end{aligned}$$

By using Lemma 1, we get the desired result. ∎

The *quadrilateral snake* Q_n is obtained from the path P_n by replacing each edge of the path by a quadrilateral C_4.

Theorem 20. If Q_n is a quadrilateral snake of order $3n-2$, then

$$P_{DSS(Q_n)}(\mu) = (\mu+8)^{2n-1}(\mu+32)^{n-3}\{(\mu-8(2n-1))(\mu-32(n-3))-800n(n-2)\}.$$

Proof. The quadrilateral snake Q_n is a graph, which has two types of vertices. The $2n$ vertices have degree 2 and the remaining $n-2$ vertices have degree 4. Hence,

$$DSS(Q_n) = \begin{bmatrix} 8(J_{2n}-I_{2n}) & 20J_{2n\times(n-2)} \\ 20J_{(n-2)\times 2n} & 32(J_{n-2}-I_{n-2}) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(Q_n)}(\mu) &= |\mu I - DSS(Q_n)| \\ &= \begin{vmatrix} (\mu+8)I_{2n}-8J_{2n} & -20J_{2n\times(n-2)} \\ -20J_{(n-2)\times 2n} & (\mu+32)I_{n-2}-32J_{n-2} \end{vmatrix}. \end{aligned}$$

By using Lemma 1, we get the desired result. ∎

The *alternate quadrilateral snake* $A(Q_n)$ is obtained from a path $v_1v_2...v_n$ by joining v_iv_{i+1} (alternatively) to new vertices v_i, w_i respectively and then joining v_i and w_i. That is every alternate edge of a path is replaced by a cycle C_4.

Theorem 21. If $A(Q_{2n}), (n \geq 2)$ is a quadrilateral snake of order $4n$, then

$$P_{DSS(A(Q_{2n}))}(\mu) = (\mu+8)^{2n+1}(\mu+18)^{2n-3}\{(\mu-8(2n+1))(\mu-18(2n-3))-169(2n+1)(2n-3)\}.$$

Proof. The alternate quadrilateral snake $A(Q_{2n})$ is a graph, which has two types of vertices. The $2n+2$ vertices with degree 2 and the remaining $2n-2$ vertices are of degree 3. Hence,

$$DSS(A(Q_{2n})) = \begin{bmatrix} 8(J_{2n+2}-I_{2n+2}) & 13J_{(2n+2)\times(2n-2)} \\ 13J_{(2n-2)\times(2n+2)} & 18(J_{(2n-2)}-I_{(2n-2)}) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(A(Q_{2n}))}(\mu) &= |\mu I - DSS(A(Q_{2n}))| \\ &= \begin{vmatrix} (\mu+8)I_{2n+2}-8J_{2n+2} & -13J_{(2n+2)\times(2n-2)} \\ -13J_{(2n-2)\times(2n+2)} & (\mu+18)I_{2n-2}-18J_{2n-2} \end{vmatrix}. \end{aligned}$$

By using Lemma 1, we get the desired result. ∎

DEGREE SQUARE SUM POLYNOMIALS OF GENERALIZED xyz-POINT-LINE TRANSFORMATION GRAPHS $T^{xyz}(G)$

The procedure of obtaining a new graph from a given graph by using incidence (or nonincidence) relation between vertex and an edge and an adjacency (or nonadjacency) relation between two vertices or two edges of a graph is known as *Graph Transformation* and the graph obtained by doing so is called a *Transformation graph*. For a graph $G=(V,E)$, let G^0 be the graph with $V(G^0)=V(G)$ and with no edges, G^1 the complete graph with $V(G^1)=V(G)$, $G^+=G$, and $G^-=\overline{G}$. Let $\mathcal{G}$ denotes the set of simple graphs. The following graph operations depending on $x,y,z \in \{0,1,+,-\}$ induce functions $T^{xyz}:\mathcal{G}\to\mathcal{G}$. These operations were introduced by Deng et al. in [9]. They called these resulting graphs as xyz-transformations of G, denoted by $T^{xyz}(G) = G^{xyz}$ and studied the Laplacian characteristic polynomials and some other Laplacian parameters of xyz-transformations of an r-regular graph G. In [2], Wu Bayoindureng et al. introduced the total transformation graphs and studied the basic properties of total transformation graphs. Motivated by this, Basavanagoud [3] studied the basic properties of the xyz-transformation graphs and called them xyz-point-line transformation graphs by changing the notation of xyz-transformations of a graph G as $T^{xyz}(G)$ to avoid confusion between parent graph G and its xyz-transformations.

Definition 22. [9] Given a graph $G(n,m)$ with vertex set $V(G)$ and edge set $E(G)$ and three variables $x, y, z \in \{0, 1, +, -\}$, the xyz-point-line transformation graph $T^{xyz}(G)$ of G is the graph with vertex set $V(T^{xyz}(G)) = V(G) \cup E(G)$ i.e $|V(G)| = n + m$ and the edge set $E(T^{xyz}(G)) = E((G)^x) \cup E((L(G))^y) \cup E(W)$ where $W = S(G)$ if $z = +$, $W = \overline{S}(G)$ if $z = -$, W is the graph with $V(W) = V(G) \cup E(G)$ and with no edges if $z = 0$ and W is the complete bipartite graph with parts $V(G)$ and $E(G)$ if $z = 1$.

Since there are 64 distinct 3-permutations of $\{0, 1, +, -\}$, thus obtained 64 kinds of generalized xyz-point-line transformation graphs. There are 16 different graphs for each case when $z = 0, z = 1, z = +, z = -$.

The following Propositions are useful for proving forthcoming theorems.

Proposition 23. [5] If G is a graph of order n and size m and let v be a point-vertex of $T^{xyz}(G)$ corresponding to a vertex v of G, then

(i)

$$d_{T^{xy+}(G)}(v) = \begin{cases} d_G(v) & if\ x = 0, y \in \{0,1,+,-\} \\ n-1+d_G(v) & if\ x = 1, y \in \{0,1,+,-\} \\ 2d_G(v) & if\ x = +, y \in \{0,1,+,-\} \\ n-1 & if\ x = -, y \in \{0,1,+,-\} \end{cases}$$

(ii)

$$d_{T^{xy-}(G)}(v) = \begin{cases} m-d_G(v) & if\ x = 0, y \in \{0,1,+,-\} \\ n+m-1-d_G(v) & if\ x = 1, y \in \{0,1,+,-\} \\ m & if\ x = +, y \in \{0,1,+,-\} \\ n+m-1-2d_G(v) & if\ x = -, y \in \{0,1,+,-\} \end{cases}$$

Proposition 24. [5] If G is a graph of order n and size m and let e be the line-vertex of $T^{xyz}(G)$ corresponding to an edge e of G, then

(i)

$$d_{T^{xy+}(G)}(e) = \begin{cases} 2 & if\ y = 0, x \in \{0,1,+,-\} \\ m+1 & if\ y = 1, x \in \{0,1,+,-\} \\ 2+d_G(e) & if\ y = +, x \in \{0,1,+,-\} \\ m+1-d_G(e) & if\ y = -, x \in \{0,1,+,-\} \end{cases}$$

(ii)

$$d_{T^{xy-}(G)}(e) = \begin{cases} n-2 & if\ y = 0, x \in \{0,1,+,-\} \\ n+m-3 & if\ y = 1, x \in \{0,1,+,-\} \\ n-2+d_G(e) & if\ y = +, x \in \{0,1,+,-\} \\ n+m-3-d_G(e) & if\ y = -, x \in \{0,1,+,-\} \end{cases}$$

Theorem 25. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{10+}(G))}(\mu) &= [\mu + 2(n-1+r)^2]^{n-1}[\mu+8]^{m-1}([\mu - 2(n-1)(n-1+r)^2][\mu - 8(m-1)] \\ &\quad -mn((n-1+r)^2+2^2)^2). \end{aligned}$$

Proof. The graph $T^{10+}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $n-1+r$ and m vertices have degree 2. Hence,

$$DSS(T^{10+}(G)) = \begin{bmatrix} 2(n-1+r)^2(J_n - I_n) & ((n-1+r)^2+4)J_{n\times m} \\ ((n-1+r)^2+4)J_{m\times n} & 8(J_m - I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{10+}(G))}(\mu) &= |\mu I - DSS(T^{10+}(G))| \\ &= \begin{vmatrix} (\mu+2(n-1+r)^2)I_n - 2(n-1+r)^2 J_n & -((n-1+r)^2+4)J_{n\times m} \\ -((n-1+r)^2+4)J_{m\times n} & (\mu+8)I_m - 8J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 26. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{-0+}(G))}(\mu) &= [\mu+2(n-1)^2]^{n-1}[\mu+8]^{m-1}([\mu-2(n-1)^3][\mu-8(m-1)] \\ &\quad -mn((n-1)^2+2^2)^2). \end{aligned}$$

Proof. The graph $T^{-0+}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $n-1$ and m vertices have degree 2. Hence,

$$DSS(T^{-0+}(G)) = \begin{bmatrix} 2(n-1)^2(J_n - I_n) & ((n-1)^2+4)J_{n\times m} \\ ((n-1)^2+4)J_{m\times n} & 8(J_m - I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{-0+}(G))}(\mu) &= |\mu I - DSS(T^{-0+}(G))| \\ &= \begin{vmatrix} (\mu+2(n-1)^2)I_n - 2(n-1)^2 J_n & -((n-1)^2+4)J_{n\times m} \\ -((n-1)^2+4)J_{m\times n} & (\mu+8)I_m - 8J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 27. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{01+}(G))}(\mu) &= [\mu+2r^2]^{n-1}[\mu+2(m+1)^2]^{m-1}([\mu-2(n-1)r^2][\mu-2(m-1)(m+1)^2] \\ &\quad -mn(r^2+(m+1)^2)^2). \end{aligned}$$

Proof. The graph $T^{01+}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree r and m vertices have degree $m+1$. Hence,

$$DSS(T^{01+}(G)) = \begin{bmatrix} 2r^2(J_n - I_n) & (r^2+(m+1)^2)J_{n\times m} \\ (r^2+(m+1)^2)J_{m\times n} & 2(m+1)^2(J_m - I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{01+}(G))}(\mu) &= |\mu I - DSS(T^{01+}(G))| \\ &= \begin{vmatrix} (\mu+2r^2)I_n - 2r^2 J_n & -(r^2+(m+1)^2)J_{n\times m} \\ -(r^2+(m+1)^2)J_{m\times n} & (\mu+2(m+1)^2)I_m - 2(m+1)^2 J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 28. If G is an r-regular graph of order n and size m, then

$$P_{DSS(T^{11+}(G))}(\mu) = [\mu+2(n-1+r)^2]^{n-1}[\mu+2(m+1)^2]^{m-1}([\mu-2(n-1)(n-1+r)^2][\mu -2(m-1)(m+1)^2]-mn((n-1+r)^2+(m+1)^2)^2).$$

Proof. The graph $T^{11+}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $n-1+r$ and m vertices have degree $m+1$. Hence,

$$DSS(T^{11+}(G)) = \begin{bmatrix} 2(n-1+r)^2(J_n-I_n) & ((n-1+r)^2+(m+1)^2)J_{n\times m} \\ ((n-1+r)^2+(m+1)^2)J_{m\times n} & 2(m+1)^2(J_m-I_m) \end{bmatrix}$$

Therefore,

$$P_{DSS(T^{11+}(G))}(\mu) = |\mu I - DSS(T^{11+}(G))| = \begin{vmatrix} (\mu+2(n-1+r)^2)I_n - 2(n-1+r)^2J_n & -((n-1+r)^2+(m+1)^2)J_{n\times m} \\ -((n-1+r)^2+(m+1)^2)J_{m\times n} & (\mu+2(m+1)^2)I_m-2(m+1)^2J_m \end{vmatrix}.$$

Now using Lemma 1, we get the desired result. ■

Theorem 29. If G is an r-regular graph of order n and size m, then

$$P_{DSS(T^{+1+}(G))}(\mu) = [\mu+8r^2]^{n-1}[\mu+2(m+1)^2]^{m-1}([\mu-8(n-1)r^2][\mu-2(m-1)(m+1)^2] -mn(4r^2+(m+1)^2)^2).$$

Proof. The graph $T^{+1+}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $2r$ and m vertices have degree $m+1$. Hence,

$$DSS(T^{+1+}(G)) = \begin{bmatrix} 8r^2(J_n-I_n) & (4r^2+(m+1)^2)J_{n\times m} \\ (4r^2+(m+1)^2)J_{m\times n} & 2(m+1)^2(J_m-I_m) \end{bmatrix}$$

Therefore,

$$P_{DSS(T^{+1+}(G))}(\mu) = |\mu I - DSS(T^{+1+}(G))| = \begin{vmatrix} (\mu+8r^2)I_n-8r^2J_n & -(4r^2+(m+1)^2)J_{n\times m} \\ -(4r^2+(m+1)^2)J_{m\times n} & (\mu+2(m+1)^2)I_m-2(m+1)^2J_m \end{vmatrix}.$$

Now using Lemma 1, we get the desired result. ■

Theorem 30. If G is an r-regular graph of order n and size m, then

$$P_{DSS(T^{1++}(G))}(\mu) = [\mu+2(n-1+r)^2]^{n-1}[\mu+2r^2]^{m-1}([\mu-2(n-1)(n-1+r)^2][\mu-2(m-1)r^2] -mn((n-1+r)^2+r^2)^2).$$

Proof. The graph $T^{1++}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $n-1+r$ and m vertices have degree r. Hence,

$$DSS(T^{1++}(G)) = \begin{bmatrix} 2(n-1+r)^2(J_n-I_n) & ((n-1+r)^2+r^2)J_{n\times m} \\ (r^2+(n-1+r)^2)J_{m\times n} & 2r^2(J_m-I_m) \end{bmatrix}$$

Therefore,

$$P_{DSS(T^{1++}(G))}(\mu) = |\mu I - DSS(T^{1++}(G))| = \begin{vmatrix} (\mu+2(n-1+r)^2)I_n-2(n-1+r)^2J_n & -((n-1+r)^2+r^2)J_{n\times m} \\ -((n-1+r)^2+r^2)J_{m\times n} & (\mu+2r^2)I_m-2r^2J_m \end{vmatrix}.$$

Now using Lemma 1, we get the desired result. ■

Theorem 31. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{-++}(G))}(\mu) &= [\mu+2(n-1)^2]^{n-1}[\mu+2r^2]^{m-1}([\mu-2(n-1)^3][\mu-2(m-1)r^2] \\ &\quad -mn((n-1)^2+r^2)^2). \end{aligned}$$

Proof. The graph $T^{-++}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $n-1$ and m vertices have degree r. Hence,

$$DSS(T^{-++}(G)) = \begin{bmatrix} 2(n-1)^2(J_n-I_n) & ((n-1)^2+r^2)J_{n\times m} \\ (r^2+(n-1)^2)J_{m\times n} & 2r^2(J_m-I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{-++}(G))}(\mu) &= |\mu I - DSS(T^{-++}(G))| \\ &= \begin{vmatrix} (\mu+2(n-1)^2)I_n - 2(n-1)^2J_n & -((n-1)^2+r^2)J_{n\times m} \\ -((n-1)^2+r^2)J_{m\times n} & (\mu+2r^2)I_m - 2r^2J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 32. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{0-+}(G))}(\mu) &= [\mu+2r^2]^{n-1}[\mu+(m-2r+3)^2]^{m-1}([\mu-2(n-1)r^2][\mu-2(m-1)(m-2r+3)^2] \\ &\quad -mn(r^2+(m-2r+3)^2)^2). \end{aligned}$$

Proof. The graph $T^{0-+}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree r and m vertices have degree $m-2r+3$. Hence,

$$DSS(T^{0-+}(G)) = \begin{bmatrix} 2r^2(J_n-I_n) & (r^2+(m-2r+3)^2)J_{n\times m} \\ (r^2+(m-2r+3)^2)J_{m\times n} & 2(m-2r+3)^2(J_m-I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{0-+}(G))}(\mu) &= |\mu I - DSS(T^{0-+}(G))| \\ &= \begin{vmatrix} (\mu+2r^2)I_n - 2r^2J_n & -(r^2+(m-2r+3)^2)J_{n\times m} \\ -(r^2+(m-2r+3)^2)J_{m\times n} & (\mu+2(m-2r+3)^2)I_m - 2(m-2r+3)^2J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 33. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{1-+}(G))}(\mu) &= [\mu+2(n-1+r)^2]^{n-1}[\mu+(m-2r+3)^2]^{m-1}([\mu-2(n-1)(n-1+r)^2][\mu \\ &\quad -2(m-1)(m-2r+3)^2] - mn((n-1+r)^2+(m-2r+3)^2)^2). \end{aligned}$$

Proof. The graph $T^{1-+}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $n-1+r$ and m vertices have degree $m-2r+3$. Hence,

$$DSS(T^{1-+}(G)) = \begin{bmatrix} (n-1+r)^2(J_n-I_n) & ((n-1+r)^2+(m-2r+3)^2)J_{n\times m} \\ ((n-1+r)^2+(m-2r+3)^2)J_{m\times n} & 2(m-2r+3)^2(J_m-I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{1-+}(G))}(\mu) &= |\mu I - DSS(T^{1-+}(G))| \\ &= \begin{vmatrix} (\mu+2(n-1+r)^2)I_n - 2(n-1+r)^2J_n & -((n-1+r)^2+(m-2r+3)^2)J_{n\times m} \\ -((n-1+r)^2+(m-2r+3)^2)J_{m\times n} & (\mu+2(m-2r+3)^2)I_m - 2(m-2r+3)^2J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 34. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{+-+}(G))}(\mu) &= [\mu+8r^2]^{n-1}[\mu+(m-2r+3)^2]^{m-1}([\mu-8(n-1)r^2][\mu-2(m-1)(m-2r+3)^2] \\ &\quad -mn(4r^2+(m-2r+3)^2)^2). \end{aligned}$$

Proof. The graph $T^{+-+}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $2r$ and m vertices have degree $m-2r+3$. Hence,

$$DSS(T^{+-+}(G)) = \begin{bmatrix} 8r^2(J_n-I_n) & (4r^2+(m-2r+3)^2)J_{n\times m} \\ (4r^2+(m-2r+3)^2)J_{m\times n} & 2(m-2r+3)^2(J_m-I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{+-+}(G))}(\mu) &= |\mu I - DSS(T^{+-+}(G))| \\ &= \begin{vmatrix} (\mu+8r^2)I_n - 8r^2J_n & -(4r^2+(m-2r+3)^2)J_{n\times m} \\ -(4r^2+(m-2r+3)^2)J_{m\times n} & (\mu+2(m-2r+3)^2)I_m - 2(m-2r+3)^2J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 35. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{--+}(G))}(\mu) &= [\mu+2(n-1)^2]^{n-1}[\mu+(m-2r+3)^2]^{m-1}([\mu-2(n-1)^3][\mu \\ &\quad -2(m-1)(m-2r+3)^2] - mn((n-1)^2+(m-2r+3)^2)^2). \end{aligned}$$

Proof. The graph $T^{--+}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $n-1$ and m vertices have degree $m-2r+3$. Hence,

$$DSS(T^{--+}(G)) = \begin{bmatrix} 2(n-1)^2(J_n-I_n) & ((n-1)^2+(m-2r+3)^2)J_{n\times m} \\ ((n-1)^2+(m-2r+3)^2)J_{m\times n} & 2(m-2r+3)^2(J_m-I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{--+}(G))}(\mu) &= |\mu I - DSS(T^{--+}(G))| \\ &= \begin{vmatrix} (\mu+2(n-1)^2)I_n - 2(n-1)^2J_n & -((n-1)^2+(m-2r+3)^2)J_{n\times m} \\ -((n-1)^2+(m-2r+3)^2)J_{m\times n} & (\mu+2(m-2r+3)^2)I_m - 2(m-2r+3)^2J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

The degree square sum polynomials of the transformation graphs $T^{00+}(G)$(subdivision graph $S(G)$), $T^{+0+}(G)$ (semitotal point graph $T_2(G)$), $T^{0++}(G)$ (semitotal line graph $T_1(G)$), $T^{+++}(G)$ (total graph $T(G)$) are already computed in [4].

Theorem 36. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{00-}(G))}(\mu) &= [\mu+2(m-r)^2]^{n-1}[\mu+2(n-2)^2]^{m-1}([\mu-2(n-1)(m-r)^2][\mu \\ &\quad -2(m-1)(n-2)^2] - mn((m-r)^2+(n-2)^2)^2). \end{aligned}$$

Proof. The graph $T^{00-}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $m-r$ and m vertices have degree $n-2$. Hence,

$$DSS(T^{00-}(G)) = \begin{bmatrix} 2(m-r)^2(J_n-I_n) & ((m-r)^2+(n-2)^2)J_{n\times m} \\ ((m-r)^2+(n-2)^2)J_{m\times n} & 2(n-2)^2(J_m-I_m) \end{bmatrix}$$

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Therefore,

$$P_{DSS(T^{00-}(G))}(\mu) = |\mu I - DSS(T^{00-}(G))| = \begin{vmatrix} (\mu+2(m-r)^2)I_n - 2(m-r)^2J_n & -((m-r)^2+(n-2)^2)J_{n\times m} \\ -((m-r)^2+(n-2)^2)J_{m\times n} & (\mu+2(n-2)^2)I_m - 2(n-2)^2J_m \end{vmatrix}.$$

Now using Lemma 1, we get the desired result. ■

Theorem 37. If G is an r-regular graph of order n and size m, then

$$P_{DSS(T^{10-}(G))}(\mu) = [\mu+2(n+m-1-r)^2]^{n-1}[\mu+2(n-2)^2]^{m-1}([\mu-2(n-1)(n+m-1-r)^2][\mu -2(m-1)(n-2)^2] - mn((n+m-1-r)^2+(n-2)^2)^2).$$

Proof. The graph $T^{10-}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $n+m-1-r$ and m vertices have degree $n-2$. Hence,

$$DSS(T^{10-}(G)) = \begin{bmatrix} 2(n+m-1-r)^2(J_n-I_n) & ((n+m-1-r)^2+(n-2)^2)J_{n\times m} \\ ((n+m-1-r)^2+(n-2)^2)J_{m\times n} & 2(n-2)^2(J_m-I_m) \end{bmatrix}$$

Therefore,

$$P_{DSS(T^{10-}(G))}(\mu) = |\mu I - DSS(T^{10-}(G))| = \begin{vmatrix} (\mu+2(n+m-1-r)^2)I_n - 2(n+m-1-r)^2J_n & -((n+m-1-r)^2+(n-2)^2)J_{n\times m} \\ -((n+m-1-r)^2+(n-2)^2)J_{m\times n} & (\mu+2(n-2)^2)I_m - 2(n-2)^2J_m \end{vmatrix}.$$

Now using Lemma 1, we get the desired result. ■

Theorem 38. If G is an r-regular graph of order n and size m, then

$$P_{DSS(T^{+0-}(G))}(\mu) = [\mu+2m^2]^{n-1}[\mu+2(n-2)^2]^{m-1}([\mu-2(n-1)m^2][\mu-2(m-1)(n-2)^2] -mn(m^2+(n-2)^2)^2).$$

Proof. The graph $T^{+0-}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree m and m vertices have degree $n-2$. Hence,

$$DSS(T^{+0-}(G)) = \begin{bmatrix} 2m^2(J_n-I_n) & (m^2+(n-2)^2)J_{n\times m} \\ (m^2+(n-2)^2)J_{m\times n} & 2(n-2)^2(J_m-I_m) \end{bmatrix}$$

Therefore,

$$P_{DSS(T^{+0-}(G))}(\mu) = |\mu I - DSS(T^{+0-}(G))| = \begin{vmatrix} (\mu+2m^2)I_n - 2m^2J_n & -(m^2+(n-2)^2)J_{n\times m} \\ -(m^2+(n-2)^2)J_{m\times n} & (\mu+2(n-2)^2)I_m - 2(n-2)^2J_m \end{vmatrix}.$$

Now using Lemma 1, we get the desired result. ■

Theorem 39. If G is an r-regular graph of order n and size m, then

$$P_{DSS(T^{-0-}(G))}(\mu) = [\mu+2(n+m-1-2r)^2]^{n-1}[\mu+2(n-2)^2]^{m-1}([\mu -2(n-1)(n+m-1-2r)^2][\mu-2(m-1)(n-2)^2] -mn((n+m-1-2r)^2+(n-2)^2)^2).$$

Proof. The graph $T^{-0-}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $n+m-1-2r$ and m vertices have degree $n-2$. Hence,

$$DSS(T^{-0-}(G)) = \begin{bmatrix} 2(n+m-1-2r)^2(J_n - I_n) & ((n+m-1-2r)^2+(n-2)^2)J_{n\times m} \\ ((n+m-1-2r)^2+(n-2)^2)J_{m\times n} & 2(n-2)^2(J_m - I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{-0-}(G))}(\mu) &= |\mu I - DSS(T^{-0-}(G))| \\ &= \begin{vmatrix} (\mu + 2(n+m-1-2r)^2)I_n - 2(n+m-1-2r)^2 J_n & -((n+m-1-2r)^2+(n-2)^2)J_{n\times m} \\ -((n+m-1-2r)^2+(n-2)^2)J_{m\times n} & (\mu+2(n-2)^2)I_m - 2(n-2)^2 J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 40. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{01-}(G))}(\mu) &= [\mu + 2(m-r)^2]^{n-1}[\mu + 2(n+m-3)^2]^{m-1}([\mu - 2(n-1)(m-r)^2][\mu \\ &\quad -2(m-1)(n+m-3)^2] - mn((m-r)^2 + (n+m-3)^2)^2). \end{aligned}$$

Proof. The graph $T^{01-}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $m-r$ and m vertices have degree $n+m-3$. Hence,

$$DSS(T^{01-}(G)) = \begin{bmatrix} 2(m-r)^2(J_n - I_n) & ((m-r)^2+(n+m-3)^2)J_{n\times m} \\ ((m-r)^2+(n+m-3)^2)J_{m\times n} & 2(n+m-3)^2(J_m - I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{01-}(G))}(\mu) &= |\mu I - DSS(T^{01-}(G))| \\ &= \begin{vmatrix} (\mu + 2(m-r)^2)I_n - 2(m-r)^2 J_n & -((m-r)^2+(n+m-3)^2)J_{n\times m} \\ -((m-r)^2+(n+m-3)^2)J_{m\times n} & (\mu+2(n+m-3)^2)I_m - 2(n+m-3)^2 J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 41. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{11-}(G))}(\mu) &= [\mu + 2(n+m-1-r)^2]^{n-1}[\mu + 2(n+m-3)^2]^{m-1}([\mu \\ &\quad -2(n-1)(n+m-1-r)^2][\mu - 2(m-1)(n+m-3)^2] \\ &\quad -mn((n+m-1-r)^2 + (n+m-3)^2)^2). \end{aligned}$$

Proof. The graph $T^{11-}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $n+m-1-r$ and m vertices have degree $n+m-3$. Hence,

$$DSS(T^{11-}(G)) = \begin{bmatrix} 2(n+m-1-r)^2(J_n - I_n) & ((n+m-1-r)^2+(n+m-3)^2)J_{n\times m} \\ ((n+m-1-r)^2+(n+m-3)^2)J_{m\times n} & 2(n+m-3)^2(J_m - I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{11-}(G))}(\mu) &= |\mu I - DSS(T^{11-}(G))| \\ &= \begin{vmatrix} (\mu + 2(n+m-1-r)^2)I_n - 2(n+m-1-r)^2 J_n & -((n+m-1-r)^2+(n+m-3)^2)J_{n\times m} \\ -((n+m-1-r)^2+(n+m-3)^2)J_{m\times n} & (\mu+2(n+m-3)^2)I_m - 2(n+m-3)^2 J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 42. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{+1-}(G))}(\mu) &= [\mu+2m^2]^{n-1}[\mu+2(n+m-3)^2]^{m-1}([\mu-2(n-1)m^2][\mu \\ &\quad -2(m-1)(n+m-3)^2]-mn(m^2+(n+m-3)^2)^2). \end{aligned}$$

Proof. The graph $T^{+1-}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree m and m vertices have degree $n+m-3$. Hence,

$$DSS(T^{+1-}(G)) = \begin{bmatrix} 2m^2(J_n-I_n) & (m^2+(n+m-3)^2)J_{n\times m} \\ (m^2+(n+m-3)^2)J_{m\times n} & 2(n+m-3)^2(J_m-I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{+1-}(G))}(\mu) &= |\mu I - DSS(T^{+1-}(G))| \\ &= \begin{vmatrix} (\mu+2m^2)I_n-2m^2J_n & -(m^2+(n+m-3)^2)J_{n\times m} \\ -(m^2+(n+m-3)^2)J_{m\times n} & (\mu+2(n+m-3)^2)I_m-2(n+m-3)^2J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 43. If G is an r-regular graph of order n and size m, then

$$P_{DSS(T^{-1-}(G))}(\mu) = [\mu+2R_1^2]^{n-1}[\mu+2R_2^2]^{m-1}([\mu-2(n-1)R_1^2][\mu-2(m-1)R_2^2]-mn(R_1^2+R_2^2)^2).$$

Proof. The graph $T^{-1-}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $R_1=n+m-1-2r$ and m vertices have degree $R_2=n+m-3$. Hence,

$$DSS(T^{-1-}(G)) = \begin{bmatrix} 2R_1^2(J_n-I_n) & (R_1^2+R_2^2)J_{n\times m} \\ (R_1^2+R_2^2)J_{m\times n} & 2R_2^2(J_m-I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{-1-}(G))}(\mu) &= |\mu I - DSS(T^{-1-}(G))| \\ &= \begin{vmatrix} (\mu+2R_1^2)I_n-2R_1^2J_n & -(R_1^2+R_2^2)J_{n\times m} \\ -(R_1^2+R_2^2)J_{m\times n} & (\mu+2R_2^2)I_m-2R_2^2J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 44. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{0+-}(G))}(\mu) &= [\mu+2(m-r)^2]^{n-1}[\mu+2(n+2(r-2))^2]^{m-1}([\mu-2(n-1)(m-r)^2][\mu \\ &\quad -2(m-1)(n+2(r-2))^2]-mn((m-r)^2+(n+2(r-2))^2)^2). \end{aligned}$$

Proof. The graph $T^{0+-}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $m-r$ and m vertices have degree $n+2(r-2)$. Hence,

$$DSS(T^{0+-}(G)) = \begin{bmatrix} 2(m-r)^2(J_n-I_n) & ((m-r)^2+(n+2(r-2))^2)J_{n\times m} \\ ((m-r)^2+(n+2(r-2))^2)J_{m\times n} & 2(n+2(r-2))^2(J_m-I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{0+-}(G))}(\mu) &= |\mu I - DSS(T^{0+-}(G))| \\ &= \begin{vmatrix} (\mu+2(m-r)^2)I_n-2(m-r)^2J_n & -((m-r)^2+(n+2(r-2))^2)J_{n\times m} \\ -((m-r)^2+(n+2(r-2))^2)J_{m\times n} & (\mu+2(n+2(r-2))^2)I_m-2(n+2(r-2))^2J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 45. If G is an r-regular graph of order n and size m, then

$$P_{DSS(T^{1+-}(G))}(\mu) = [\mu + 2R_3^2]^{n-1}[\mu + 2R_4^2]^{m-1}([\mu - 2(n-1)R_3^2][\mu - 2(m-1)R_4^2] - mn(R_3^2 + R_4^2)^2).$$

Proof. The graph $T^{1+-}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $R_3 = n + m - 1 - r$ and m vertices have degree $R_4 = n + 2(r - 2)$. Hence,

$$DSS(T^{1+-}(G)) = \begin{bmatrix} 2R_3^2(J_n - I_n) & (R_3^2 + R_4^2)J_{n\times m} \\ (R_3^2 + R_4^2)J_{m\times n} & 2R_4^2(J_m - I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{1+-}(G))}(\mu) &= |\mu I - DSS(T^{1+-}(G))| \\ &= \begin{vmatrix} (\mu + 2R_3^2)I_n - 2R_3^2 J_n & -(R_3^2 + R_4^2)J_{n\times m} \\ -(R_3^2 + R_4^2)J_{m\times n} & (\mu + 2R_4^2)I_m - 2R_4^2 J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 46. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{++-}(G))}(\mu) &= [\mu + 2m^2]^{n-1}[\mu + 2(n + 2(r-2))^2]^{m-1}([\mu - 2(n-1)m^2][\mu \\ &\quad -2(m-1)(n + 2(r-2))^2] - mn(m^2 + (n + 2(r-2))^2)^2). \end{aligned}$$

Proof. The graph $T^{++-}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree m and m vertices have degree $n + 2(r - 2)$. Hence,

$$DSS(T^{++-}(G)) = \begin{bmatrix} 2m^2(J_n - I_n) & (m^2 + (n + 2(r-2))^2)J_{n\times m} \\ (m^2 + (n + 2(r-2))^2)J_{m\times n} & 2(n + 2(r-2))^2(J_m - I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{++-}(G))}(\mu) &= |\mu I - DSS(T^{++-}(G))| \\ &= \begin{vmatrix} (\mu + 2m^2)I_n - 2m^2 J_n & -(m^2 + (n + 2(r-2))^2)J_{n\times m} \\ -(m^2 + (n + 2(r-2))^2)J_{m\times n} & (\mu + 2(n + 2(r-2))^2)I_m - 2(n + 2(r-2))^2 J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 47. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{-+-}(G))}(\mu) &= [\mu + 2R_5^2]^{n-1}[\mu + 2R_6^2]^{m-1}([\mu - 2(n-1)R_5^2][\mu - 2(m-1)R_6^2] \\ &\quad -mn(R_5^2 + R_6^2)^2). \end{aligned}$$

Proof. The graph $T^{-+-}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $R_5 = n + m - 1 - 2r$ and m vertices have degree $R_6 = n + 2(r - 2)$. Hence,

$$DSS(T^{-+-}(G)) = \begin{bmatrix} 2R_5^2(J_n - I_n) & (R_5^2 + R_6^2)J_{n\times m} \\ (R_5^2 + R_6^2)J_{m\times n} & 2R_6^2(J_m - I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{-+-}(G))}(\mu) &= |\mu I - DSS(T^{-+-}(G))| \\ &= \begin{vmatrix} (\mu + 2R_5^2)I_n - 2R_5^2 J_n & -(R_5^2 + R_6^2)J_{n\times m} \\ -(R_5^2 + R_6^2)J_{m\times n} & (\mu + 2R_6^2)I_m - 2R_6^2 J_m \end{vmatrix}. \end{aligned}$$

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Now using Lemma 1, we get the desired result. ■

Theorem 48. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{0--}(G))}(\mu) &= [\mu+2(m-r)^2]^{n-1}[\mu+2(n+m-2r-1)^2]^{m-1}([\mu-2(n-1)(m-r)^2][\mu \\ &\quad -2(m-1)(n+m-2r-1)^2]-mn((m-r)^2+(n+m-2r-1)^2)^2). \end{aligned}$$

Proof. The graph $T^{0--}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $m-r$ and m vertices have degree $n+m-2r-1$. Hence,

$$DSS(T^{0--}(G)) = \begin{bmatrix} 2(m-r)^2(J_n-I_n) & ((m-r)^2+(n+m-2r-1)^2)J_{n\times m} \\ ((m-r)^2+(n+m-2r-1)^2)J_{m\times n} & 2(n+m-2r-1)^2(J_m-I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{0--}(G))}(\mu) &= |\mu I - DSS(T^{0--}(G))| \\ &= \begin{vmatrix} (\mu+2(m-r)^2)I_n-2(m-r)^2J_n & -((m-r)^2+(n+m-2r-1)^2)J_{n\times m} \\ -((m-r)^2+(n+m-2r-1)^2)J_{m\times n} & (\mu+2(n+m-2r-1)^2)I_m-2(n+m-2r-1)^2J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 49. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{1--}(G))}(\mu) &= [\mu+2R_7^2]^{n-1}[\mu+2R_8^2]^{m-1}([\mu-2(n-1)R_7^2][\mu-2(m-1)R_8^2] \\ &\quad -mn(R_7^2+R_8^2)^2). \end{aligned}$$

Proof. The graph $T^{1--}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree $R_7=n+m-1-r$ and m vertices have degree $R_8=n+m-2r-1$. Hence,

$$DSS(T^{1--}(G)) = \begin{bmatrix} 2R_7^2(J_n-I_n) & (R_7^2+R_8^2)J_{n\times m} \\ (R_7^2+R_8^2)J_{m\times n} & 2R_8^2(J_m-I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{1--}(G))}(\mu) &= |\mu I - DSS(T^{1--}(G))| \\ &= \begin{vmatrix} (\mu+2R_7^2)I_n-2R_7^2J_n & -(R_7^2+R_8^2)J_{n\times m} \\ -(R_7^2+R_8^2)J_{m\times n} & (\mu+2R_8^2)I_m-2R_8^2J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 50. If G is an r-regular graph of order n and size m, then

$$\begin{aligned} P_{DSS(T^{+--}(G))}(\mu) &= [\mu+2m^2]^{n-1}[\mu+2(n+m-2r-1)^2]^{m-1}([\mu-2(n-1)m^2][\mu \\ &\quad -2(m-1)(n+m-2r-1)^2]-mn(m^2+(n+m-2r-1)^2)^2). \end{aligned}$$

Proof. The graph $T^{+--}(G)$ of an r-regular graph G has two types of vertices. From Propositions 23 and 24, the n vertices have degree m and m vertices have degree $n+m-2r-1$. Hence,

$$DSS(T^{+--}(G)) = \begin{bmatrix} 2m^2(J_n-I_n) & (m^2+(n+m-2r-1)^2)J_{n\times m} \\ (m^2+(n+m-2r-1)^2)J_{m\times n} & 2(n+m-2r-1)^2(J_m-I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T^{+--}(G))}(\mu) &= |\mu I - DSS(T^{+--}(G))| \\ &= \begin{vmatrix} (\mu+2m^2)I_n-2(n+m-2r-1)^2J_n & -(m^2+(n+m-2r-1)^2)J_{n\times m} \\ -(m^2+(n+m-2r-1)^2)J_{m\times n} & (\mu+2(n+m-2r-1)^2)I_m-2(n+m-2r-1)^2J_m \end{vmatrix}. \end{aligned}$$

Now using Lemma 1, we get the desired result. ■

Theorem 51. If G is an r-regular graph of order n and size m, then

$$P_{DSS(T^{---}(G))}(\mu) = [\mu - 2R_9^2(n+m-1)][\mu + 2R_9^2]^{n+m-1}.$$

Proof. All the vertices of a graph $T^{---}(G)$ of an r-regular graph G have degree $R_9 = n+m-2r-1$ from Propositions 23 and 24. Hence, the result follows from Eq. (1). ■

Similarly one can compute the degree square sum polynomial of transformation graphs $T^{xy1}(G)$. The degree square sum polynomial of transformation graphs $T^{xy0}(G)$ can be omitted since they are disconnected graphs and no chemical compound corresponds to those graphs and hence there is no scope for application.

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