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Algebraic Properties of Plus Weighted Finite state Machines

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Abstract: The purpose of this paper is to introduce and study the concepts of semigroups, homomorphism and admissible relations on plus weighted finite state machine(pwfm). We establish their basic properties for the algebraic study on plus weighted finite state machine.

Keywords: Plus weighted Finite state Machine, semigroups, homomorphism, admissible relations.

INTRODUCTION

Mathematical models in classical computation, automata have been an important area in theoretical computer science [5]. It started from the seminal papers of Kleene [7] and within a few years developed into a rich mathematical research topic. From the beginning finite automata constituted a core of computer science. Part of the reason is that they capture something very fundamental as it is witnessed by a numerous different characterizations of the family of rational languages, i.e., languages defined by finite automata [5]. In fact, the interrelation of finite automata and their applications in computer science is a splendid example of really fruitful connection of theory and practical which will accept regular language [13]. Finite automata plays a crucial role in the theory of programming languages, compiler constructions, switching circuit designing, computer controllers,neuron net, text editor and lexical analyzer [1].

Weighted automata were introduced by Schutzenberger (1961). Weighted finite automata are classical nondeterministic finite automata in which the transition carry weights. These weights may model e.g., the cost involved when executing a transition, the amount of resources or time needed for this, or probability or reliability of its successful execution. The behavior of weighted finite automata can then be considered as the function (suitably defined) associating with each word the weight of its execution. Clearly, weights can also be added to classical automata with infinite state sets like pushdown automata; this extension constitutes the general concept of weighted automata.

The main aim of this paper is to apply algebraic concepts on plus weighted finite state machine. Section 1 comprises of preliminaries. It presents some basic definition which are necessary for the succeeding sections. Section 2 introduces plus weighted finite state machine (pwfm), two different semigroups namely S_P and $E(P)$ respectively. Homomorphism and strong homomorphism on pwfms are defined in Section 3. If there is a homomorphism between two pwfms then the existence of homomorphism (strong homomorphism) for a string in X_1^* is also proved. In Section 4 Admissible relations on the set of states is defined and some results are obtained.

PRELIMINARIES

In this section, we provide some basic definition concerning semigroups [10] that are needed for the development of the succeeding sections on hand. Also definition for weighted automata [2] and finite state machine are discussed.

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Definition 1. Let X be a non-empty set. Then, any function from $X \times X$ into X is called a binary operation on X. If $*$ is a binary operation on X, then the pair $(X, *)$ is called an algebraic system. If $(X, *)$ is an algebraic system such that $\forall a, b, c \in X$, $(a * b) * c = a * (b * c)$ then $*$ is called associative and $(X, *)$ is called a semigroup.

Definition 2. Let $(X, *)$ be an algebraic system. If there exists $e \in X$ such that $\forall a \in X$, $a*e = e*a$, then e is called an identity element and $(X, *)$ is said to have an identity. If $(X, *)$ has an identity, then it is easily seen that the identity is unique. A semigroup $(X, *)$ is called a monoid if it has an identity.

Definition 3. Let $(X, *)$ be a semigroup and $\equiv$ be an equivalence relation on X. Then, $\equiv$ is called a right (left) congruence relation on X if $\forall a, b, c \in X$, $a \equiv b$ implies $a * c \equiv b * c\,(c * a \equiv c * b)$. A right and left congruence relation $\equiv$ on X is called a congruence relation. The number of congruence classes of $\equiv$, i.e., equivalence classes of $\equiv$, is called the index of $\equiv$.

Definition 4. Let $(X, *)$ and $(Y, \circ)$ be semigroups. A function f from X into Y is called a homomorphism if $\forall\ a, b \in X$, $f(a*b) = f(a) \circ f(b)$. Let f be a homomorphism of X into Y. If f is one-one, then f is called a monomorphism. If f is onto Y, then f is called an epimorphism. If f is both a monomorphism and epimorphism, then f is called an isomorphism and X and Y are said to be isomorphic.

Definition 5. Let $(X, *)$ and $(Y, \circ)$ be monoids. A function f from X into Y is called a homomorphism if $\forall\ a, b \in X$, $f(a * b) = f(a) \circ f(b)$ and $f(e) = e'$ where e and e' are the identity elements of X and Y respectively.

Definition 6. A semiring is a structure S $= (S, +, \cdot, 0, 1)$ where

(i) $(S, +, 0)$ is a commutative monoid,
(ii) $(S, \cdot, 1)$ is a monoid,
(iii) "$\cdot$" is distributive over "$+$",
(iv) 0 is an annihilator w.r.to "$\cdot$."

Definition 7. A weighted Automaton over semiring S is a structure A $= (Q, A, \lambda, \mu, \gamma)$ where

(i) Q is a finite non-empty set of states
(ii) A is the input alphabet.
(iii) $\mu : Q \times A \times Q \to S$ is the transition weight function.
(iv) $\lambda : Q \to S$ is the initial-weight function.
(v) $\gamma : Q \to S$ is the finial-weight function.

Definition 8. A six tuple $M = (Q, X, Y, f, g, s)$ is called a finite- state machine if,

(i) Q is a finite non-empty set of states
(ii) X is a finite non-empty set of input symbols.
(iii) Y is a finite non-empty set of output symbols.
(iv) $f : Q \times X \to Q$ is called the state transition function.
(v) $g : Q \times X \to Y$ is called the output function.
(vi) The state $s \in Q$ is called the initial state.

SEMIGROUPS ON PLUS WEIGHTED FINITE STATE MACHINE

This section Introduces plus weighted finite state machine(pwfm), two different semigroups namely S_P and $E(P)$ respectively.

Definition 9. A plus weighted finite state machine is a four tuple (pwfm)
$P = (Q, X, W, \mu)$, where

(i) Q is a finite non-empty set of states.
(ii) X is a finite non-empty set of input symbols.
(iii) W is a weighting space. *i.e.*, weighting space $W = ([0, \infty), +, \cdot)$ where + usual addition . usual multiplication.
(iv) The plus weighted subset $\mu : Q \times X \times Q \to [0, \infty)$ is a function called the weighted transition function.

The extension of μ^* to $Q \times X^* \times Q \to [0, \infty)$ is defined by

(i) $\mu^*(q, \lambda, p) = \begin{cases} 1, \text{if } q = p \\ 0, \text{if } q \neq p \end{cases}$
(ii) $\mu^*(q, xa, p) = \sum_{r \in Q} \mu^*(q, x, r).\mu(r, a, p)\ \forall x \in X^*\ a \in X.$

Let $X^+ = X^* \setminus \lambda$. Then X^+ is a semigroup. For μ^* given in above definition, we let $\mu = \mu^*$ restricted to $Q \times X^+ \times Q$.

Theorem 10. Let $P = (Q, X, W, \mu)$ be an pwfm. Define a relation $\equiv$ on X^* by $\forall x, y \in X^*, x \equiv y$ if and only if $\mu^*(q, x, p) = \mu^*(q, y, p)\,\forall q, p \in Q$. Then $\equiv$ is a congruence relation on X^*

Proof. Let $x \in X^*$, then $\mu^*(q, x, q) = \mu^*(q, x, q)$. Therefore $\equiv$ is reflexive.

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Let $x, y \in X^*$, If $x \equiv y$, then $\mu^*(q, x, p) = \mu^*(q, y, p)$ implies that $\mu^*(q, y, p) = \mu^*(q, x, p)$, then we have $y \equiv x$. Therefore $\equiv$ is symmetric.
$x, y, z \in X^*$, If $x \equiv y$, then $\mu^*(q, x, p) = \mu^*(q, y, p)$ and if $y \equiv z$, then $\mu^*(q, y, p) = \mu^*(q, z, p)$, implies that $\mu^*(q, x, p) = \mu^*(q, y, p) = \mu^*(q, z, p)$.
Now $\mu^*(q, x, p) = \mu^*(q, z, p)$, implies that $x \equiv z$, Thus $\equiv$ is transitive. Therefore $\equiv$ is an equivalence relation. Let $z \in X^*$ and let $x \equiv y$, Then $\forall q, p \in Q$,

$$\begin{aligned}\mu^*(q, xz, p) &= \sum_{r \in Q} \mu^*(q, x, r).\mu(r, z, p) \\ &= \sum_{r \in Q} \mu^*(q, y, r).\mu(r, z, p) \quad \text{(Since } x \equiv y) \\ &= \mu^*(q, yz, p).\end{aligned}$$

Thus $xz = yz$. Similarly $zx = zy$. Thus $\equiv$ is a congruence relation on X^*. ■

Theorem 11. Let $P = (Q, X, W, \mu)$ be a pwfm. Let $x \in X^*$, $[x] = \{y \in X^* | x \equiv y\}$, and $E(P) = \{[x] | x \in X^*\}$. Then $(E(P), *)$ is a semigroup with identity.

Proof. Define a binary operation on $E(P)$ by $\forall [x], [y] \in E(P), [x] * [y] = [xy]$.
Clearly $*$ is well defined. Let $[x], [y], [z] \in X^*$. Now,

$$\begin{aligned}[x] * ([y] * [z]) = [x] * [yz] = [xyz] = [xy] * [z] \\ = ([x] * [y]) * [z].\end{aligned}$$

Therefore * is associative. Consider,

$$[x] * [\lambda] = [x\lambda] = [\lambda x] = [\lambda] * [x] \quad \forall \quad [x] \in E(P).$$

Thus λ is the identity of $(E(P), *)$. Hence $(E(P), *)$ is a semigroup with identity. ■

Example 12. Consider the pwfm $P = (Q, X, W, \mu)$, where $Q = \{q\}$, $X = \{a, b\}$, $\mu : Q \times X \times Q \to [0, \infty)$ is defined as follows:

$$\mu(q, a, q) = 2 \qquad \mu(q, b, q) = 2.$$

Then $\forall \quad x, y \in X^+, \mu^*(q, x, q) = 2^n$, where n is the length of x
$\mu^*(q, y, q) = 2^n$ where n is the length of y. Here, $\mu^*(q, x, q) = \mu^*(q, y, q) = 2^n$, where $|x| = |y| = n$, then, $x \equiv y$. $[\lambda] * [x] = [x] * [\lambda] = [x]$. Hence $[\lambda]$ is the identity of $E(P)$. Thus $E(P) = \{[\lambda], [x]\}$ is a semigroup with identity.

Example 13. Consider the pwfm $P = (Q, X, W, \mu)$, where $Q = \{q\}$, $X = \{a, b\}$, $\mu : Q \times X \times Q \to [0, \infty)$ is defined as follows:

$$\mu(q, a, q) = 2; \qquad \mu(q, b, q) = 3.$$

Then $\forall \quad x, y \in X^+, \mu^*(q, x, q) = 2^m \cdot 3^n \ \forall m, n \geq 0$. Now, $[x] = \{ \cup_{i \in I} [x_i]\}$ where $[x_i] = \{y_i \mid |x_i|_a = |y_i|_a$ and $|x_i|_b = |y_i|_b\}$ Here $[\lambda] * [x] = [x] * [\lambda] = [x]$. Hence $[\lambda]$ is the identity of $E(P)$. Hence $E(P) = \{[\lambda], [x]\}$ is a semigroup with identity.

Theorem 14. Let $P = (Q, X, W, \mu)$ be an pwfm. Let $x, y \in X^*$. Define a relation $\approx$ on X^* by $x \approx y$ iff $\forall \ s, t \in Q, \mu^*(s, x, t) > 0$ iff $\mu^*(s, y, t) > 0$. Then $\approx$ is a congruence relation on X^*.

Proof. Let $x \in X^*$, then $\mu^*(q, x, q) > 0$. Therefore $\approx$ is reflexive. Let $x, y \in X^*$, If $x \approx y$, then $\mu^*(q, x, p) > 0$ iff $\mu^*(q, y, p) > 0$. implies that $\mu^*(q, y, p) > 0$ iff $\mu^*(q, x, p) > 0$, then we have $y \approx x$. Therefore $\approx$ is symmetric. $x, y, z \in X^*$, If $x \approx y$, then $\mu^*(q, x, p) > 0$ iff $\mu^*(q, y, p)$ and if $y \approx z$, then $\mu^*(q, y, p) > 0$ iff $\mu^*(q, z, p)$, implies that $\mu^*(q, x, p) > 0$ iff $\mu^*(q, y, p) > 0$ iff $\mu^*(q, z, p) > 0$.
Now $\mu^*(q, x, p) > 0$ iff $\mu^*(q, z, p) > 0$, implies that $x \approx z$, Thus $\approx$ is transitive. Therefore $\equiv$ is an equivalence relation. Let $x, y \in X^*$. Then $x \approx y$ iff $\forall \ s, t \in Q$, $\mu^*(s, x, t) > 0$ iff $\mu^*(s, y, t) > 0$. Let $z \in X^*$. Then $\forall \ s, t \in Q \ \mu^*(s, zx, t) = \sum_{r \in Q} \mu^*(s, z, r) \cdot \mu^*(r, x, t) > 0$
$\Leftrightarrow \ \exists \ u \in Q$ such that $\mu^*(s, z, u) \cdot \mu^*(u, x, t) > 0$
$\Leftrightarrow \ \exists \ u \in Q$ such that $\mu^*(s, z, u) \cdot \mu^*(u, y, t) > 0$
$\Leftrightarrow \mu^*(s, zy, t) = \sum_{r \in Q} \mu^*(s, z, r) \cdot \mu^*(r, y, t) > 0$.
Hence $zx \approx zy$. Similarly $xz \approx yz$. Thus $\approx$ is a congruence relation on X^*. ■

Theorem 15. Let $P = (Q, X, W, \mu)$ be an pwfm. Let $x \in X^*$,
$[[x]] = \{y \in X^* | x \approx y\}$, and $\widetilde{E(P)} = \{[[x]] | x \in X^*\}$. Then $(\widetilde{E(P)}, \tilde{*})$ is a semigroup with identity and $[x] \to [[x]]$ is a homomorphism of $E(P)$ onto $\widetilde{E(P)}$.

Proof. Define a binary operation $\tilde{*}$ on $\widetilde{E(P)}$ by $\forall \quad [[x]], [[y]] \in \widetilde{E(P)}$,
$[[x]] \tilde{*} [[y]] = [[xy]]$. Clearly $(\widetilde{E(P)}, \tilde{*})$ is a semigroup with identity.
Define $f : E(P) \to \widetilde{E(P)}$ by $f([x]) = [[x]] \ \ \forall x \in E(P)$. We show that f is well defined.

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Let $x, y \in X^*$ and $[x] = [y]$. Then $\forall\ s,t \in Q$, $\mu^*(s,x,t) = \mu^*(s,y,t)$. Thus $\forall\ s,t \in Q, \mu^*(s,x,t) > 0 \Leftrightarrow \mu^*(s,y,t) > 0$.Hence $x \approx y$ or $[[x]] = [[y]]$. Thus f is well defined. Next we show that f is a homomorphism. Let $f([x]), f([y]) \in f(E(P))$. Now $f([xy]) = [[xy]] = [[x]]\tilde{*}[[y]] = f([x])\tilde{*}f([y])$. Thus f is a homomorphism. Finally we prove f is onto. Let $[[x]] \in \widetilde{E(P)}$., for some $x \in X^*$. Therefore $[x] \in E(P)$ and we have $f([x]) = [[x]]$. Hence f is onto. Therefore f is an onto homomorphism. ■

Example 16. Consider the pwfm $P = (Q, X, W, \mu)$, where $Q = \{q\}$,
$X = \{a, b\}, \mu : Q \times X \times Q \to [0,\infty)$ is defined as follows:

$$\mu(q,a,q) = 2 \qquad \mu(q,b,q) = 2.$$

Then $\forall\ x, y \in X^*, \mu^*(q,x,q) = 2^n$, where $|x| = n$
$\mu^*(q,y,q) = 2^n$ where $|y| = n$.
Now, $[[x]] = \{y \in X^* \mid \mu^*(q,x,q) > 0\ and\ \mu^*(q,y,q) > 0\}$
Here $[[\lambda]] \,\widetilde{*}\, [[x]] = [[x]] \,\widetilde{*}\, [[\lambda]] = [[x]]$.
Hence $\widetilde{E(P)} = \{[[x]], [[\lambda]]\}$ where $x \in X^+$.

Definition 17. Let $P = (Q, X, W, \mu)$ be an pwfm. For all $x \in X^*$ define the plus weighted subset $x^P of\, Q \times Q$ by $x^P(s,t) = \mu^*(s,x,t)\ \forall s,t \in Q$.

Theorem 18. Let $P = (Q, X, W, \mu)$ be an pwfm. Let $S_P = \{x^P | x \in X^*\}$. Then
(1) $x^P \circ y^P = (xy)^P\ \ \forall x, y \in X^*$,
(2) $(S_P, \circ)$ is a semigroup with identity,
where $\circ$ is defined as $(x^P \circ y^P)(s,t) = \sum_{q\in Q} x^P(s,q) \cdot y^P(q,t)$.

Proof. (1) Let $s,t \in Q$. Then $(xy)^P(s,t)$

$$\begin{aligned} &= \mu^*(s,xy,t) \\ &= \sum_{q\in Q} \mu^*(s,x,q) \cdot \mu^*(q,y,t) \\ &= \sum_{q\in Q} x^P(s,q) \cdot y^P(q,t) \\ &= x^P \circ y^P(s,t). \end{aligned}$$

Thus $(xy)^P = x^P \circ y^P$.
(2) By (1)S_P is closed under $\circ$. we show that associative law is satisfied.

$$(x^P \circ y^P) \circ z^P = (xy)^P \circ z^P = (xyz)^P = x^P \circ (y^P \circ z^P)$$

$\lambda \in S_P and (x^P \circ \lambda^P) = (x\lambda)^P = x^P = (\lambda x)^P = (\lambda \circ x^P)$.
Therefore λ^P is the identity element which is in S_P. ■

Theorem 19. Let $P = (Q, X, W, \mu)$ be an pwfm. Then $S_P \simeq E(P)$, $i.e., S_P$ and $E(P)$ are isomorphic as semigroups.

Proof. Define $f : S_P \to E(P)$ by $f(x^P) = [x]\ \forall x^P \in S_P$. First we show that f is well defined. Let $x^P, y^P \in S_P$. Then $x^P = y^P$ iff $x^P(s,t) = x^P(s,t)\ \forall s,t \in Q$ iff $\mu^*(s,x,t) = \mu^*(s,y,t)\ \forall s,t \in Q$
iff $[x] = [y]$. Thus f is well defined. Next we show that f is a homomorphism. Consider $f(x^P \circ y^P)$

$$\begin{aligned} &= f((xy)^P) = [xy] = [x] * [y] \\ &= f(x^P) * f(y^P). \end{aligned}$$

Thus f is a homomorphism. Now we prove that f is one to one. Consider $f(x^P) = f(y^P)$ implies that $[x] = [y]$ implies that $\mu^*(s,x,t) = \mu^*(s,y,t)$ implies that $x^P = y^P$. Therefore f is one to one. Finally we show that f is onto. Let $[x] \in E(P)\ \forall x \in X^*$. Therefore $x^P \in S_P$. Thus we have $f(x^P) = [x]$. Thus f is onto. Hence f is an isomorphism. ■

Definition 20. Let $P = (Q, X, W, \mu)$ be an pwfm. The index of an equivalence relation is the number of distinct equivalence classes. Let $\sim$ be a congruence relation of finite index on X^*. Let $x \in X^*$ and

$$\prec x \succ = \{y \in X^* \mid x \sim y\}.\ \ Let\ \tilde{Q} = \{\prec x \succ \mid x \in X^*\}.$$

Define $\sigma : \tilde{Q} \times X \times \tilde{Q} \to [0,\infty)$ by $\forall\ \prec x \succ \in \tilde{Q}$ and $\forall\ a \in X$,
$\sigma(\prec x \succ, a, \prec w \succ)$ an arbitrary fixed element in $[0,\infty)$ and
$\forall\ \prec x \succ, \prec w \succ \in \tilde{Q}$,

$$\sigma(\prec x \succ, a, \prec w \succ) = \begin{cases} \sigma(\prec x \succ, a, \prec xa \succ) \text{if } w \sim xa \\ 0, \text{otherwise.} \end{cases}$$

Let $\prec x \succ, \prec y \succ, \prec z \succ \in \tilde{Q}$ and $a, b \in X$.
Suppose that $(\prec x \succ, a, \prec u \succ) = (\prec y \succ, b, \prec v \succ)$.
Then $\prec x \succ = \prec y \succ \Rightarrow x \sim y$ then $xa \sim yb$.
If $a = b$ then $xa \sim ya$.
we know that $\sigma(\prec x \succ, a, \prec u \succ) = \sigma(\prec x \succ, a, \prec xa \succ)$ if $u \sim xa$ Now $\prec u \succ = \prec v \succ$ then $u \sim v$ implies that $v \sim xa$ implies that $v \sim ya$. Thus $\sigma(\prec x \succ, a, \prec u \succ) = \sigma(\prec y \succ, b, \prec v \succ) u \sim xa$ iff $v \sim ya$. Thus σ is a singled valued function.

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Theorem 21. Let $\tilde{P} = (\tilde{Q}, X, W, \sigma)$ be a pwfm and let $\prec z \succ, \prec w \succ \in \tilde{Q}$, then the following assertion holds:
(1) $\forall\ x \in X^*$, if $\sigma^*(\prec z \succ, x, \prec w \succ) > 0$, then $\prec zx \succ = \prec w \succ$
(2) $\sigma^*(\prec z \succ, x, \prec zx \succ) > 0\ \forall\ z, x \in X^*$.

Proof. (1) Let $x \in X^*$. We prove the result by induction on $|x| = n$. If $n = 0$ then $x = \lambda$. Hence if $\sigma^*(\prec z \succ, \lambda, \prec w \succ) > 0$ we have $\sigma^*(\prec z \succ, \lambda, \prec w \succ) = 1 > 0$ iff $\prec z \succ = \prec w \succ$ or $\prec zx \succ = \prec z\lambda \succ = \prec z \succ = \prec w \succ$. Thus the result is true for $n = 0$. Suppose the result is true for all $|x| = n - 1, n > 0$. Now we prove the result is true for $|x| = n$. Consider $x = ya$, where $|y| = n - 1, n > 0$. Now, $\sigma^*(\prec z \succ, x, \prec w \succ)$

$$= \sigma^*(\prec z \succ, ya, \prec w \succ) > 0$$
$$= \sum_{\prec q \succ \in \tilde{Q}} \sigma^*(\prec z \succ, y, \prec q \succ)$$
$$\cdot\, \sigma^*(\prec q \succ, a, \prec w \succ) > 0$$

Hence $\sigma^*(\prec z \succ, y, \prec q \succ) > 0$ and $\sigma^*(\prec q \succ, a, \prec w \succ) > 0$ for some $\prec q \succ \in \tilde{Q}$ $\prec zy \succ = \prec q \succ$ implies that $zy \sim q$ implies that $zya \sim qa$ implies that $\prec zya \succ = \prec qa \succ = \prec w \succ$. Hence $\prec zx \succ = \prec w \succ$. Thus the result is true for $|x| = n$.
(2) Let $z, x \in X^*$. We the result by induction on$|x| = n$. If $n = 0$, then $x = \lambda$. Now, $\sigma^*(\prec z \succ, x, \prec zx \succ)$

$$= \sigma^*(\prec z \succ, \lambda, \prec z\lambda \succ)$$
$$= \sigma^*(\prec z \succ, \lambda, \prec z \succ) = 1 > 0.$$

Thus the result is true for $n = 0$. Suppose the result is true for $|x| = n-1$. Consider $x = ya$, where $|y| = n-1, n > 0$. Now, $\sigma^*(\prec z \succ, x, \prec zx \succ)$

$$= \sigma^*(\prec z \succ, ya, \prec zya \succ)$$
$$= \sum_{\prec q \succ \in \tilde{Q}} \sigma^*(\prec z \succ, y, \prec q \succ) \cdot \sigma^*(\prec q \succ, a, \prec zya \succ)$$
$$> 0$$

Thus the result is true for $|x| = n$. ■

HOMOMORPHISM

Definition for homomorphism and strong homomorphism with suitable examples and relevant theorems are given in this section.

Definition 22. Let $P_1 = (Q_1, X_1, W, \mu_1)$ and $P_2 = (Q_2, X_2, W, \mu_2)$ be two pwfms. A pair (α, β) of mappings, $\alpha : Q_1 \to Q_2$ and $\beta : X_1 \to X_2$, called a homomorphism, written as $(\alpha, \beta) : P_1 \to P_2$, if

$$\mu_1(q, x, p) \leq \mu_2(\alpha(q), \beta(x), \alpha(p)) \forall\ p, q \in Q_1\ and\ \forall x \in X_1.$$

The pair (α, β) is called a strong homomorphism if

$$\mu_2(\alpha(q), \beta(x), \alpha(p)) = \sum_{t \in Q_1} \{\mu_1(q, x, t)\}\ where\ \alpha(t) = \alpha(p)$$

$\forall\ p, q \in Q_1\ and\ \forall\ x \in X_1$.

Example 23. Let $P_1 = (Q_1, X_1, W, \mu_1)$ and $P_2 = (Q_2, X_2, W, \mu_2)$ an pwfm. where $Q_1 = \{q_0, q_1, q_2\}, X_1 = \{a, b\}, Q_1 = \{q_0', q_1', q_2'\}, X_2 = \{a, b\}$, and μ_1, μ_2 are defined as follows:
$\mu_1 : Q_1 \times X_1 \times Q_1 \to [0, \infty)$ $\mu_2 : Q_2 \times X_2 \times Q_2 \to [0, \infty)$

$\mu_1(q_0, a, q_0) = 4;$	$\mu_2(q_0', a, q_0') = 5$
$\mu_1(q_0, a, q_1) = 5;$	$\mu_2(q_0', a, q_1') = 6$
$\mu_1(q_0, a, q_1) = 7;$	$\mu_2(q_0', a, q_1') = 7$
$\mu_1(q_1, a, q_0) = 6;$	$\mu_2(q_1', a, q_0') = 8$
$\mu_1(q_1, b, q_1) = 4;$	$\mu_2(q_1', b, q_1') = 6$
$\mu_1(q_1, b, q_2) = 6;$	$\mu_2(q_1', b, q_2') = 7$
$\mu_1(q_2, a, q_1) = 5;$	$\mu_2(q_2', a, q_1') = 5$
$\mu_1(q_2, b, q_2) = 6;$	$\mu_2(q_2', b, q_2') = 7$

Define $\alpha : Q_1 \to Q_2$ by $\alpha(q_0) = \alpha(q_1) = q_0', \alpha(q_2) = q_2'$, and $\beta : X_1 \to X_2$ by $\beta(a) = a, \beta(b) = b$.
Clearly the above (α, β) is a homomorphism.

Example 24. Let $P_1 = (Q_1, X_1, W, \mu_1)$ and $P_2 = (Q_2, X_2, W, \mu_2)$ be pwfms. where $Q_1 = \{q_1, q_2, q_3\}, X_1 = \{a, b\}, Q_1 = \{q_0', q_1'\}, X_2 = \{0, 1\}$, and μ_1, μ_2 are defined as follows:
$\mu_1 : Q_1 \times X_1 \times Q_1 \to [0, \infty)$ $\mu_2 : Q_2 \times X_2 \times Q_2 \to [0, \infty)$

$\mu_1(q_0, a, q_0) = 3;$	$\mu_2(q_0', 0, q_0') = 14$
$\mu_1(q_0, a, q_1) = 5;$	$\mu_2(q_0', 1, q_0') = 11$
$\mu_1(q_0, b, q_0) = 7;$	$\mu_2(q_0', 1, q_2') = 6$
$\mu_1(q_1, a, q_0) = 6;$	$\mu_2(q_2', 0, q_0') = 5$
$\mu_1(q_1, b, q_1) = 4;$	$\mu_2(q_2', 1, q_2') = 6$
$\mu_1(q_1, b, q_2) = 6;$	
$\mu_1(q_2, a, q_1) = 5;$	
$\mu_1(q_2, b, q_2) = 6;$	

Define $\alpha : Q_1 \to Q_2$ by $\alpha(q_0) = \alpha(q_1) = q_0', \alpha(q_2) = q_2'$, and $\beta : X_1 \to X_2$ by $\beta(a) = 0, \beta(b) = 1$. Clearly the above$(\alpha, \beta)$ is a strong homomorphism.

Theorem 25. Let $P_1 = (Q_1, X_1, W, \mu_1)$ and $P_2 = (Q_2, X_2, W, \mu_2)$ be two pwfms. Let $(\alpha, \beta) : P_1 \to P_2$ be a strong homomorphism. Then $\forall q, r \in Q_1, \forall x \in X_1$, if $\mu_2(\alpha(q), \beta(x), \alpha(r)) > 0$, then $\exists t \in Q_1$ such that $\mu_1(q, x, t) > 0$ and $\alpha(t) = \alpha(r)$. Furthermore, $\forall p \in Q_1$ if $\alpha(p) = \alpha(q)$, then $\mu_1(q, x, t) \geq \mu_1(p, x, r)$.

Proof. Let $\forall p, q, r \in Q, x \in X_1$ and $\mu_2(\alpha(q), \beta(x), \alpha(r)) > 0$. But

$$\mu_2(\alpha(q), \beta(x), \alpha(r)) = \sum_{s \in Q_1} \{\mu_1(q, x, s) | \alpha(s) = \alpha(r)\} > 0$$

Since Q is finite, $\exists t \in Q_1$ such that $\alpha(t) = \alpha(r)$ and

$$\mu_1(q, x, t) = \sum_{s \in Q_1} \{\mu_1(q, x, s) | \alpha(t) = \alpha(r)\} > 0.$$

Suppose $\alpha(p) = \alpha(q)$, then

$$\begin{aligned} \mu_1(q, x, t) &= \mu_2(\alpha(q), \beta(x), \alpha(r)) \\ &= \mu_2(\alpha(p), \beta(x), \alpha(r)) \end{aligned}$$

$\geqslant \mu_1(p, x, r)$. ■

Definition 26. $P_1 = (Q_1, X_1, W, \mu_1)$ and $P_2 = (Q_2, X_2, W, \mu_2)$ be two pwfms. Let $(\alpha, \beta) : P_1 \to P_2$ be a strong homomorphism. Define $\beta^* : X_1^* \to X_2^*$ by 1)$\beta^*(\lambda) = \lambda$ 2)$\beta^*(ua) = \beta^*(u)\beta(a)$ $\forall$ $u \in X_1^*, a \in X_1$.

Lemma 27. $\beta^*(uv) = \beta^*(u)\beta^*(v)$ $\forall$ $u, v \in X_1^*$.

Proof. Let $u, v \in X_1^*$ we prove the result by induction on $|v| = n$. If $n = 0$, then $v = \lambda$ and hence $\beta^*(uv) = \beta^*(u\lambda) = \beta^*(u)\beta^*(v)$. Thus the result is true for $n = 0$. Suppose the result is true for $|v| = n-1, n > 0$. Now we prove the result is true for $|v| = n$. Let $v = ya$ where $y \in X_1^*$, $a \in X_1$ and $|y| = n - 1$. Then,

$$\begin{aligned} \beta^*(uv) &= \beta^*(uya) \\ &= \beta^*(uy)\beta(a) \\ &= \beta^*(u)\beta^*(y)\beta(a) \\ &= \beta^*(u)\beta^*(ya) \\ &= \beta^*(u)\beta^*(v). \end{aligned}$$

Thus the result is true for $|v| = n$. ■

Theorem 28. Let $P_1 = (Q_1, X_1, W, \mu_1)$ and $P_2 = (Q_2, X_2, W, \mu_2)$ be two pwfms. Let $(\alpha, \beta) : P_1 \to P_2$ be a homomorphism. Then $\mu_1^*(q, x, p) \leq \mu_2^*(\alpha(q), \beta^*(x), \alpha(p)) \forall$ $p, q \in Q_1 and$ $\forall x \in X_1^*$.

Proof. Let $p, q \in Q_1 and$ $\forall x \in X_1^*$.We prove the result by induction on $|x| = n$.
If $n = 0$, then $x = \lambda$. Then $\beta^*(x) = \beta^*(\lambda) = \lambda$.
If $q = p, \mu_1^*(q, \lambda, p) = 1 = \mu_2^*(\alpha(q), \lambda, \alpha(p))$.
If $q \neq p, \mu_1^*(q, \lambda, p) = 0 \leq \mu_2^*(\alpha(q), \lambda, \alpha(p))$.
Suppose the result is true for all $x \in X_1^*$ such that $|x| \leq n - 1, n > 0$.
Let $|x| = n, x = ya$, where $y \in X_1^*$ and $|y| = n - 1$. Now

$$\begin{aligned} \mu_1^*(p, x, q) &= \mu_1^*(q, ya, p) \\ &= \sum_{r \in Q_1} \{\mu_1^*(q, y, r) \cdot \mu_1(r, a, p)\} \\ &\leq \sum_{r \in Q_1} \{\mu_2^*(\alpha(q), \beta^*(y), \alpha(r)) \cdot \mu_2(\alpha(r), \beta(a), \alpha(p))\} \\ &\leq \sum_{r' \in Q_2} \{\mu_2^*(\alpha(q), \beta^*(y), r') \cdot \mu(r', \beta(a), \alpha(p))\} \end{aligned}$$

where $\alpha(r) = r'$

$$\begin{aligned} &= \mu_2^*(\alpha(q), \beta^*(y)\beta(a), \alpha(p)) \\ &= \mu_2^*(\alpha(q), \beta^*(ya), \alpha(p)) \\ &= \mu_2^*(\alpha(q), \beta^*(x), \alpha(p)) \end{aligned}$$

Thus the result is true for $|x| = n$. ■

Theorem 29. Let $P_1 = (Q_1, X_1, W, \mu_1)$ and $P_2 = (Q_2, X_2, W, \mu_2)$ be two pwfms. Let $(\alpha, \beta) : P_1 \to P_2$ be a strong homomorphism. Then α is one-one if and only if $\mu_1^*(q, x, p) = \mu^*(\alpha(p), \beta^*(x), \alpha(p))$ $\forall$ $q, p \in Q_1$, and $x \in X_1^*$.

Proof. Suppose α is one-one. Let $p, q \in Q_1$ and $x \in X_1^*$. Let $|x| = n$. We prove the result by induction on n. If $n = 0$, then $x = \lambda$. Then $\beta^*(\lambda) = \lambda$. Now $\alpha(q) = \alpha(p) \Leftrightarrow q = p$. Hence $\mu_1^*(q, \lambda, p) = 1 \Leftrightarrow \mu_2^*(\alpha(q), \lambda, \alpha(p)) = 1$. If $\alpha(q) \neq \alpha(p) \Leftrightarrow q \neq p$. Hence $\mu_1^*(q, \lambda, p) = 0 \Leftrightarrow \mu_2^*(\alpha(q), \lambda, \alpha(p)) = 0$. Thus the result is true for $n = 0$. Suppose the result is true for all $x \in X_1^*$ such that $|x| \leq n - 1, n > 0$. Let $|x| = n, x = ya$, where $y \in X_1^*, a \in X_1$ and $|y| = n - 1, n > 0$. Then,

$$\begin{aligned} &\mu_2^*(\alpha(q), \beta^*(x), \alpha(p)) \\ &= \mu_2^*(\alpha(q), \beta^*(ya), \alpha(p)) \\ &= \mu_2^*(\alpha(q), \beta^*(y)\beta(a), \alpha(p)) \\ &= \sum_{r \in Q_1} \{\mu_2^*(\alpha(q), \beta^*(y), \alpha(r)) \cdot \mu_2(\alpha(r), \beta(a), \alpha(p))\} \end{aligned}$$

$$= \sum_{r\in Q_1}\{\mu_1^*(q,y,r)\cdot\mu_1(r,a,p)\}$$
$$= \mu_1^*(q,ya,p)$$
$$= \mu_1^*(q,x,p).$$

Conversely, let $q,p\in Q_1$ and $\alpha(q)=\alpha(p)$. Then,

$$1=\mu_2^*(\alpha(q),\lambda,\alpha(p))=\mu_1^*(\alpha(q),\lambda,\alpha(p)).$$

Hence $q=p, i.e., \alpha$ is one-one. ■

ADMISSIBLE RELATIONS

This section deals with admissible relations over plus weighted finite state machines. $\sim$ an equivalence relation is taken and proved as an admissible relation. Here $ker\alpha$ is defined and related theorems are discussed.

Definition 30. Let $P=(Q,X,W,\mu)$ be a pwfm and let $\sim$ be an equivalence relation on Q. Then $\sim$ is called an admissible relation if and only if $\forall\ p,q,r\in Q,\ \forall\ a\in X$, if $p\sim q$ and $\mu(p,a,r)>0$, then $\exists t\in Q$ such that $\mu(q,a,t)\geq\mu(p,a,r)$ and $t\sim r$.

Theorem 31. Let $P=(Q,X,W,\mu)$ be a pwfm and let $\sim$ be an equivalence relation on Q. Then $\sim$ is an admissible relation if and only if $\forall\ p,q,r\in Q,\ \forall\ x\in X^*$, if $p\sim q$ and $\mu^*(p,x,r)>0$, then $\exists t\in Q$ such that $\mu^*(q,x,t)\geq\mu^*(p,x,r)$ and $t\sim r$.

Proof. Suppose $\sim$ is admissible. Let $p,q\in Q$ such that $p\sim q$. Let $x\in X^*, r\in Q$ be such that $\mu^*(p,x,r)>0$. We pove the result by induction on $|x|=n$. If $n=0$, then $x=\lambda$. Thus $\mu^*(p,x,r)>0\Longrightarrow p=r$ and $\mu^*(p,x,p)=1$. Now $\mu^*(q,x,q)=1=\mu^*(p,x,p)$ and $q\sim p$.
Thus the result is true for $n=0$. Suppose the result is true for all $x\in X_1^*$ such that $|x|\leq n-1, n>0$.
We prove the result is true for $|x|=n, x=ya$, where $y\in X_1^*, a\in X_1$ and $|y|=n-1, n>0$. Now,

$$\mu^*(p,x,r)=\mu^*(p,ya,r)$$
$$=\sum_{q_1\in Q}\{\mu^*(p,y,q_1)\cdot\mu_1(q_1,a,r)\}>0.$$

Since Q is finite, $\exists s\in Q$ be such that

$$\mu^*(p,y,s)\cdot\mu(s,a,r)=\sum_{q_1\in Q}\{\mu(p,y,q_1)\cdot\mu(q_1,a,r)\}.$$

Then $\mu^*(p,y,s)>0$, and $\mu(s,a,r)>0$. By induction hypothesis, $\exists\ t_s\in Q$ such that $\mu^*(q,y,t_s)\geq\mu^*(p,y,s)$ and $t_s\sim s$. Since $\sim$ is admissible, $\exists\ t\in Q$ such that $\mu(t_s,a,t)\geq\mu(s,a,r)$ and $t\sim r$.
Thus $\exists\ t\in Q$ such that $t\sim r$ and $\mu^*(q,y,t_s)\cdot\mu(t_s,a,t)\geq\mu^*(p,y,s)\cdot\mu(s,a,r)$. Thus

$$\mu^*(p,x,r)=\mu^*(p,y,s)\cdot\mu(s,a,r)$$
$$\leq\mu^*(q,y,t_s)\cdot\mu(t_s,a,t)$$
$$\leq\sum_{r_1\in Q}\{\mu^*(q,y,r_1)\cdot\mu_1(r_1,a,t)\}$$
$$=\mu^*(q,ya,t)$$
$$=\mu^*(q,x,t),\ and\ t\sim r.$$

Therefore the result is true for $|x|=n$. The converse is trivial. ■

Theorem 32. Let $P=(Q,X,W,\mu)$ be a pwfm and let $\sim$ be an admissible relation on Q.

i) Then there exists a plus weighted subset $\tilde{\mu}:\tilde{Q}\times X\times\tilde{Q}\to[0,\infty)$, where $\tilde{Q}=Q/\sim$ such that $\tilde{\mu}$ is one-one.
ii) (α_1,β) is a homomorphism where $\alpha_1:Q\to\tilde{Q}$ and $\beta:X\to X$ is an identity map.

Proof. Let $q\in Q$ and $[q]$ be the equivalence class of q. $i.e., [q]=\{p\in Q|q\sim p\}$. Let $\tilde{Q}=Q/\sim=\{[q]|q\in Q\}$.
Define $\tilde{\mu}:\tilde{Q}\times X\times\tilde{Q}\to[0,\infty)$ by
$\tilde{\mu}([q],x,[p])=\sum_{t\in[p]}\{\mu(q,x,t)\}\ \forall\ q,p\in Q, x\in X$.
Suppose $([q],a,[p])=([q'],b,[p'])$.
Therefore $[q]=[q'], x=y, [p]=[p']$.
Implies that $q\sim q', p\sim p'$. Now $\tilde{\mu}([q],x,[p])=\sum_{r\in[p]}\{\mu(q,x,r)\}$ and
$\tilde{\mu}([q'],y,[p'])=\tilde{\mu}([q'],x,[p'])=\sum_{t\in[p']}\{\mu(q',x,t)\}$.
Let $r\in[p]$ be such that $\mu(q,x,r)>0$. Then since $\sim$ is admissible, $\exists t\in Q$ such that $\mu(q',x,t)\geq\mu(q,x,r)>0$ and $t\sim r$.
Now since $t\sim r, t\in[p]=[p']$. Thus $\exists\ t\in[p']$ such that $mu(q',x,t)\geq\mu(q,x,r)>0$.
Similarly if $\mu(q',x,t)>0$ for some $t\in[p']$, then $\exists r\in[p]$ such that
$\mu(q,x,r)\geq\mu(q',x,t)>0$. Hence
$\tilde{\mu}([q],x,[p])=\tilde{\mu}([q'],x,[p'])$.Thus $\tilde{\mu}$ is single valued.
Now we have to prove that $\tilde{\mu}$ is one-one.
Consider $\tilde{\mu}([q],x,[p])=\tilde{\mu}([q'],y,[p'])\ \forall x,y\in X, q,p,q',p'\in Q$. Then there exists $r\in[p]$ and $t\in[p']$ such that

$\tilde{\mu}([q], x, [p]) = \sum_{r \in [p]} \{\mu(q, x, r)\}$ and
$\tilde{\mu}([q'], y, [p']) = \sum_{t \in [p']} \{\mu(q', y, t)\}$
$\Longrightarrow \mu(q', y, t) \geq \mu(q, x, r) > 0$ (since $\sim$ is admissible)
$\Longrightarrow q' \sim q$ and $r \sim t$.
$\Longrightarrow x = y$.
Thus $\tilde{\mu}$ is one-one.
Finally we show that (α_1, β) is a homomorphism.
Clearly α_1 is onto. Let $\forall\ q, t \in Q$ and $x \in X$. Then,

$$\begin{aligned}\tilde{\mu}(\alpha_1(q), x, \alpha_1(t)) &= \tilde{\mu}([q], x, [t]) \\ &= \sum_{r \in [t]} \{\mu(q, x, r)\} \\ &\geq \mu(q, x, t).\end{aligned}$$

Hence (α_1, β) is a homomorphism. ■

Definition 33. Let $P_1 = (Q_1, X_1, W, \mu_1)$ and $P_2 = (Q_2, X_2, W, \mu_2)$ be two pwfms. Let $\alpha : P_1 \to P_2$ be a strong homomorphism. The kernel of α, denoted $Ker\alpha$, is defined to be the set
$Ker\alpha = \{(p, q) | \alpha(p) = \alpha(q)\}$.

Theorem 34. Let $P_1 = (Q_1, X_1, W, \mu_1)$ and $P_2 = (Q_2, X_2, W, \mu_2)$ be two pwfms. Let $\alpha : P_1 \to P_2$ be a strong homomorphism. Then $Ker\alpha$ is an admissible relation.

Proof. By definition, $Ker\alpha = \{(p, q) | \alpha(p) = \alpha(q)\}$.
Let $(p, p) \in Ker\alpha$ implies that $\alpha(p) = \alpha(p)$. Therefore $Ker\alpha$ is reflexive.
Consider $(p, q) \in Ker\alpha$. Then $\alpha(p) = \alpha(q)$, implies that $\alpha(q) = \alpha(p)$, implies that $(q, p) \in Ker\alpha$. Thus $Ker\alpha$ is symmetric.
Let $(p, q) \in Ker\alpha$ implies that $\alpha(p) = \alpha(q)$. Let $(q, r) \in Ker\alpha$ implies that $\alpha(q) = \alpha(r)$. Then $\alpha(p) = \alpha(q) = \alpha(r)$ implies that $\alpha(p) = \alpha(r)$. Then $(p, r) \in Ker\alpha$. Thus $Ker\alpha$ is transitive. Therefore $Ker\alpha$ is an equivalence relation.
Let $a \in X, r \in Q_1,\ and\ \mu_1(p, a, r) > 0$. Then,

$$\begin{aligned}\mu_2(\alpha(q), a, \alpha(r)) &= \mu_2(\alpha(p), a, \alpha(r)) \\ &\geq \mu_1(p, a, r) > 0\ \ and\ \ \alpha(t) = \alpha(r).\end{aligned}$$

Since $\alpha(t) = \alpha(r), (t, r) \in Ker\alpha$ is admissible. ■

Theorem 35. Let $P_1 = (Q_1, X_1, W, \mu_1)$ and $P_2 = (Q_2, X_2, W, \mu_2)$ be two pwfms. Let $\alpha : P_1 \to P_2$ be an onto strong homomorphism. Then $\exists$ an isomorphism $\gamma : (Q_1/(Ker\alpha), X, W, \tilde{\mu_1}) \to (Q_2, X, W, \mu_2)$ such that $\gamma \circ \alpha_1$.

Proof. Let P_1, P_2 be two pwfms. Define $\gamma : Q_1/Ker\alpha \to Q_2$ by $\gamma([q]) = \alpha(q)$. Let $p, q \in Q_1$ be such that $[p] = [q]$. Then
$(p, q) \in Ker\alpha$. Hence $\alpha(p) = \alpha(q)$. Thus γ is well defined. Now we prove that γ is one-one. Consider $\gamma([q]) = \gamma([p]) \Longrightarrow \alpha(q) = \alpha(p)$
$\Longrightarrow (q, p) \in Ker\alpha \Longrightarrow [q] = [p]$
Thus γ is one-one. Next we show that γ is onto. Let $\alpha(q) \in Q_2$ for some $q \in Q_1/Ker\alpha$. Therefore we have $\gamma([q]) = \alpha(q)$. Hence γ is onto. Finally we prove that γ is a homomorphism. Now, let $q, p \in Q_1$ and $x \in X$. Then

$$\begin{aligned}\tilde{\mu_1}([q], x, [p]) &= \sum_{r \in [p]} \{\mu_1(q, x, r)\} \\ &= \sum_{r \in Q_1} \{\mu_1(q, x, r) | \alpha(r) = \alpha(p).\} \\ &= \mu_2(\alpha(q), x, \alpha(p)) \\ &= \mu_2(\gamma([q]), x, \gamma([p])).\end{aligned}$$

Thus γ is a homomorphism. ■

CONCLUSION

In this paper an attempt is made to introduce the algebraic concepts such as semigroup, homomorphism and admissible relation on pwfm. These concepts can be extended towards other algebraic structures which will give fruitful results in the area of weighted automata.

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