International Journal of Applied Engineering Research ISSN 0973-4562 Volume 13, Number 1 (2018) pp. 545-549
© Research India Publications. http://www.ripublication.com

An Analytical Study on Fractional Partial Differential Equations by Laplace Transform Operator Method

S.K.Elagan[1,2], M. Sayed[1,3] and M.Higazy[1,3]

[1]*Mathematics Department, Faculty of Science,Taif University, Taif, Saudi Arabia.*
[2]*Mathematics Department, Faculty of Science, Menofiya University, Shebin Elkom, Egypt.*
[3]*Department of Engineering Mathematics, Faculty of Electronic Engineering, Menoufia University, Menouf 32952, Egypt.*

Abstract

Although a very extensive literature including papers on Laplace transform of a function of a single variable, but a very little is a vailable on the double Laplace transform. This paper deals with the double Laplace transforms and their application to obtain an exact analytic solution of nonhomogeneous space-time-fractional telegraph equation. We used Titchmarsh and Residue theorems to get new exact solutions. The main idea of this paper is to develop the method of the double Laplace transform method to solve initial and boundary value problems in mathematical physics..

Keywords Riemann-Liouville fractional integrals; double Laplace transform ; Titchmarsh theorem; Residue theorem..

Mathematics Subject Classification: 26A33; 33C10; 34A05.

INTRODUCTION

In the past two decades, the widely investigated subject of fractional calculus has remark ablygained importance and popularity due to its demonstrated applications in numerous diverse fields of science and engineering. These contributions to the fields of science and engineering are based on the mathematical analysis. It covers the widely known classical fields such as Abel' s integral equation and viscoelasticity. Also, including the analysis of feedback amplifiers, capacitor theory, generalized voltage dividers, fractional-order Chua-Hartley systems, electrode-electrolyte interface models, electric conductance of biological systems, fractional-order models of neurons, fitting of experimental data, and the fields of special functions [1-6]. Several methods have been used to solve fractional differential equations, fractional partial differential equations, fractional integro-differential equations and dynamic systems containing fractional derivatives, such as Adomian's decomposition method [7–11], He's variational iteration method [12–16], homotopy perturbation method [17–19], homotopy analysis method [20], spectral methods [21–24], and other methods [25–27]. This paper is organized as follows: We begin by introducing some necessary definitions and mathematical preliminaries of the fractional calculus theory. In section 3, the double Laplace transform and the inverse of double Laplace transform is demonstrated. In section 4, the existence conditions of the double Laplace transform is proposed. In section 5, we introduced the convolution theorem of double Laplace transform , In the last section, we apply the method of double Laplace transform to solve the nonhomogeneous space- time-fractional telegraph equation.

PRELIMINARIES AND NOTATIONS

In this section, we give some basic definitions and properties of fractional calculus theory which are further used in this paper.

Definition 2.1. A real function $h(t),\ t>0$, is said to be in the space $C_\mu,\ \mu\in\mathbb{R}$ if there exists a real number $p>\mu,$ such that $h(t)=t^p h_1(t),$

where $h_1(t)\in C[0,\infty),$ and it is said to be in the space C_μ^n if and only if $h^{(n)}\in C_\mu,\ n\in\mathbb{N}$.

Definition 2.2. The Riemann-Liouville fractional integral operator (J^α) of order $\alpha\geq 0,$ of a function $h\in C_\mu,$ $\mu\geq -1,$ is defined as

$$J^\alpha h(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}h(\tau)d\tau \quad (\alpha>0),$$
$$J^0 h(t)=h(t),$$

$\Gamma(\alpha)$ is the well- known Gamma function. Some of the properties of the operator J^α , are as follows:

(1) $J^\alpha J^\beta h(t)=J^{\alpha+\beta}h(t),$

(2) $J^\alpha J^\beta h(t)=J^\beta J^\alpha h(t),$

(3) $J^\alpha t^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma}$, where $\beta\geq 0,$ and $\gamma\geq -1.$

Definition 2.3. The fractional derivative (D^α) of $h(t)$ in the Caputo's sense is defined as

$$D^\alpha h(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}h^{(n)}(\tau)d\tau \ ,$$

for $n-1<\alpha\leq n, n\in\mathbb{N}$ ıd $h\in C_{-1}^n.$

The following are three basic properties of Caputo's fractional Derivative [4]:

(1) Let $h\in C_{-1}^n,\ n\in\mathbb{N}$ Then $D^\alpha h,\ 0\leq\alpha\leq n$ is well defined and $D^\alpha h\in C_{-1}.$

(2) Let $n-1<\alpha\le n$, $n\in\mathbb{N}$ and $h\in C_{\mu}^{n},\ \mu\ge -1$. Then

$$(J^{\alpha}D^{\alpha})h(t)=h(t)-\sum_{k=0}^{n-1}h^{(k)}(0^{+})\frac{t^{k}}{k!}.$$

(3) The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^{\alpha}f(t)=J^{m-\alpha}D^{m}f(t),$$

for $m-1<\alpha\le m, m\in N, t>0$ and $f\in C_{-1}^{n}$.

DEFINITION OF THE DOUBLE LAPLACE TRANSFORM

In [28], the double Laplace transform of a function $f(x,y)$ of two variables x and y defined in the first quadrant of the $x-y$ plane is defined by the double integral in the form

$$\bar{\bar{f}}(p,q)=L_2[f(x,y)]=L[L\{f(x,y);x\to p\}]$$
$$=L\left[\bar{f}(p,y);y\to q\right]=\int_0^{\infty}\int_0^{\infty}f(x,y)e^{(px+qy)}dxdy,$$

provided the integral exists, where

$$\bar{f}(p)=L\{f(x);x\to p\}$$

denotes the Laplace transform of $f(x)$ and to define by

$$\bar{f}(p)=L\{f(x)\}=\int_0^{\infty}e^{-px}f(x)dx, \operatorname{Re}(p)>0,$$

and $L\equiv L_2$ is used throughout this paper. Similarly, $L^{-1}\equiv L_1^{-1}$ is used to denote the inverse Laplace transformation of $\bar{f}(p)$ and to define by

$$f(x)=L^{-1}\left\{\bar{f}(p)\right\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{px}\bar{f}(p)dp, c\ge 0.$$

The inverse double Laplace transform

$$L_2^{-1}\left[\bar{\bar{f}}(p,q)\right]=f(x,y)$$

is defined by the complex double integral formula

$$L_2^{-1}\left[\bar{\bar{f}}(p,q)\right]=f(x,y)$$
$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{px}dp\ \frac{1}{2\pi i}\int_{dc-i\infty}^{d+i\infty}e^{qpx}\bar{\bar{f}}(p,q)dq$$

where $\bar{\bar{f}}(p,q)$ must be analytic function for all p and q in the region defined by the inequalities $\operatorname{Re}p\ge c$ and $\operatorname{Re}q\ge d$, where c and d are real constants to be chosen suitably. It is easy to see that L_2 is linear integral transformation , so $L_2^{-1}\left[\bar{\bar{f}}(p,q)\right]$ satisfies the linear property

$$L_2^{-1}\left[a\bar{\bar{f}}(p,q)+b\bar{\bar{g}}(p,q)\right]=aL_2^{-1}\left[\bar{\bar{f}}(p,q)\right]+bL_2^{-1}\left[\bar{\bar{g}}(p,q)\right],$$

Where a and b are constants. This shows that L_2^{-1} is also a linear transformation .

EXISTANCE CONDITIONS FOR THE DOUBLE LAPLACE TRANSFORM

The $f(x,y)$ is said to be of exponential order $a(>0)$ and $b(>0)$ on $0\le x<\infty, 0\le y<\infty$, if there exists a positive constant K such that for all $x>X$ and $y>Y$

$$|f(x,y)|\le Ke^{aX+bY}$$

and we write

$$|f(x,y)|=O\left(e^{aX+bY}\right) \text{ as } x\to\infty, y\to\infty.$$

Or, equivalently,

$$\lim_{x\to\infty,y\to\infty}e^{-\alpha x-\beta y}|f(x,y)|$$
$$=K\lim_{x\to\infty,y\to\infty}e^{-(\alpha-a)x}e^{-(\beta-b)y}=0, \alpha>a, \beta>b.$$

Such a function $f(x,y)$ is simply called an exponential order as $x\to\infty, y\to\infty$ and clearly, it does not grow faster than $K\exp(ax+by)$ as $x\to\infty, y\to\infty$.

The following theorem gives the conditions which make the double Laplace transform of $f(x,y)$ exists.

Theorem 4.1 If the function $f(x,y)$ is a continuous function in every finite intervals $(0,X)$ and $(0,Y)$ and exponential order $\exp(ax+by)$, then the double Laplace transform of $f(x,y)$ exists for all p and q provided $\operatorname{Re} p > a$ and $\operatorname{Re} q > b.$

The following theorem computes the inverse Laplace transform for some complicated functions.

Theorem 4.2 (**Titchmarsh Theorem)** Let $F(p)$ be an analytic function having no singularities in the cut plane $\mathbb{C}\ \mathbb{R}$. Assuming that $\overline{F(p)} = F(\overline{p})$ and the limiting values

$$F^{\pm}(t) = \lim_{\phi \to \pi^-} F\left(te^{\pm i\phi}\right),\ F^{+}(t) = \overline{F^{-}(t)}$$

exist for almost all

(i) $F(p) = o(1)\ for\ |p| \to \infty\ and\ F(p) = o\left(|p|^{-1}\right)\ for\ |p| \to 0$

uniformly in any sector

$$|\arg p| < \pi - \eta, \pi > \eta > 0,\ \ \pi - \varepsilon < \phi \leq \pi.$$

(ii) There exists $\varepsilon > 0$ such that for every

$$\frac{F\left(re^{\pm i\phi}\right)}{1+r} \in L_1(\mathbb{R} \qquad e^{\pm i\phi}\Big)\Big| \leq a(r).$$

Where $a(r)$ doesn't depend on ϕ and $a(r)e^{-\delta r} \in L_1(\mathbb{R}$ for any $\delta > 0$.

Then

$$f(t) = L^{-1}\left[F(s)\right] = \frac{1}{\pi}\int_0^{\infty} \operatorname{Im}\left[F^{-}(\eta)\right]e^{-t\eta}\,d\eta\ .$$

COVOLUTION THEOREM OF THE DOUBLE LAPLACE TRANSFORM

Definition 5.1 [29]. The convolution of $f(x,y)$ and $g(x,y)$ is denoted by $(f**g)(x,y)$ and defined by

$$(f**g)(x,y) = \int_0^x\int_0^y f(x-\xi, y-\eta)\,g(\xi,\eta)\,d\xi\, d\eta.$$

We observe that the convolution is commutative, that is .

$$(f**g)(x,y) = (g**f)(x,y)$$

Theorem 5.1 (Convolution Theorem).

If $L_2[f(x,y)] = \bar{f}(p,q)$ and $L_2[g(x,y)] = \bar{g}(p,q)$, then

$$L_2\left[(f**g)(x,y)\right] = L_2\{f(x,y)\}L_2\{g(x,y)\} = \bar{f}(p,q)\bar{g}(p,q).$$

Or equivalenty,

$$L_2^{-1}\left[\bar{f}(p,q)\,\bar{g}(p,q)\right] = (f**g)(x,y).$$

NONHOMOGENOUS SPACE-TIME FRACTIONAL TELEGRAPH EQUATION

In this section, we consider the following nonhomogeneous space- time fractional telegraph equation

$$\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}} = \frac{\partial^{2\beta}u}{\partial t^{2\beta}} + \frac{\partial^{\beta}u}{\partial t^{\beta}} + u, \quad 0 < \alpha, \beta \leq 1$$

With the initial and boundary conditions

$$u(0,t) = e^{-t}, u_x(0,t) = e^{-t}, u(\mathrm{x},0) = e^{x}, u_t(\mathrm{x},0) = e^{x},$$

Now applying two dimensional Laplace transform with respect to x,t we get

$$p^{2\alpha}U(p,q) - p^{2\alpha-1}\bar{U}(0,q) - p^{2\alpha-2}\bar{U}x(0,q) = q^{2\beta}U(p,q)$$
$$-q^{2\beta-1}\bar{U}(\mathrm{p},0) - q^{2\beta-2}\bar{U}_t(\mathrm{p},0) + q^{\beta}U(p,q) - q^{\beta-1}\bar{U}(\mathrm{p},0) + U(p,q).$$

In which

$$U(p,q) = L_r^{p,q}\{u(x,t): x \to p, \mathrm{t} \to q\}, \bar{U}(0,q) = L\{u(0,t): \mathrm{t} \to q\}$$
$$\bar{U}_x(0,q) = L\{u_x(0,t): \mathrm{t} \to q\}, \bar{U}(\mathrm{p},0) = L\{u(\mathrm{x},0): \mathrm{x} \to p\}$$
$$\bar{U}t(\mathrm{p},0) = L\{u_t(\mathrm{x},0): \mathrm{x} \to p\}.$$

After simplifying we have:

$$U(p,q) = \frac{1}{p^{2\alpha} - q^{2\beta} - q^{\beta} - 1}\left(\frac{p^{2\alpha-1}}{q+1} + \frac{p^{2\alpha-2}}{q+1} - \frac{q^{2\beta-1}}{p-1} - \frac{q^{2\beta-2}}{p-1} - \frac{q^{\beta-1}}{p-1}\right).$$

Taking inverse Laplace transform first with respect to p and then q, we have

$$u(x,t) = L_{p,q}^{-1}\left\{\frac{1}{p^{2\alpha} - q^{2\beta} - q^{\beta} - 1}\left(\frac{p^{2\alpha-1} + p^{2\alpha-2}}{q+1} - \frac{q^{2\beta-1} + q^{2\beta-2} + q^{\beta-1}}{p-1}\right)\right\}.$$

By using Titchmarsh Theorem, we have

$$L_{p,q}^{-1}\left\{\frac{p^{2\alpha-1}}{q+1}, p\to x, q\to t\right\}=\frac{e^{-t}}{\pi}\int_0^\infty e^{-rx}\, r^{2\alpha-1}\sin((1-2\alpha)r)dr.$$

Similarly we have:

$$L_{p,q}^{-1}\left\{\frac{p^{2\alpha-1}+p^{2\alpha-2}}{q+1}-\frac{q^{2\beta-1}+q^{2\beta-2}+q^{\beta-1}}{p-1}\right\}=$$

$$\left\{\begin{array}{l}\left(\frac{e^{-t}}{\pi}\int_0^\infty e^{-rx}\, r^{2\alpha-1}\sin((1-2\alpha)r)dr\right)+\left(\frac{e^{-t}}{\pi}\int_0^\infty e^{-rx}\, r^{2\alpha-2}\sin((2-2\alpha)r)dr\right)-\\ \left(\frac{e^{x}}{\pi}\int_0^\infty e^{-rt}\, r^{2\beta-1}\sin((1-2\beta)r)dr\right)-\left(\frac{e^{x}}{\pi}\int_0^\infty e^{-rt}\, r^{2\beta-2}\sin((2-2\beta)r)dr\right)-\\ \left(\frac{e^{x}}{\pi}\int_0^\infty e^{-rt}\, r^{\beta-1}\sin((1-\beta)r)dr\right)\end{array}\right\}$$

$$=e^{-t}\left(\frac{x^{-2\alpha}}{\Gamma(1-2\alpha)}+\frac{x^{1-2\alpha}}{\Gamma(2-2\alpha)}\right)-e^{x}\left(\frac{t^{-2\beta}}{\Gamma(1-2\beta)}+\frac{t^{1-2\beta}}{\Gamma(2-2\beta)}+\frac{t^{-\beta}}{\Gamma(1-\beta)}\right).$$

.

Let $\alpha\to\frac{1}{2}$ and $\beta\to\frac{1}{2}$,

then

$$e^{-t}\left(\frac{x^{-2\alpha}}{\Gamma(1-2\alpha)}+\frac{x^{1-2\alpha}}{\Gamma(2-2\alpha)}\right)-e^{x}\left(\frac{t^{-2\beta}}{\Gamma(1-2\beta)}+\frac{t^{1-2\beta}}{\Gamma(2-2\beta)}+\frac{t^{-\beta}}{\Gamma(1-\beta)}\right)$$

$$=e^{-t}-e^{t}\left(1+\frac{1}{\sqrt{\pi x}}\right)=J(x,t)$$

Now we find the double inverse Laplace transform for the function

$$I=L_{p,q}^{-1}\left\{\frac{1}{p^{2\alpha}-q^{2\beta}-q^{\beta}-1}\right\}.$$

when $\alpha=\beta=\frac{1}{2}$. So the double integral becomes

$$I=\frac{1}{(2\pi i)^2}\int_{c-i\infty}^{c+i\infty}\int_{c-i\infty}^{c-i\infty}\frac{e^{px+qy}}{p-q-\sqrt{q}-1}dp\,dq,$$

and the $p-$integral has a pole at $p_0=q+\sqrt{q}+1$. Hence assuming that c has been a suitable chosen, with $\Re p_0<c$, we can evaluate the $p-$integral by taking the residue and obtain

$$I=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{p_0x+qt}dq=\frac{e^{x}}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{(x+t)q+x\sqrt{q}}dq.$$

Observe that this cannot be seen as a Laplace inversion interval because of $e^{x\sqrt{q}}$ when $x>0$. Anyhow , we proceed, writing

$$(x+t)q+x\sqrt{q}=(x+t)\left(\sqrt{q}+\frac{x}{2(x+t)}\right)^2-\frac{x^2}{4(x+t)},$$

and substituting

$$z=\sqrt{q}+\frac{x}{2(x+t)},$$

to obtain

$$I=\frac{e^{\frac{x(3x+t)}{4(x+t)}}}{\pi i}\int_{c-i\infty}^{c+i\infty}e^{(x+t)z^2}\left(z-\frac{x}{2(x+t)}\right)dz.$$

Take $c=0$ which is allowed because there are no singular points in the integrand. Then we observe that the exponential function in the integrand is even in z . Hence the odd term z in the integrand will not give a contribution, and

$$I=-\frac{x}{(x+t)}\frac{e^{\frac{x(3x+t)}{4(x+t)}}}{\pi i}\int_0^{i\infty}e^{(x+t)z^2}dz$$

$$=-\frac{x}{(x+t)}\frac{e^{\frac{x(3x+t)}{4(x+t)}}}{\pi i}\frac{i\sqrt{\pi}}{2\sqrt{x+t}}$$

$$=-\frac{xe^{\frac{x(3x+4t)}{4(x+t)}}}{2\sqrt{\pi}(x+t)^{\frac{3}{2}}}.$$

Then the double convolution $(I**J)(x,t)$ reads

$$(I**J)(x,t)=u(x,t)=\int_0^x\int_0^t I(x-\xi,t-\tau)J(\xi,\tau)d\xi d\tau$$
$$=\int_0^x\int_0^t I(\xi,\tau)J(x-\xi,t-\tau)d\xi d\tau.$$

So, in the present case,

$$u(x,t)=-\frac{1}{2\sqrt{\pi}}\int_0^x\int_0^t\frac{\xi e^{\frac{\xi(3\xi+\tau)}{4(\xi+\tau)}}}{(\xi+\tau)^{\frac{3}{2}}}\times\left(e^{-(t-\tau)}-e^{x-\xi}\left(1+\frac{1}{\sqrt{\pi(x-\xi)}}\right)\right)d\xi d\tau.$$

CONCLUSIONS

We have applied double Laplace transform method to obtain the exact solutions of nonhomogeneous space- time-fractional telegraph equation. Our example shows that double Laplace transform method is capable of reducing the volume of computational work as compared to other methods.

REFERENCES

[1] I. Podlubny, Fractional differential equations, Academic Press, San Diego, 1999.

[2] I. Podlubny, Geometric and physical interpretation of fractional integration and fractional differentiation, Fract.Calculus. Appl. Anal, 5(2002):367-386.

[3] J. He, Nonlinear oscillation with fractional derivative and its applications, International Conference on Vibrating Engineering, Dalian, China, (1998):288-291.

[4] J. He, Some applications of nonlinear fractional differential equations and their approximations, Bull. Sci. Technol, 15(1999):86-90.

[5] J. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods. Appl. Mech. Eng, 167(1998):57-68.

[6] I. Grigorenko, E. Grigorenko, Chaotic dynamics of the fractional lorenz system, Phys. Rev. Lett, 91(2003):034101.

[7] S. Momani, N. T. Shawagfeh, Decomposition method for solving fractional Riccati differential equations, Appl.Math. Comput. 182(2006):1083-1092.

[8] S. Momani, M. A. Noor, Numerical methods for fourth-order fractional integrodifferential equations, Appl. Math. Comput, 182(2006):754-760.

[9] V. D. Gejji, H. Jafari, Solving a multi-order fractional differential equation, *Appl. Math. Comput*, 189(2007):541-548.

[10] S. S. Ray, K. S. Chaudhuri, R. K. Bera, Analytical approximate solution of nonlinear dynamic system containing fractional derivative by modified decomposition method, *Appl. Math. Comput*, 182(2006):544-552.

[11] Q.Wang, Numerical solutions for fractional KdV-Burgers equation by Adomian decomposition method, *Appl. Math.Comput*, 182(2006):1048-1055.

[12] M. Inc, The approximate and exact solutions of the space- and time-fractional Burgers equations with initial conditions by variational iteration method, *J.Math. Anal. Appl*, 345(2008):476-484.

[13] S. Momani, Z. Odibat, Analytical approach to linear fractional partial differential equations arising in fluid mechanics, *Phys. Lett. A*, 355(2006):271-279.

[14] Z. Odibat, S. Momani, Application of variational iteration method to nonlinear differential equations of fractional order, *Int. J. Nonlinear Sci. Numer. Simul*, 7 (2006):271-279.

[15] S. Momani, Z. Odibat, Homotopy perturbation method for nonlinear partial differential equations of fractional order, Phys. Lett. A, 365(2007):345-350.

[16] N. H. Sweilam, M. M. Khader, R. F. Al-Bar, Numerical studies for a multi-order fractional differential equation, Phys. Lett. A, 371(2007):26-33.

[17] K. A. Gepreel, The homotopy perturbation method applied to the nonlinear fractional Kolmogorov-Petrovskii-PISkunov equations, Applied Mathematics Letters, 24(8)(2011):1428–1434.

[18] M. A. Herzallah and K. A. Gepreel, Approximate solution to time—space fractional cubic nonlinear Schrodinger equation, Applied Mathematical Modeling, 36(11)(2012): 5678–5685.

[19] M. S. Mohamed, Analytical treatment of Abel integral equations by optimal homotopy analysis transform method, Journal of Information and Computing Science; 10(1) (2015): 19-28.

[20] Mohamed S. Mohamed, Khaled A. Gepreel, Faisal Al-Malki, Maha Al-humyani, Approximate solutions of the generalized Abel's integral equations using the extension Khan's homotopy analysis transformation method, Journal of Applied Mathematics. (Hindawi Publishing Corporation) (Volume 2015, Article ID 357861, 9 pages).

[21] K. A. Gepreel and M. S. Mohamed. Analytical approximate solution for nonlinear space-time fractional Klein Gordon equation, Chinese physics B, 22(1)(2013):010201-6.

[22] K. Hemida, M. S. Mohamed. Numerical simulation of the generalized Huxley equation by homotopy analysis method, Journal of applied functional analysis, 5(4)(2010):344-350.

[23] K. A. Gepreel and M. S. Mohamed, An optimal homotopy analysis method nonlinear fractional differential equation, Journal of Advanced Research in Dynamical and Control Systems, 6(1)(2014) : 1–10.

[24] K. A. Gepreel and S. Omran, Exact solutions for nonlinear partial fractional differential equations, Chinese Physics B, 21(11)(2012): 110204–110211.

[25] R. Churchill, Operational Mathematics (3rd edition), McGraw-Hill,New York, 1972.

[26] D.G. Duffy. Transform Methods for Solving Partial Differential. Chapman and Hall/CRC, 2004.

[27] G. Doetsch, Introduction to the Theory and Applications of the Laplace Transformation, Springer-Verlag, Berlin 1974.

[28] S.K.Elagan , On the Invalidity of Semigroup Property for the Mittag-Leffler Function with Two Parameters", Egyptian Journal of Mathematical Society 24 (2), (2016),: 200-203.

[29] Lokenath Debnath, The Double Laplace Transforms and Their Properties with Applications to Functional, Integral and Partial Differential Equations, Int. J. Appl. Comput. 2 (2016) 23–241.