International Journal of Applied Engineering Research ISSN 0973-4562 Volume 13, Number 21 (2018) pp. 15179-15182
© Research India Publications. http://www.ripublication.com

Weakly 2-Absorbing ideals in Non-Commutative Rings

Malik Bataineh[1], Rashid Abu-Dawwas [2], Odai Masoud [1]

[1]*Mathematics Department, Jordan University of Science and Technology, Irbid 22110, Jordan.*

[2]*Mathematics Department, Yarmouk University*

[3]*Mathematics Department, Jordan University of Science and Technology*

Abstract

Let R be a commutative ring with identity. Various generalizations of 2-absorbing ideals have been studied. For example, a proper ideal I of R is weakly 2-absorbing (resp., almost 2-absorbing ideal) if $a,b,c \in R$ with $0 \neq abc \in I$ (resp., $abc \in I - I^2$), then either $ab \in I$ or $ac \in I$, or $bc \in I$. In this paper we only consider non-commutative rings and introduce weakly 2-absorbing ideals and show that it enjoys analogs of many of the properties of 2-absorbing ideals.

Mathematics Subject Classification: 13A05, 13F05.

Keywords: 2-absorbing ideal, weakly 2-absorbing ideal, weakly prime ideal, almost prime ideal.

INTRODUCTION

Various generalizations of 2-absorbing ideals over commutative rings have been studied. A proper ideal P of a commutative ring R with unity is called 2-absorbing ideal if whenever $a,b,c \in R$ and $abc \in P$, then $ab \in P$ or $ac \in P$, or $bc \in P$ (See [1]). Also A proper ideal P of a commutative ring R with unity is called weakly 2-absorbing ideal if whenever $a,b,c \in R$ and $0 \neq abc \in P$, then $ab \in P$ or $ac \in P$, or $bc \in P$ (see [2]). A proper ideal I of R is called almost 2-absorbing ideal if for all $a,b,c \in R$, $abc \in I - I^2$ implies that $ab \in I$ or $ac \in I$ or $bc \in I$ (see [5]). The following statements are equivalent for an ideal P of a commutative ring R:

(*i*) P is 2-absorbing ideal.

(*ii*) For ideals A, B and C of R with $ABC \subseteq P$, then either $AB \subseteq P$ or $AC \subseteq P$, or $BC \subseteq P$. For rings that are not necessarily commutative, it is obvious that (*ii*) does not imply (*i*), so the standard definition of a 2-absorbing ideal P of non-commutative ring R is that, for ideals A, B and C of R and $ABC \subseteq P$, then either $AB \subseteq P$ or $AC \subseteq P$, or $BC \subseteq P$. Badawi and Darani have closed their article with the question; for the weakly 2-absorbing ideal P of a commutative ring which is not 2-absorbing ideal and $0 \neq ABC \subseteq P$, for some ideals A, B and C of R, does it imply that $AB \subseteq P$ or $AC \subseteq P$, or $BC \subseteq P$?

Note that, every 2-absorbing ideal is weakly 2-absorbing, but the converse need not to be true (see [1]). For non-commutative rings, we define that a proper ideal P of R is said to be weakly 2-absorbing ideal if whenever, A, B and C of R with $ABC \subseteq P$, then either $AB \subseteq P$ or $AC \subseteq P$, or $BC \subseteq P$, for some ideals A, B and C of R.

WEAKLY 2-ABSORBING IDEALS

Throughout, R will be non-commutative ring.

Definition 2.1 *A proper ideal P of a ring R is said to be weakly 2-absorbing if whenever, A, B and C of R with* $ABC \subseteq P$*, then either* $AB \subseteq P$ or $AC \subseteq P$, or $BC \subseteq P$*, for some ideals A, B and C of R.*

Theorem 2.2 *Let P be an ideal of a ring R. If P is weakly 2-absorbing but not 2-absorbing, then* $P^3=0$.

proof Since P is not 2-absorbing, then there exist ideals A,B and C of R such that $AB \not\subset P$, $AC \not\subset P$, $BC \not\subset P$ and $ABC \subseteq P$. If $P^3 \neq 0$, then
$0 \neq P^3 \subseteq (A+P)(B+P)(C+P) \subseteq P$ since P is weakly 2-absorbing then
$AB \subseteq (A+P)(B+P) \subseteq P$ or $AC \subseteq (A+P)(C+P) \subseteq P$ or $BC \subseteq (B+P)(C+P) \subseteq P$ which is contradiction. Hence $P^3=0$.▪

Theorem 2.3 *Let* P *be a proper ideal of a ring* R *with unity, then P is weakly 2-absorbing ideal of R if and only if* $0 \neq aRbRc \subseteq P$ *implies that* $a \in P$ *or* $b \in P$ or $c \in P$ for any $a,b,c \in R$

proof (⟹) Suppose P is weakly 2-absorbing of R , let $a,b,c \in R$ such that 0□*aRbRc*□*P*, since R has a unity $0 \neq (aR)(bR)(cR) \subseteq P$ and P is weakly 2-absorbing then $ab \in (aR)(bR) \subseteq P$ or $ac \in (aR)(cR) \subseteq P$ or $bc \in (bR)(cR) \subseteq P$.

(⟸) Let A,B and C be ideal of R such that $0 \neq ABC \subseteq P$. Suppose that $AB \not\subset P$, $AC \not\subset P$, $BC \not\subset P$, then there exists $x \in AB - P$, $\in AC - P$ and $z \in BC - P$. Assume that $\alpha \in AB \cap P$, $\beta \in AC \cap P$ and $\gamma \in BC \cap P$ then $x+\alpha, y+\beta$*, and* $z+\gamma \notin P$, so $(x+\alpha)R(y+\beta)R(z+\gamma)=0$. Taking all combinations when α and(or) β and(or) γ equal zero proves that

$$0 = xyz = xy\alpha = xy\beta = xy\gamma = x\alpha z$$
$$= x\beta z = x\gamma z = \alpha yz = \beta yz = \gamma yz = \alpha\beta y = ...$$

Hence, ABC=0 which is a contradiction, so P is weakly 2-absorbing ideal of R. ▪

Now we are interested in the structure of ring in which every ideal is weakly 2-absorbing. Note that it is not possible that every ideal of a ring is weakly 2-absorbing. However, a ring whose zero ideal is 2-absorbing is called a 2-absorbing ring. In the same since, every ring is a weakly 2-absorbing ring since the zero ideal is always weakly 2-absorbing.

So, every ideal of a ring is weakly 2-absorbing, when every proper ideal of the ring is a weakly 2-absorbing ideal.

If R^3=0, then it is evident that every ideal of R is weakly 2-absorbing. In particular; if an ideal P of a ring R is weakly 2-absorbing but not 2-absorbing, then every ideal of P as a ring is weakly 2-absorbing by Theorem 2.2.

Theorem 2.4 *Every ideal of a ring R is weakly 2-absorbing ideal if and only if for any ideals A,B and C of R, AB=ABC or AC=ABC or BC=ABC or ABC=0 .*

Proof ($\Longrightarrow$) Let A,B and C be ideals of R. If $ABC \neq R$, then ABC is weakly 2-absorbing. Suppose that $ABC \neq 0$, then $0 \neq ABC \subseteq ABC$ and $AB \subseteq ABC$ or $AC \subseteq ABC$ or $BC \subseteq ABC$ and hence $AB=ABC$ or $AC=ABC$ or $BC=ABC$. If $ABC=R$, then $A=B=C=R$. Therefore, $R^3=R$.

($\Longleftarrow$) Let P be a proper ideals of R, suppose that $0 \neq ABC \subseteq P$ for some ideals$A$,B and C of R then $AB=ABC \subseteq P$ or $AC=ABC \subseteq P$ or $BC=ABC \subseteq P$. Hence, P is weakly 2-absorbing ideal of R.▪

Theorem 2.5 *If every ideal of a ring R is weakly 2-absorbing and $R^3=R^2=R$, then R has at most three maximal ideal.*

Proof Suppose that M_1,M_2,M_3,M_4 are distinct maximal ideal of R. Let $P = M_1 \cap M_2 \cap M_3$. Hence P is a weakly 2-absorbing ideal of R that is not a 2-absorbing ideal of R. Thus $P^3=\{0\}$ by Theorem 2.2. Hence,
$P^3 = M_1{}^3 M_2{}^3 M_3{}^3 = \{0\} \subset M_4$, and thus one of the $M_i s$, $1 \leq i \leq 3$, is contained in M_4, which is a contradiction.
Hence R has at most three maximal ideals. ▪

Corollary 2.6 *Let R be a Ring in which every ideal of R is weakly 2-absorbing ideal , then for any ideal P of R, $P^3=P^2$ or $P^3=0$.*

The following example shows that the condition $R^3=R^2=R$ in Theorem 2.5 cannot be dropped.

Example 2.7 [3] *Let R be the unique maximal ideal of Z_4. then $S=R \oplus R \oplus R \oplus R$ is an example of a ring all of whose ideals are weakly 2-absorbing and have more than three maximal ideals.*

Theorem 2.8 *Suppose that every ideal of a ring R is weakly 2-absorbing , if R has three maximal ideals then there product is zero.*

proof Let X_1, X_2 and X_3 be the maximal ideal of R. Then since $X_1 \cap X_2 \cap X_3$ is weakly 2-absorbing and $X_1 X_2 X_3 \subseteq X_1 \cap X_2 \cap X_3$, we have $X_1 X_2 X_3 = 0$ ▪

Theorem 2.9 *Suppose that every ideal of a ring R is weakly 2-absorbing ,if R has three maximal ideal and R has identity element, then R is direct sum of three simple rings.*

proof Let A,B,C be distinct maximal of R, if R has identity element, then

$$A \cap B \cap C = (A \cap B \cap C)R$$
$$= (A \cap B \cap C)(A + B + C) \subseteq CA + BA + AC = 0$$

and this implies that $R \cong R / A \oplus R / B \oplus R / C$. ▪

Theorem 2.10 *Suppose that every ideal of a ring R is a weakly 2-absorbing, then every non-zero ideal P of R such that $P^3=P^2$ is 2-absorbing.*

proof Suppose that every ideal of a ring R is weakly 2-absorbing by Theorem 2.2 and Corollary 2.6, any nontrivial idempotent ideal of R is 2-absorbing ideal. ▪

Recall that the intersection of all 2-absorbing ideals of a ring R is called the 2-absorbing radical of R, we denote the 2-absorbing radical of R by $P(R)$, and the sum of all ideals whose triple is zero by $N(R)$.

Theorem 2.11 *Suppose that every ideal of a ring R is weakly 2-absorbing and $R^3=R^2=R$, then P(R)=N(R) and $(P(R))^3=(N(R))^3=0$.*

proof Take $a_1, a_2, a_3 \in N(R)$ then there are finitely many triple-zero ideals $I_1, I_2, I_3, ..., I_m$ such that $a_1, a_2, a_3 \in I_1 + I_2 + I_3 + ... + I_m$ since I_j^3=0 for each j , $(I_1 + I_2 + I_3 + ... + I_m)^k = 0$ for some k but then $(I_1 + I_2 + I_3 + ... + I_m)^3 = 0$ by Corollary 2.6 , Hence $(N(R))^3$=0. This implies that if P is any 2-absorbing ideal of R, $N(R) \subseteq P$ and consequently $N(R) \subseteq P(R)$ Note that R contains at least one 2-absorbing ideal. If R contains a nonzero idempotent ideal then by above it must be 2-absorbing. If every ideal is nilpotent then since $R^3=R^2=R$, $N(R) \neq R$ is 2-absorbing ideal.

If P(R) is not 2-absorbing, then $(P(R))^3$=0 this implies that $P(R) \subseteq N(R)$ by the definition of N(R), and hence the result follows.

Suppose P(R) is 2-absorbing, In this case we will show that N(R) must be 2-absorbing, this implies that $P(R)\subseteq N(R)$ by definition of P(R), and hence the result follows.

To show that N(R) is also 2-absorbing, suppose that $IJK\subseteq N(R)$ for ideals I and J and K of R. Since N(R) is weakly 2-absorbing if $IJK\neq 0$ then $IJ\subseteq N(R)$ or $IK\subseteq N(R)$ or $JK\subseteq N(R)$, suppose that $IJK=0$ if $I^3=0$ or $J^3=0$ or $K^3=0$ then $IJ\subseteq N(R)$ or $IK\subseteq N(R)$ or $JK\subseteq N(R)$ if I,J,K are not triple zero then they are 2-absorbing, but then either $I=I^3\subseteq IJK=0$ or $J=J^3\subseteq IJK=0$ or $K=K^3\subseteq IJK=0$ contradiction. This shows that N(R) is 2-absorbing ideal. ▪

Corollary 2.12 *Suppose that every ideal of a right Noetherian ring R with identity is weakly 2-absorbing and $R^3=R^2=R$ then P(R)=N(R)=J(R) and $(J(R))^3=(P(R))^3=(N(R))^3=0$, where J(R) is the Jacobson radical of R.*

proof If $J(R)^3=(J(R))^2=J(R)$ then $J(R)=0$ since $P(R)\subseteq J(R)$, the result follows if $(J(R))^3=0$ then $J(R)\subseteq P$ for every 2-absorbing ideal P in R, Hence $J(R)\subseteq P(R)$.▪

Corollary 2.13 *Suppose that every ideal of a ring R is weakly 2-absorbing , then every non zero ideal of R/N(R) is 2-absorbing ideal.*

Corollary 2.14 *Suppose that every ideal of a ring R is weakly 2-absorbing , then $(N(R))^3=0$ and every 2-absorbing ideal contains N(R) There are three possibilities:*

1. *N(R)=R.*
2. *N(R)=P(R) is the smallest 2-absorbing ideal and all other 2-absorbing ideal are idempotent if N(R) $\neq 0$, then it is the only non-idempotent 2-absorbing.*
3. *N(R)=P(R) is not a 2-absorbing ideal in this case, there exist three nonzero minimal 2-absorbing ideal J_1,J_2 and J_3 with $N(R)=J_1\cap J_2\cap J_3$ and $J_1J_2J_3=J_3J_2J_1=0$ all other ideal containing N(R) also contains $J_1+J_2+J_3$.*

proof If $R^3=0$ then $N(R)=R$, so clearly $(N(R))^3=0$, and there are no 2-absorbing ideal. If $R^3=R^2=R$ then by Theorem 2.11, $P(R)=N(R)$ and $(P(R))^3=(N(R))^3=0$ by definition of P(R), every 2-absorbing g ideal contains $N(R)=P(R)$ if $N(R)=P(R)$ is 2-absorbing, it is the smallest 2-absorbing ideal and all other 2-absorbing ideals are idempotent. If $N(R)=P(R)$ is not 2-absorbing then since every ideal of R/N(R) except {0} is 2-absorbing thus there exist three nonzero minimal 2-absorbing ideal J_1,J_2 and J_3 with $N(R)=J_1\cap J_2\cap J_3$ moreover, all other ideals containing N(R) also contain $J_1+J_2+J_3$, since $N(R)=J_1\cap J_2\cap J_3$ is weakly 2-absorbing must have $J_1J_2J_3=J_3J_2J_1=0$ otherwise $J_1J_2J_3\subseteq J_1\cap J_2\cap J_3$, $J_3J_2J_1\subseteq J_1\cap J_2\cap J_3$ implies either

$J_1J_2\subseteq J_1\cap J_2\cap J_3\subseteq J_3$ or

$J_1J_3\subseteq J_1\cap J_2\cap J_3\subseteq J_2$ or

$J_2J_3\subseteq J_1\cap J_2\cap J_3\subseteq J_1$ contradiction. ▪

Definition 2.15 [4] *A nonzero proper ideal P of R is called almost prime ideal if for all A,B ideals of R ,with $AB\subseteq P-P^2$ implies $A\subseteq P$ or $B\subseteq P$.*

Definition 2.16 *A nonzero proper ideal P of R is called almost 2-absorbing ideal if for all A,B,C ideals of R with $ABC\subseteq P-P^2$ implies that $AB\subseteq P$ or $AC\subseteq P$ or $BC\subseteq P$.*

Next proposition gives a relationship between weakly 2-absorbing and almost 2-absorbing ideals.

Proposition 17 *Let R be a ring and I be a proper ideal of R. Then*

1. *If P is a 2-absorbing ideal, then P is a almost 2-absorbing ideal.*
2. *P weakly 2-absorbing $\Longrightarrow$ P almost 2-absorbing .*

proof (1) Suppose that $ABC\subseteq P-P^2$ where $A,B,C\subseteq R$. Since $\phi\subset P^2$, we have $ABC\subseteq P$ and therefore $AB\subseteq P$ or $AC\subseteq P$ or $BC\subseteq P$ thus P is almost 2-absorbing ideal

(2) Suppose that $ABC\subseteq P-P^2$ where $A,B,C\subseteq R$. Since $\{0\}\subseteq P^2$, we have $ABC\subseteq P$ and therefore $AB\subseteq P$ or $AC\subseteq P$ or $BC\subseteq P$ thus P is almost 2-absorbing ideal. ▪

Theorem 2.18 *Let R be a ring with identity such that P is almost 2-absorbing that is not 2-absorbing. Then $P^3\cap P^2\neq\phi$.*

proof Suppose that $P^3\cap P^2\neq\phi$, we show that P is 2-absorbing. Let A,B,C are ideal of R ,such that $ABC\subseteq P$, If $ABC\cap P^2=\phi$, then P is almost 2-absorbing ideal gives $AB\subseteq P$ or $AC\subseteq P$ or $BC\subseteq P$. So let $ABC\cap P^2\neq\phi$ suppose that $ABP\cap P^2=\phi$ then $ABP_0\cap P^2=\phi$ where $P_0\subseteq P$. Then $AB\left(C+P_0\right)\subseteq P-P^2$. So $AB\subseteq P$ or $A\left(C+P_0\right)\subseteq P$ or $B\left(C+P_0\right)\subseteq P$, and hence $AB\subseteq P$ or $AC\subseteq P$ or $BC\subseteq P$. So we can assume $ABP\cap P^2\neq\phi$. Likewise $ACP\cap P^2\neq\phi$, $BCP\cap P^2\neq\phi$. Since $P^3\cap P^2=\phi$, there exist $P_1,P_2,P_3\subseteq P$ with $P_1P_2P_3\cap P^2=\phi$ then $\left(A+P_1\right)(B+P_2)(C+P_3)\subseteq P-P^2$ so P is almost 2-absorbing ideal gives $\left(A+P_1\right)\left(B+P_2\right)\subseteq P$ or

$(A+P_1)(C+P_3) \subseteq P$ or $(B+P_2)(C+P_3) \subseteq P$ then $AB \subseteq P$ or $AC \subseteq P$ or $BC \subseteq P$, So P is 2-absorbing ideal. ▪

Corollary 19 *Let P be almost 2-absorbing ideal and* $P^3 \cap P^2 = \phi$. *Then P is a 2-absorbing ideal.*

REFERENCES

[1] A Badawi, On 2-absorbing ideals of commutative ringsâ, Bulletin of Australian Mathematical Society, vol. 75, pp. 417-429, 2007.

[2] A Badawi and Y Darani , On weakly 2-absorbing ideals of commutative ringsâ, Houston Journal of Mathematics, vol. 39, pp. 441-452, 2013.

[3] Yasuyuki Hirano, Edward Poon, and Hisaya Tsutsui, on rings in which every ideal is weakly prime, Bull. Korean Math. Soc., 47(5), pp. 1077-1087, 2010.

[4] D. D. Anderson and M. Bataineh. Generalizations of Prime Ideals. Communications in Algebra, vol. 36, pp.686-696, 2008.

[5] A. Khaksari, φ 2-absorbing ideals International Journal of Pure and Applied Mathematics, 99(1), 1-10, 2015.