Optimization of the Regularization of the Solution of the Plate Vibration Problem

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Abstract

In this paper optimization of the regularization of the Fourier series in case of the plate vibration problem studied and the regularization of the series solutions at a fixed point of the plate studied at initial time and critical index.

INTRODUCTION

Fourier series and Integral are widely used in the solution of engineering problem. For example in [1] an analytical model is proposed to compute the three-dimensional temperature distribution in a solid, subjected to a moving rectangular heat source with surface cooling. In [2] Fourier sine series solution used for the rectangular plate's vibration problem where the plates were elastic restrained alongside all the edges. The rectangular plate's vibration problems with elastically restrained boundary conditions are solved using Fourier series method [3].

In [4] buckling problem of thin rectangular functionally graded plates with various edge conditions are solved by using Fourier series expansion. In this study, the plate displacement function is considered to be in the form of a double Fourier series. Here the accuracy of solution is achieved for the number of truncated terms from the infinitive Fourier series and a good convergence is achieved for the Reisz method of summation.

The solutions of all above mentioned problem are based on the application of the Fourier series and transformations. This methods requires regularization in case the input data has some singularities. For example when input data expressed by singular distributions. In this paper we classify the singularity in term of the Sobolev spaces and consider the Reisz method of summation as regularization of the Fourier series solutions of the problems.

PRELIMINARIES

Localization is dependence of the convergence or divergence of the Fourier series at the given point only from the behaviour of the function in arbitrary small neighbourhood of that point. Equiconvergence means convergence and divergence of the Fourier series and Fourier integral at the same time and the same term.

For any arbitrary self-adjoint elliptic operators the sufficient condition for localization in the Liouville space $L_2^l(T^N)$ has proved by Alimov [5]. He showed that the sufficient localization condition for Reisz means of order s of multiple Fourier series and integrals is,

$$l + s \geq \max\left\{\frac{N-1}{2}, \frac{N-1}{p}\right\}, \qquad 1 \leq p \leq \infty$$

Moreover, $l + s \geq \dfrac{N-1}{p}$ be the exact localization condition for any elliptic operator [see in 5]. The sufficient condition for the localization of the expansion in multiple Fourier integrals is,

$$l + s \geq \frac{N-1}{p} - r\left(\frac{1}{p} - \frac{1}{2}\right), \qquad 1 \leq p \leq 2$$

But for the expansion in multiple Fourier series this condition is true for $l = 0$ [6]. In N-dimension, equiconvergence of the Fourier series and integral is not valid for the rectangular partial sums. For three dimensional case, the spherical problem for the special sequence of partial sums of the expansions in the Fourier series and integral is discussed in [7]. Equiconvergence in summation of the Fourier series and integral of the linear continuous functional which is associated with an elliptic polynomial discussed in [6]. Moreover the general expanded expansions of distributions are discussed in [8-18].

Let us consider the space of infinitely differentiable function φ: $T^N \rightarrow$ C which is defined by $\varepsilon(T^N)$. For any compact subset K of $T^N = [-\pi, \pi]^N$, the system of semi norm is,

$$P_{k,\gamma}(\phi) = \sup_{x \in k} |D^\gamma \phi(x)|$$

where $\gamma = (\gamma_1, \gamma_{2,} ......\gamma_N)$ is N dimensional vector with the non-negative integer components γ_j (j = 1, 2 ...., N). the multi-index γ is denoted by $|\gamma| = (\gamma_1 + \gamma_2 + ... + \gamma_N)$.

For instance, $D^\gamma = D_1^{\gamma_1} D_2^{\gamma_2} .... D_N^{\gamma_N}$, where

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j}, j = 1,2,.....N .$$

Let $\varepsilon'(T^N)$ be the conjugate space of the locally convex topological space $\varepsilon(T^N)$. For any functional $f \in \varepsilon'(T^N)$, we can write

$$f = (2\pi)^{-\frac{N}{2}} \sum_{n \in Z^N} f_n e^{inx}, \qquad (1)$$

where Z^N is the set of all vectors with integer components, f_n is the Fourier coefficient which is defined as the value of f on the test function on $f = (2\pi)^{-N/2} e^{-inx}$ and $x \in T^N$.

The Riesz means of order s (s is non-negative real number) of the Fourier series (1) is define as,

$$\sigma_\lambda^s f(x) = (2\pi)^{-\frac{N}{2}} \sum_{A(n)<\lambda} \left(1 - \frac{A(n)}{\lambda}\right)^s f_n e^{inx}. \qquad (2)$$

Now, we extend a distribution f from N - dimensional torus T^N to the whole space R^N by zero. Then the Bochner-Riesz means of order s of the Fourier integral of f is,

$$R_\lambda^s f(x) = (2\pi)^{-\frac{N}{2}} \int_{A(\xi)<\lambda} \left(1 - \frac{A(\xi)}{\lambda}\right)^s \hat{f}(y) e^{iA(\xi)x} d\xi, \quad (3)$$

where, $\hat{f}(y) = < f, (2\pi)^{-N/2} e^{-iA(\xi)x} >$ is the Fourier transformation of the extended functional f and it acts on , $(2\pi)^{-N/2} e^{-iA(\xi)x}$ via x.

Let l be any real number and $L_2^l(T^N)$ denote the Sobolev space of distributions

$$L_2^l(T^N) = \{f \in \varepsilon' : \sum_{n \in Z^N} (1 + |n|^2)^l f_n^2 < \infty\}.$$

We have $\varepsilon(T^N) \subset \bigcup_{l=-\infty}^{\infty} L_2^{-l}$. For any $\delta \in L_2^{-l}$ (δ is a Dirac delta function) and $l > N/2$, (where N = 2) and

$$s > \max\{\frac{(N-r-1)(1-1/2m)}{2} + \frac{r}{2}, \frac{N-1}{2}\} + l$$
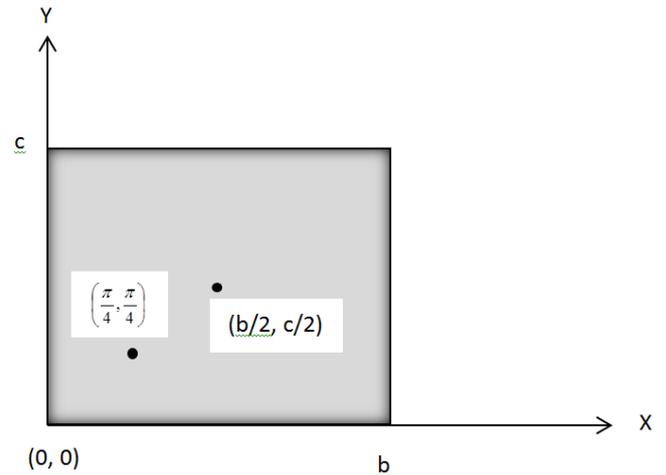
.Then for any $f \in L_2^{-l}(T^N)$

$$\sigma_\lambda^s f(x) = R_\lambda^s f(x) + O(1)\|f\|_{-l},$$

where $\|.\|_{-l}$ is a norm in $L_2^{-l}(T^N)$:

$$\|f\|_{-l} = (2\pi)^{-\frac{N}{2}} \sqrt{\sum_{n \in Z^N} (1 + |n|^2)^{-l} f_n^2} .\text{ [See in 6]}$$

In the section 3 and 4 we will verify this numerically for square plate vibration problem subjected to different boundary conditions. As it is mentioned Regular summation method will use for excellent accuracy and convergence.

Description of the Problem

Figure 1: A square plate of dimensions b × c in the xy-plane

Consider a square aluminium plate of dimensions b × c in the xy-plane as shown on Fig. 1 where b = c = π. The vibration equation subjected to a boundary condition is

$$u_{tt} = a^2 (u_{xx} + u_{yy}), \qquad 0<x<b, \quad 0<y<c \qquad (4)$$

$$u(0, y, t) = 0, \qquad (5)$$

$$u(b, y, t) = 0, \qquad (6)$$

$$u(x, 0, t) = 0, \qquad (7)$$

$$u(x, c, t) = 0, \qquad (8)$$

where, t is a time and 'a' is a characteristic of homogeneous material. Here we take aluminium (Al). From equation (5-8) it is clear that all the boundary of the square membrane is fixed. The initial condition of the vibration equation is,

$$u(x, y, 0) = \delta\left(x - \frac{b}{2}, y - \frac{c}{2}\right), \qquad (9)$$

$$u_t(x, y, 0) = 0, \qquad (10)$$

where, δ is a Dirac delta function. After putting the boundary condition we find,

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(A_{nm}\,Cos\,a\sqrt{\left(\frac{n\pi}{b}\right)^2+\left(\frac{m\pi}{c}\right)^2}\,t + B_{nm}\,Sin\,a\sqrt{\left(\frac{n\pi}{b}\right)^2+\left(\frac{m\pi}{c}\right)^2}\,t \right)Sin\,\frac{n\pi x}{b}\,Sin\,\frac{m\pi y}{c},.. \quad (11)$$

where, A_{nm} and B_{nm} are constant coefficient. Putting the initial condition in equation (11) we can find

$$B_{nm}=0, \quad (12)$$

$$A_{nm}=Sin\,\frac{n\pi}{2}\,Sin\,\frac{m\pi}{2}, \quad (13)$$

Putting the value of A_{nm} and B_{nm} in equation (11) we can find the solution of the square plate vibration problem. The solution is

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}Sin\,\frac{n\pi}{2}\,Sin\,\frac{m\pi}{2}\,Cos\,a\sqrt{n^2+m^2}\,t\,Sin\,nx\,Sin\,my , \quad (14)$$

Now the question is when the series is convergent. We know that in the point ($\frac{\pi}{2}$, $\frac{\pi}{2}$), the value of the series is infinity.

Now choose a point in the square plate except the Dirac delta function. Here we choose $x = \frac{\pi}{4}$ and $y = \frac{\pi}{4}$.

It should not be convergent. The series need regularization. The Riesz means of order s (s is non-negative real number) of the Fourier series (14) is define as

$$E_\lambda^s u(x,y,t)= \sum_{|L|<\lambda}\left(1-\frac{|L|}{\lambda}\right)^s Sin\,\frac{n\pi}{2}\,Sin\,\frac{m\pi}{2}\,Cos\,a\sqrt{n^2+m^2}\,t\,Sin\,nx\,Sin\,my,$$

$$(15)$$

Regularization of Square Plate Vibration at Initial Time

In this section we will test the theorem 1 for different value of s. To find the regularization of the series at the point ($\frac{\pi}{4}$, $\frac{\pi}{4}$) for different value of λ here we take s = 0, 1, 2 and time t = 0.

Table 1: Solution of the Reisz means for different s (s=0, s=1, s=2) and time t=0.

λ	$E_\lambda^0(\pi/4,\pi/4,0)$	$E_\lambda^1(\pi/4,\pi/4,0)$	$E_\lambda^2(\pi/4,\pi/4,0)$
850	7.00000000000111	-0.00411526008122465	0.00315287084334459
2000	7.05258074162887e-12	0.00437619755418917	0.00224139519878630
2500	-5.99999999999709	-0.00424875153686566	0.00139881698091759
3500	-11.9999999999978	0.00265330270521225	0.000670850331949691
4000	-1.49999999998671	-0.00301893616894222	0.000496940178414063
6000	-14.4999999999866	0.00390290242268027	0.000192221516156804
6500	8.00000000000999	-0.00459805620623655	0.000158396465790134
7000	4.50000000002167	0.00370161678701597	0.000129784596000590
8000	17.5000000000110	-0.00293904530899636	0.0000919562578425501
8500	-21.9999999999889	0.00475047959718943	0.0000777342813546005
9000	9.99999999997983	-0.00392974892915876	0.0000669218950743560

From the table it is clear that for s = 0 the Fourier series diverges. After regularization the Reisz means of order s = 1 of the Fourier series also diverges. Finally for s = 2 the series converges. The value of the Riesz means of order s = 2 of the Fourier series is approximately near to zero.

Regularization in Critical Index

In Table 1 the results clearly show that when s = 2 the series converges. We know that the series will converge when

$s>\frac{N-1}{2}+l$. So s = 3/2 is the critical point for the series.

Finally, we take a more complicated point s=1.5. To understand the difference between below critical point and above critical point we take two points 1.5 − ε and 1.5 + ε (ε is very small number). Here, we choose ε = 0.1. Where all the other parameters are kept the same.

Table 2: Solution of the Reisz means for different s (s=1.4, s=1.5, s=1.6) and time t=0.

λ	$E_\lambda^{1.4}(\pi/4,\pi/4,0)$	$E_\lambda^{1.5}(\pi/4,\pi/4,0)$	$E_\lambda^{1.6}(\pi/4,\pi/4,0)$
850	0.00803593046444506	-0.00115524562312308	0.00142023240665223
2000	0.000875761946008247	0.000963512443736381	0.00112013058002347
2500	-0.000784349026601738	-0.000780214420703640	0.000834852932308561
3500	-0.000499871318010436	0.000780214420703640	0.000476428457911737
4000	-0.00204231619672313	-0.000286399500720402	0.00033969719526239
6000	-0.000187415490236209	0.000177104174169329	0.000175328908221639
6500	0.000152049739436416	-0.000152735642328927	0.000152411243847878
7000	0.000248924682223761	0.000767207453907792	0.000102762495466868
8000	0.00508716709242962	-0.000739754672066731	0.0000843421787644696
8500	-0.000859976145705169	0.000796355406602819	0.0000775983948437184
9000	0.000776983513026165	-0.000771428936626843	0.0000731753987851646

It can be seen that after critical point the series converges and below critical point the series diverges. Now, in the critical point the answer of the sine series is so close to zero but it diverges.

CONCLUSION

In this paper we studied optimization of the regularization of the Fourier series of square plate vibration problem and a numerical method is used to find the series solution at a fixed point of the plate at initial time and critical index. As it is expected, we achieved the good convergence after critical point.

ACKNOWLEDGMENT

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