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Numerical Solution for the time-fractional Fokker-Planck equation Using Fractional Power Series Method and The shifted Chebyshev polynomials of the third kind

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Abstract

This paper presents a numerical method for solving the time-fractional Fokker-Planck equation. We use a fractional power series method (FPSM) and the properties of the Chebyshev polynomials of the third kind to obtain the approximate solution of the fractional Fokker-Planck equation. The obtained results of the FFPE show the simplicity and the efficiency of the proposed methods.

Keywords: Fractional Fokker-planck equation; Caputo derivative; Fractional power series method; Chebyshev polynomials of third kind.

INTRODUCTION

Fractional differential equations (FDEs) have been the focus of many studies due to their frequent appearance in various applications such as in fluid mechanics, viscoelasticity, biology, physics and engineering applications ([2], [3]). Consequently, considerable attention has been given to the efficient numerical solutions of the FDEs of physical interest ([13]-[19]), because it is difficult to find the exact solutions for it. Different numerical methods have been proposed in the literature for solving the FDEs ([4]-[11]).

The time-fractional Fokker-Planck equation serves as a mathematical model for a number of problems in physical and biological sciences ([1], [15], [19], [21]-[23]). It arises from a diffusion approximation of some stochastic processes regarcted as Markovian and continuous. It is a generalized diffusion equation governing the evolution of the probability density in time. For the two-variable case, to which attention is restricted here, the equation can be written in the form [1]:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = [-\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} A_i(x,t,u) + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_i x_j} B_i(x,y,u)]u. \tag{1.1}$$

where $u := u(x,t)$, α is a paramerer describing the order of the fractional derivative $(0 < \alpha \le 1)$, $A_i(x,t,u)$ and $B_i(x,t,u)$ are arbitrary constants and n is integer such that $n \neq 0$.

In this analysis, we intend the application of FPSM to provide numerical, analytical solutions for a class of nonlinear partial differential equations included some well-known the time-fractional Fokker-Planck equation [5]. The FPSM ([11], 15]) has several advantages for dealing directly with suggested equations; it needs a few iterations to get high accuracy, it is very simple for obtaining analytical approximate solutions in rapidly convergent formulas [19], it allows better significantly information in providing continuous representation of these approximations and it has the ability for solving other problems appearing in several scientific fields [23].

In this paper, we use Chebyshev polynomials of the third kind ([8], [12]) and recall some important properties and its analytical form. Next we use these polynomials to approximate the numerical solution of (FFPE) with the aid of the Chebyshev collocation method to convert the system equations in algebraic equations that can be solved numerically.

For this purpose, organization of paper is expressed as follows. In Section 2, we provide the basic definitions fractional calculus in addition property. Which will be used throughout the paper. In Section 3, we give some properties of Chebyshev polynomials of the third kind which are of fundamental importance in what follows. In Section 4, Application power series moved on fractional Fokker-Planck equation we introduce numerical implementation. In Section 5, procedure solution of the fractional Fokker-Planck equation and numerical implementation. In Section 6, Conclusions.

BASIC DEFINITIONS

Definition ([20]):

The fractional derivative of $f(x)$ in caputo sense is defined as [2]

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^{(m)}(s)ds,$$

Definition ([20]):

A power series representation of the form

$$\sum_{n=0}^{\infty} c_n (t-t_0)^{n\alpha} = c_0 + c_1(t-t_0)^{\alpha} + c_2(t-t_0)^{2\alpha} + ..., \tag{2.1}$$

where $0 \le m-1 < \alpha \le m, m \in \mathrm{N}^{+}$ and $t>t_0$ is called a fractional power series (FPS) about t_0 ,where t is a varible and C_n are the coeffients of the series.

In addtion, we also need the following property:

Theorem 1. ([20])

Suppose that the FPSM $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ has radius of convergence $R > 0. f(t)$ is a function defined by $f(t) = \sum_{n=0}^{\infty} c_n t^{n\alpha}$ on $0 \le t < R,$ then for $m-1 < \alpha \le m,$ and $0 \le t < R,$ we have

$$D^{\alpha} f(t) = \sum_{n=1}^{\infty} c_k \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} t^{(n-1)\alpha} \qquad (2.2)$$

APPLICATION POWER SERIES MOTHED ON FRACTIONAL FOKKER-PLANCK EQUATION

Example [1]:

We consider the time-fractional Fokker planck equation:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad (3.1)$$

the initial condition

$$u(x.0) = x \qquad (3.2)$$

The exact solution to Eq.(3.1) for the non-fractional case at $\alpha = 1.$

$$u(x,t) = x + t,$$

where $x, t \ge 0.$

Equation can be rewritten as follows

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} \qquad (3.3)$$

to apply FPSM. We suppose

$$u(x,t) = \sum_{k=0}^{\infty} a_k(x) t^{\alpha k} \qquad (3.4)$$
$$= a_0(x) + a_1(x)t^{\alpha} + a_2(x)t^{2\alpha} + \dots$$

by theorem:-

$$D^{\alpha} u(x,t) = \sum_{k=1}^{\infty} a_k \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} t^{(k-1)\alpha} \qquad (3.5)$$

$$\frac{\partial u(x,t)}{\partial x} = \sum_{k=0}^{\infty} \frac{\partial a_k(x)}{\partial x} t^{\alpha k} \qquad (3.6)$$
$$= \frac{\partial a_0(x)}{\partial x} + \frac{\partial a_1(x)}{\partial x} t^{\alpha} + \frac{\partial a_2(x)}{\partial x} t^{2\alpha} + \dots$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \sum_{k=0}^{\infty} \frac{\partial^2 a_k(x)}{\partial x^2} t^{\alpha k} \qquad (3.7)$$
$$= \frac{\partial^2 a_0(x)}{\partial x^2} + \frac{\partial^2 a_1(x)}{\partial x^2} t^{\alpha} + \frac{\partial^2 a_2(x)}{\partial x^2} t^{2\alpha} + \dots$$

substituting (3.5),(3.6)and (3.7) into (3.3) and comparing the cofficients of t^{α}

$$\sum_{k=1}^{\infty} a_k \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} t^{(k-1)\alpha} = \sum_{k=0}^{\infty} \frac{\partial a_k(x)}{\partial x} t^{\alpha k} + \sum_{k=0}^{\infty} \frac{\partial^2 a_k(x)}{\partial x^2} t^{\alpha k} \qquad (3.8)$$

$$a_1(x)\Gamma(\alpha+1) + a_2(x)\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} t^{\alpha} + a_3(x)\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} t^{2\alpha} + \dots =$$
$$\left(\frac{\partial a_0(x)}{\partial x} + \frac{\partial a_1(x)}{\partial x} t^{\alpha} + \frac{\partial a_2(x)}{\partial x} t^{2\alpha} + \dots\right) \qquad (3.9)$$
$$+\left(\frac{\partial^2 a_0(x)}{\partial x^2} + \frac{\partial^2 a_1(x)}{\partial x^2} t^{\alpha} + \frac{\partial^2 a_2(x)}{\partial x^2} t^{2\alpha} + \dots\right)$$

Then

$$a_1(x)\Gamma(\alpha+1) = \frac{\partial a_0(x)}{\partial x} + \frac{\partial^2 a_0(x)}{\partial x^2} \qquad (3.10)$$

and

$$a_2(x)\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} = \frac{\partial a_1(x)}{\partial x} + \frac{\partial^2 a_1(x)}{\partial x^2} \qquad (3.11)$$

using initial condition $u(x,0) = x$

we have $u_0(x,t) = u(x.0) = x$

Next we determine the $u_k (k = 1,2,\dots).$

therefore we obtain the approximate solution of equation (3.4)

$$u(x,t) = a_0(x) + a_1(x)t^{\alpha} + a_2(x)t^{2\alpha} + \dots$$

for example, if $u_0(x,0) = x$ then form (3.10) and (3.11) we get

$$a_1(x) = \frac{1}{\Gamma(\alpha+1)}, \qquad a_2(x) = 0,$$

then

$$u(x,t)=\sum_{k=1}^{\infty}\frac{1}{\Gamma(k\alpha+1)}t^{k\alpha}=[x+\frac{1}{\Gamma(\alpha+1)}t^{\alpha}+0+...]$$

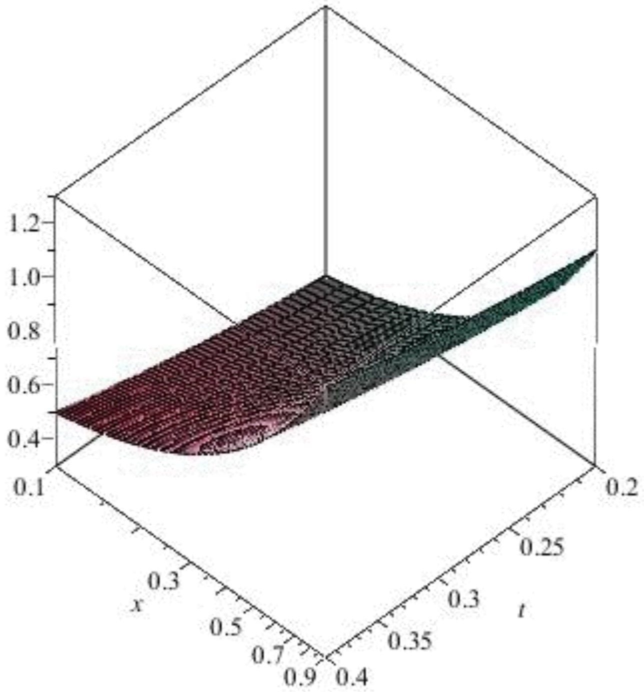

Figure 1. The exact solution of $u(x,t)$ at $\alpha=1$

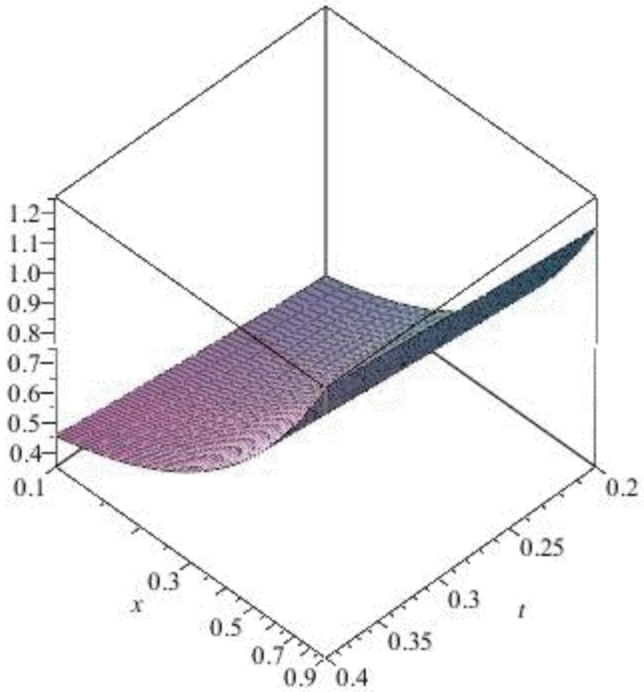

Figure2. The approximate solution of $u(x,t)$ at $\alpha=0.5$

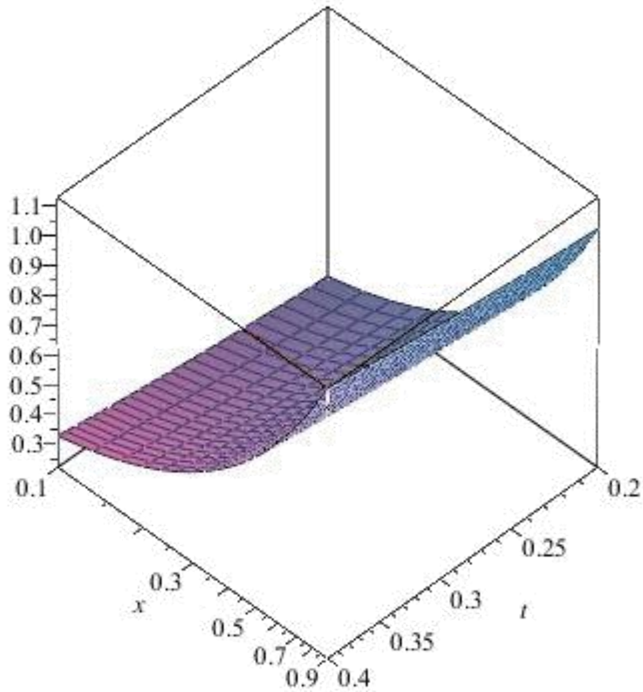

Figure 3. The approximate solution of $u(x,t)$ at $\alpha = 0.9$

SOME PROPERTIES OF CHEBYSHEV POLYNOMIALS OF THETHIRD KIND [12]

Definition:

The Chebyshev polynomials $V_n(t)$ of the third kind ([8], [12]) are orthogonal polynomials of degree n in x defined on the $[-1,1]$

$$V_n(t) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{\theta}{2}},$$

where $t = \cos 0$ and $0 \in [0,\pi]$. They can be obtained explicitly using

the Jacobi polynomials $p_k^{(\alpha,\beta)}(t)$, for the special case $\beta = -\alpha = \frac{1}{2}$. These are given by:

$$V_i(t) = \frac{2^{2i}}{\binom{2i}{i}} p_i^{(-\frac{1}{2},\frac{1}{2})}(t). \tag{4.1}$$

Also, these polynomials $V_n(x)$ are orthogonal on $[-1,1]$ with respect to the inner product:

$$\langle V_n(x), V_m(x) \rangle = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} V_n(x) V_m(x) dx = \{ {}^{0,\ n \neq m,}_{\pi,\ \ n=m,} \tag{4.2}$$

where $\sqrt{\frac{1+t}{1-t}}$ is weight function corresponding to $V_n(t)$.

The polynomials $V_n(t)$ may be generated by using the recurrence relations

$$V_{n+1}(t) = 2tV_n(t) - V_{n-1}(t), \qquad n = 1,2,...,$$

with $V_0(t) = 1,\ V_1(t) = 2t - 1.$

Using Eq.(2) and properties of Jacobi polynomials to obtain the analytical form of the Chebyshev polynomials of the third kind $V_n(x)$ of degree n, they are given as:

$$V_n(t) = \sum_{i=0}^{\lceil \frac{2n+1}{2} \rceil} (-1)^i (2)^{n-i} \frac{(2n+1)\Gamma(2n-i+1)}{\Gamma(i+1)\Gamma(2n-2i+2)} (t+1)^{n-i} \quad n \in Z^+, \tag{4.3}$$

where $\lceil \frac{2n+1}{2} \rceil$ denotes the integral part of $(2n+1)\backslash 2$.

The shifted Chebyshev polynomials of the third kind

Since the range $[0,1]$ is quite often more convenient to use than the range $[-1,1]$, we sometimes map the independent variable $t \in 0,1]$ to the variable s in $[-1,1]$ by the transformation $s = 2t-1$ or $t = \frac{(s+1)}{2}$, and this leads to a shifted Chebyshev polynomials of the third kind $V_n^*(t)$ of degree n in x on $[0,1]$ given by ([8], [12]):

$$V_n^*(t) = V_n(2t-1).$$

These polynomials are orthogonal on the support interval $[0,1]$ as the following inner product:

$$\left\langle V_n^*(x), V_m^*(x) \right\rangle = \int_0^1 \sqrt{\frac{x}{1-x}} V_n^*(x) V_m^*(x) dx = \{ \begin{smallmatrix} 0 & n \neq m, \\ \frac{\pi}{2} & n=m, \end{smallmatrix} \tag{4.4}$$

where $\sqrt{\frac{t}{1-t}}$ is weight function corresponding to $V_n^*(t)$ and normalized by the requirement that $V_n^*(1) = 1$. Also, $V_n^*(t)$ may be generated by using the recurrence relations

$$V_{n+1}^*(t) = 2(2t-1)V_n^*(t) - V_{n-1}^*(t), \quad n = 1,2,...,$$

with starting values $V_1^*(t) = 1$, $V_2^*(t) = 4t-3$.

The analytical form of the shifted Chebyshev polynomials of the third kind $V_n^*(t)$ of degree n in t is given by

$$V_n^*(t) = \sum_{i=0}^{n} (-1)^i (2)^{2n-2i} \frac{(2n+1)\Gamma(2n-i+1)}{\Gamma(i+1)\Gamma(2n-2i+2)} t^{n-i}, \ n \in Z^+. \tag{4.5}$$

In a spectral method, in contrast, the function $g(t)$ square integrable in $[0,1]$, is represented by an infinite expansion of the shifted Chebyshev polynomials of the third kind as follows:

$$g(t) = \sum_{i=0}^{m} b_i V_i^*(t), \tag{4.6}$$

where b_i is a chosen sequence of prescribed basis functions.One then proceeds somehow to estimate as many as possible of the coefficients b_i, thus approximating $g(t)$ by a finite sum of $(m+1)$ terms such as:

$$g_m(t) = \sum_{i=0}^{m} b_i V_i^*(t) \tag{4.7}$$

where the coefficients b_i, $(i = 0,1,...)$ are given by

$$b_i = \frac{1}{\pi} \int_{-1}^{1} g(\frac{t+1}{2}) \sqrt{\frac{1+t}{1-t}} V_i(t) dt \tag{4.8}$$

or

$$b_i = \frac{2}{\pi} \int_0^1 g(t) \sqrt{\frac{t}{1-t}} V_i^*(t) dt \tag{4.9}$$

Main results

the main approximate formula for the function $g_m(t)$ given in (4.7) is presented in the following theorem

Theorem $[12]$:

Let $g_m(t)$ be approximate function in terms of shifted Chebyshev polynomials of the third kind as given in (4.7) . suppose $\mu > 0$ then,we obtain:

$$D^{\alpha}(g_m(t)) = \sum_{i=[\mu]}^{m} \sum_{k=0}^{i-[\mu]} b_i N_{i.k}^{(\mu)} t^{i-k-\mu} \tag{4.10}$$

where

$$N_{i.k}^{(\alpha)} = (-1)^k 2^{(2i-2k)} \frac{(2n+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i-k+1-\mu)}. \tag{4.11}$$

NUMERICAL IMPLEMENTATION

Example :[1]

Consider the time -fractional Fokker planck equation:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{5.1}$$

the initial condition

$$u(x.0) = x \tag{5.2}$$

The exact solution to Eq.(5.1) for the non-fractional case at $\alpha = 1$.

$$u(x,t) = x + t,$$

In order ues the shifted Chebyshev polynomials of the third kind.

we apprximate $u(x.t)$ with $m = 3$ as

$$u_m(x.t) = \sum_{i=0}^{m} u_i(x) V_i^*(t) \tag{5.3}$$

$$u_3(x.t) = \sum_{i=0}^{3} u_i(x) V_i^*(t) \tag{5.4}$$

by theorem

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D^\alpha(u_m(x)) = \sum_{i=[\alpha]}^{m} \sum_{k=0}^{i-[\alpha]} b_i N_{i.k}^{(\alpha)} t^{i-k-\alpha} \tag{5.5}$$

$$D^\alpha(u_m(x)) = \sum_{i=[\mu]}^{m} b_i D^\alpha(V_i^*(t))$$

$$\frac{\partial u}{\partial x} = \sum_{i=0}^{3} u_i'(x) V_i^*(t) \tag{5.6}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \sum_{i=0i}^{3} \ddot{u}_i(x) V_i^*(t) \tag{5.7}$$

Let

substituting (5.5), (5.6) and (5.7) into (5.1) we get

$$\sum_{i=[\alpha]}^{m} \sum_{k=0}^{i-[\alpha]} u_i(x) N_{i.k}^{(\alpha)} t^{i-k-\alpha} - \sum_{i=0}^{3} u_i'(x) V_i^*(t) - \sum_{i=0i}^{3} \ddot{u}_i(x) V_i^*(t) = 0 \tag{5.8}$$

we now collcata eq.(5.8) $(m+1-[\alpha])$ points t t_p , $p = 0,1,2,3...m-[\alpha]$

$$\sum_{i=[\alpha]}^{3} \sum_{k=0}^{i-[\alpha]} u_i(x) N_{i.k}^{(\alpha)} t_p^{i-k-\alpha} - \sum_{i=0}^{3} u_i'(x) V_i^*(t_p) - \sum_{i=0i}^{3} \ddot{u}_i(x) V_i^*(t_p) = 0$$

By using (5.8), (5.9) we obtain the following nonlinear system of ODEs

$$\sum_{i=[\alpha]}^{3} \sum_{k=0}^{i-[\alpha]} u_i(x) N_{i.k}^{(\alpha)} (t_0)^{i-k-\alpha} = \sum_{i=0}^{3} u_i'(x) V_i^*(t_0) + \sum_{i=0i}^{3} \ddot{u}_i(x) V_i^*(t_0) \tag{5.9}$$

$$\sum_{i=[\alpha]}^{3} \sum_{k=0}^{i-[\alpha]} u_i(x) N_{i.k}^{(\alpha)} (t_1)^{i-k-\alpha} = \sum_{i=0}^{3} u_i'(x) V_i^*(t_1) + \sum_{i=0}^{3} \ddot{u}_i(x) V_i^*(t_1) \tag{5.10}$$

Then

$$\ddot{u}_0(x) + k_1 \ddot{u}_1(x) + k_2 \ddot{u}_2(x) = (\sum_{i=[\alpha]}^{3} \sum_{k=0}^{i-[\alpha]} u_i(x) N_{i.k}^{(\alpha)} (t_0)^{i-k-\alpha} - \sum_{i=0}^{3} u_i'(x) V_i^*(t_0)) \tag{5.11}$$

$$\ddot{u}_0(x) + k_{11} \ddot{u}_1(x) + k_{22} \ddot{u}_2(x) = (\sum_{i=[\alpha]}^{3} \sum_{k=0}^{i-[\alpha]} u_i(x) N_{i.k}^{(\alpha)} (t_1)^{i-k-\alpha} - \sum_{i=0}^{3} u_i'(x) V_i^*(t_1)) \tag{5.12}$$

Using the inital condition $u(x.0) = x$ then

$$\sum_{i=0}^{3} (-1)^i u_i(x) = x \tag{5.13}$$

where

$$k_1 = k_{11} = V_0^*(t) = 1, \quad k_2 = k_{22} = V_1^*(t) = 4t - 3$$

we use the finite difference mthod to solving system (5.11)-(5.13) ,we use the notations $t_n = n\Delta t$ to be the integration time $0 \le t_n \le T$, $\Delta t = \frac{T}{N}$, for $n = 1,2$. Then the system (5.11)-(5.13), is discretized and takes the following form

$$\frac{u_0^{n+1}-2u_0^n+u_0^{n-1}}{\Delta t^2}+k_1\frac{u_1^{n+1}-2u_1^n+u_1^{n-1}}{\Delta t^2}+k_2\frac{u_2^{n+1}-2u_2^n+u_2^{n-1}}{\Delta t^2}=(\sum_{i=[\alpha]}^{3}\sum_{k=0}^{i-[\alpha]}u_i(x)N_{i.k}^{(\alpha)}(t_0)^{i-k-\alpha}-\sum_{i=0}^{3}u_i'(x)V_i^*(t_0) \quad (5.14)$$

$$\frac{u_0^{n+1}-2u_0^n+u_0^{n-1}}{\Delta t^2}+k_1\frac{u_1^{n+1}-2u_1^n+u_1^{n-1}}{\Delta t^2}+k_2\frac{u_2^{n+1}-2u_2^n+u_2^{n-1}}{\Delta t^2}=\sum_{i=[\alpha]}^{3}\sum_{k=0}^{i-[\alpha]}u_i(x)N_{i.k}^{(\alpha)}(t_1)^{i-k-\alpha}-\sum_{i=0}^{3}u_i'(x)V_i^*(t_1) \quad (5.15)$$

$$u_0^{n+1}-u_1^{n+1}+u_2^{n+1}-u_3^{n+1}=x \quad (5.16)$$

This system presents the numerical scheme of the proposed Equation (5.1) using the fractional finite difference method. Solving this system using the Newton iteration method yields the numerical solution of the fractional Fokker Planck equation (5.1).

At $n=1$, we will evaluate the values of $u^0=(u_0^0,u_1^0,u_2^0,u_3^0)$ and $u^1=(u_0^1,u_1^1,u_2^1,u_3^1)$ using the initial conditions (5.2). Therefore, we can obtain the solutions

$$u^n=(u_0^n,u_1^n,u_2^n,u_3^n) \qquad n=2,3,...,N$$

Using the numerical scheme (5.14)-(5.16)

Then the approximate solution of the time -fractional Fokker Plank

$$u_0=x \qquad u_1=t \qquad u_2=0$$

Then

$$u_3(x.t)=\sum_{i=0}^{3}u_i(x)V_i^*(t)=u_0+u_1+u_2+...$$

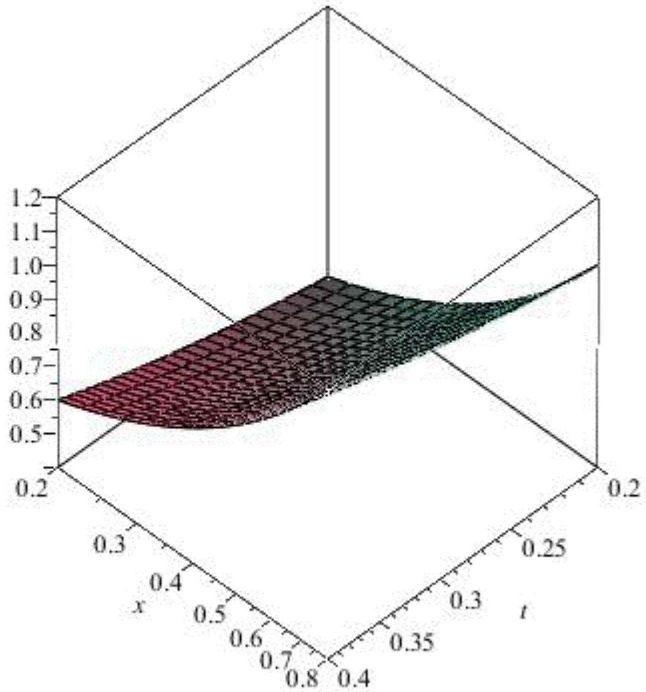

Figure 4. The exact solution of $u(x,t)$ at $\alpha=1$

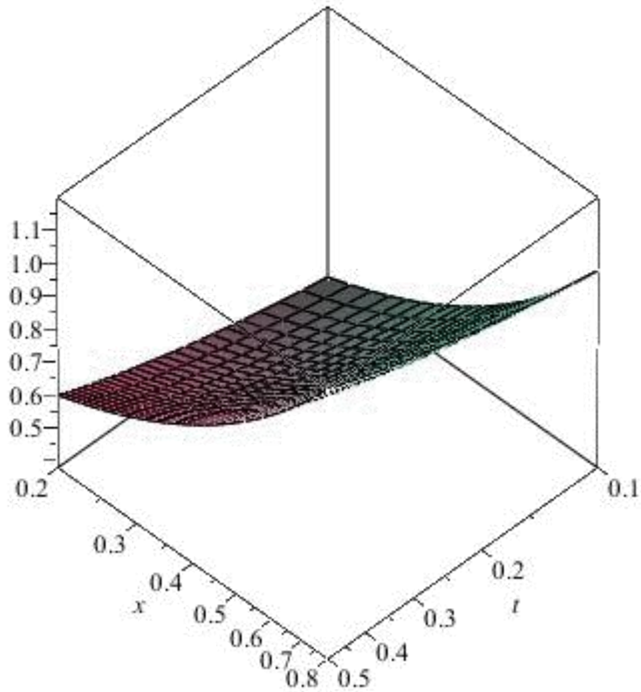

Figure 5. the approximate solution of u(x, t) at $\alpha = 0.5$

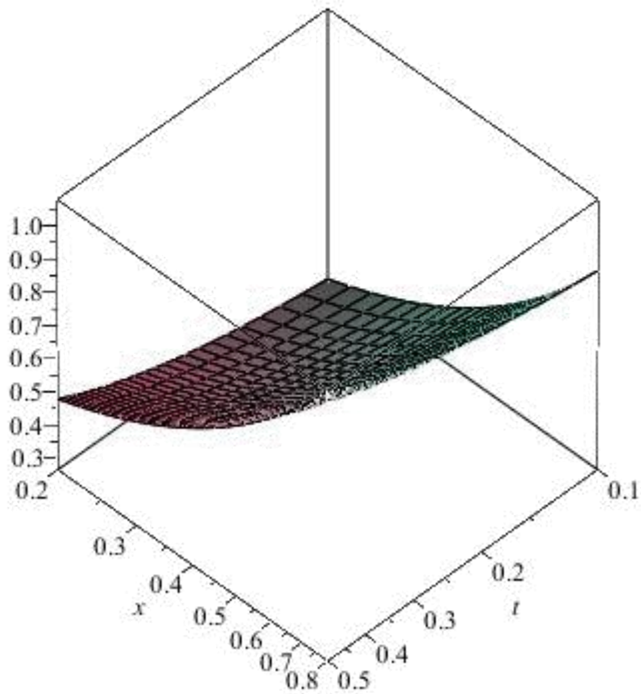

Figure 6. the approximate solution of u(x, t) at $\alpha = 0.9$

CONCLUSION

In this paper, we developed efficient, accurate methods for solving the time-fractional Fokker Planck equation by using fractional power series method and the Chebyshev polynomials of the third kind. The obtained approximate solutions using the suggested methods is in excellent agreement with the exact solution and show that these approaches can be solved the problem effectively and illustrates the validity and the great potential of the proposed technique.

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