International Journal of Applied Engineering Research ISSN 0973-4562 Volume 14, Number 13 (2019) pp. 3119-3122
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Construction of Green Matrix for the Solution of a Matrix Differential Equation

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Abstract:

In this article problem related with a matrix differential operator is considered. Problems, existence theorems are discussed in the previous article. In this article the construction of Green matrix for the solution of a Matrix Differential Equation are discussed which are useful in finding further results with related with the problem.

Keywords: Matrix differential operator, Partial Differential equation, Green's matrix.

INTRODUCTION:

We determine a particular solution of a particular ordinary differential equations with the help of Green's matrix in terms of first order in homogeneous linear system of ordinary differential equation. The concept of Green's matrix is named after George Green.

The differential equation which is considered in the problem is given below,

$$-\frac{d}{dx}\left(P_o\frac{du}{dx}\right) + pu + rv = \lambda\,(F_{11}u + F_{11}v),$$

$$i\frac{dv}{dx} + qv + ru = -\lambda\,(F_{21}u + F_{22}v) \qquad \ldots (1)$$

For a given vector

$$\phi = \begin{pmatrix} u \\ v \end{pmatrix} \qquad \ldots\ldots\ldots (2)$$

$$L = \begin{pmatrix} -\frac{d}{dx}\left(P_o\frac{d}{dx}\right) + p & r \\ r & i\,\frac{d}{dx} + q \end{pmatrix} \qquad \ldots\ldots (3)$$

Therefore, the equation 1 reduces into

$$L\,\phi = -\lambda\,F\,\phi \qquad \ldots\ldots\ldots\ldots.. (4)$$

We impose the following boundary condition on $\phi = \begin{pmatrix} u \\ v \end{pmatrix}$

u' (a) = 0 ……….. (5)

v (a) = v (b) ……….. (6) &

u' (b) = 0 ………. (7)

Now, the equation 4 together with equation 5, 6 and 7 becomes a boundary value problem. In this case Green's Formula for the problem "$\int_a^b(\phi_k^{-T}\,L\phi_j - \phi_j^T L\phi_k)\,dx = [\phi_j\,\phi_k]\,(b) - \phi_j\,\phi_k\,(a)$" is the most suitable formula for our boundary value problem and therefore for the singular solution of the problem construction of Green Matrix is necessary and important.

Here, a unique matrix exist for x ϵ [a, b] y ϵ [a, b] is given below

$$\begin{pmatrix} G_{11}\,(x, y{:}\,\lambda) & G_{21}\,(x, y{:}\,\lambda) \\ G_{12}\,(x, y{:}\,\lambda) & G_{22}\,(x, y{:}\,\lambda) \end{pmatrix} \quad \ldots\ldots\ldots (8)$$

Such that

(i) The vectors $G_1(x, y{:}\,\lambda) = \begin{pmatrix} G_{11} \\ G_{12} \end{pmatrix}(x, y{:}\,\lambda)$ and $G_2(x, y{:}\,\lambda) = \begin{pmatrix} G_{21} \\ G_{22} \end{pmatrix}(x, y{:}\,\lambda)$ satisfy the equation 4 and the boundary conditions
$[G_k\,\phi_1]$ (a) = 0, $[G_k\,\phi_2]$ (a) + $[G_k\,x_2]$ = 0, $[G_k\,x_3]$ (b) = 0
(K= 1, 2) ……… (9)

(ii) G_{11}, G_{12}, G_{21} and G_{22} are continuous in [a, b] at x= y

(iii) For x ϵ [a, y [x ϵ] y, b] : G_{11}, G_{21} posses continuous derivatives of first order and second order. G_{12}, G_{22} posses continuous derivative of first order.

(iv) G_{11}, G_{12}, G_{21} and G_{22} are such that the following satisfied at x = y

$$\lim_{h\to 0}\frac{\delta}{\delta x}G_{11}(y + h, y{:}\,\lambda) - \lim_{h\to 0}\frac{\delta}{\delta x}G_{11}(y - h, y{:}\,\lambda) = \frac{1}{P_0 y}$$

$$\lim_{h\to 0}G_{12}(y + h, y{:}\,\lambda) - \lim_{h\to 0}G_{12}(y - h, y{:}\,\lambda) = 0 \quad \ldots\ldots (10)$$

$$\lim_{h\to 0}\frac{\delta}{\delta x}G_{11}(y + h, y{:}\,\lambda) - \lim_{h\to 0}\frac{\delta}{\delta x}G_{21}(y - h, y{:}\,\lambda) = 0$$

$$\lim_{h\to 0}G_{22}(y + h, y{:}\,\lambda) - \lim_{h\to 0}G_{22}(y - h, y{:}\,\lambda) = -\frac{1}{1} \quad \ldots\ldots\ldots\ldots (11)$$

Now, the matrix $\begin{pmatrix} G_{11} & G_{21} \\ G_{12} & G_{22} \end{pmatrix}$ is the Green's Matrix for boundary value problem of equation 4. It is well known that y appears as a parameter in the functions G_{ij}, (i, j = 1, 2).

Proof:

International Journal of Applied Engineering Research ISSN 0973-4562 Volume 14, Number 13 (2019) pp. 3119-3122
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We consider following for the vectors G_1 andG_2.

G_1 (x , y: λ) = α_1 ϕ_1 (a / x, λ) + α_2 ϕ_2 (a / x, λ),

a ≤ x < y < b

G_1 (x, y :λ) = β_2 x_2 (b / x, λ) + β_3 x_2 (b / x, λ),

a < x < y < b ………….. (12)

G_2 (x, y :λ) = A_1 ϕ_1 (a / x, λ) + A_2 ϕ_2 (a / x, λ),

a < x < y < b

G_2(x, y :λ) = B_2 x_2 (b / x, λ) + B_3 x_3 (b / x, λ),

a < x < y < b ………….. (13)

Where α_1, α_2, A_1, A_2, β_2, β_3, B_2 and B_3 are real or complex constants depending on y and λ. These constant will be determined later.

The main objective is to show the existence and uniqueness of G_1 and G_2. From the definition of G_1 and G_2 the condition no. (iii) and equation no. 4 is satisfied.

According to equation 9 we say that,

$[G_1(x, y{:}\,\lambda)\ \phi_1(a/\,x, \lambda]$(a)= 0, $[G_1(x, y{:}\,\lambda)\ x_3(b/\,x, \lambda]$ (b) = 0

$[G_2(x, y{:}\,\lambda)\ \phi_1(a/\,x, \lambda]$(a)= 0, $[G_2(x, y{:}\,\lambda)\ x_3(b/\,x, \lambda]$ (b) = 0

For all values α_1, α_2, β_1, β_2, A_1, A_2, B_2 and B_3 respectively. We can also be seen that

$[G_1(x, y{:}\,\lambda)\ \phi_2(a/\,x, \lambda]$ (a) + $[G_1(x, y{:}\,\lambda)\ x_2(\mathrm{b}/\,x, \lambda]$ (b) = 0, holds true if

$\alpha_2 = -\beta_2$….. (14)

Also,

$[G_2(x, y{:}\,\lambda)\ \phi_2(a/\,x, \lambda]$ (a) + $[G_1(x, y{:}\,\lambda)\ x_2(\mathrm{b}/\,x, \lambda]$ (b) = 0, holds true for;

A_2 = - B_2 …… (15)

Now, u_2 (a/a, λ) = 0, u_2'(a/a, λ) = 0, v_1 (a/a, λ) = 1 is equivalent to

$G_{11}(x$, y: λ) = $\alpha_1 u_1(a/\,x, \lambda)$+ $\alpha_2 u_2(a/\,x, \lambda)$

$G_{12}(x$, y: λ) = $\alpha_1 v_1(a/\,x, \lambda)$+ $\alpha_2 v_2(a/\,x, \lambda)$

….. (16)

$G_{11}(x$, y: λ) = $\beta_2 u_2(b/\,x, \lambda)$+ $\beta_3 u_3(b/\,x, \lambda)$

$G_{12}(x$, y: λ) = $\beta_2 v_2(b/\,x, \lambda)$+ $\beta_3 v_3(b/\,x, \lambda)$ $a<x<y<b$

…… (17)

By differentiation yield of Equation 16 & 17, respectively we get

$G_{11}(x$, y: λ) = $\alpha_1 u_1'(a/\,x, \lambda)$+ $\alpha_2 u_2'(a/\,x, \lambda)$,

a < x < y < b; and

$G_{11}(x$, y: λ) = $\beta_2 u_2'(b/\,x, \lambda)$+ $\beta_3 u_3'(b/\,x, \lambda)$

a < x < y < b ……….. (18)

After taking limit as $x \Longrightarrow$ y (h →0)

$\alpha_1 u_1$ $(a/y\,, \lambda)$ + $\alpha_2 u_2$ $(a/y\,, \lambda)$ -$\beta_2 u_2$ (b / y, λ) - $\beta_3 u_3$ (b / y, λ) = 0

$\alpha_1 v_1$ $(a/y\,, \lambda)$ + $\alpha_2 v_2$ $(a/y\,, \lambda)$ -$\beta_2 v_2$ (b / y, λ) - $\beta_3 v_3$ (b / y, λ) = 0 &

$\alpha_1 u_1'(a/y\,, \lambda)$ + $\alpha_2 u_2'$ $(a/y\,, \lambda)$ -$\beta_2 u_2'$ (b / y, λ) - $\beta_3 u_3'$ (b / y, λ) = $\frac{1}{P_0(y)}$

$a<x<y<b$

Using; u_1(b / b, λ) = 0,u_2' = (b / b, λ) = 0, v_1 (b / b, λ) = 0 and putting α_3 = - β_3; the above expression reduce to

$\alpha_1 u_1$ $(a/y\,, \lambda)$ + $\alpha_2 u_2$ $(a/y\,, \lambda)$ + u_2 $(b/y\,, \lambda)$ + $\alpha_3 u_3$ $(b/y\,, \lambda)$ =0

$\alpha_1 v_1$ $(a/y\,, \lambda)$ + $\alpha_2 v_2$ $(a/y\,, \lambda)$ + $\beta_2 v_2$ $(b/y\,, \lambda)$ - $\beta_3 u_3$ $(b/y\,, \lambda)$ =0 &

$\alpha_1 u_1'(a/y\,, \lambda)$ + $\alpha_2 u_2'$ $(a/y\,, \lambda)$ -$\beta_2 u_2'$ (b / y, λ) - $\beta_3 u_3'$ (b / y, λ) = $\frac{1}{P_0(y)}$

Using equation 14 and putting α_3 = - β_3 the above equation reduces to

$\alpha_1 u_1$ $(a/y\,, \lambda)$ + $\alpha_2 u_2$ $(a/y\,, \lambda)$ + u_2 $(b/y\,, \lambda)$ + $\alpha_3 u_3$ $(b/y\,, \lambda)$ =0

$\alpha_1 v_1$ $(a/y\,, \lambda)$ + $\alpha_2 v_2$ $(a/y\,, \lambda)$ + v_2 $(b/y\,, \lambda)$ + $\alpha_3 u_3$ $(b/y\,, \lambda)$ =0 &

$\alpha_1 u_1'(a/y\,, \lambda)$ + $\alpha_2 u_2'$ $(a/y\,, \lambda) u_2'$ (b / y, λ) + $\alpha_3 u_3'$ (b / y, λ) = $\frac{1}{P_0(y)}$

……… (19)

Where accents denotes differentiation w.r.t. x in the respective intervals and then y is put for x using the fact that ϕ_3 and x_1 are trivial solutions.

$n_r(x$, λ) = $\phi_r(a/x\,, \lambda)$+ $x_r(b/x\,, \lambda)$ and defining n_r= $\begin{pmatrix}\xi_r\\ \xi_r\end{pmatrix}$……… (20)

$\alpha_1 \xi_1$(y, λ) + $\alpha_2 \xi_2$(y, λ) + $\alpha_3 \xi_3$(y, λ) = 0

$\alpha_1 \mu_1$(y, λ) + $\alpha_2 \mu_2$(y, λ) + $\alpha_3 \mu_3$(y, λ) = 0

$\alpha_1 \xi_1'$(y, λ) +$\alpha_2 \xi_2'$(y, λ) +$\alpha_3 \xi_3'$(y, λ) = $\frac{1}{P_0(y)}$.....(21)

Now, the determinant obtained by the coefficients of α_1, α_2 and α_3 is

$$\begin{vmatrix}\xi_1(y,\lambda) & \xi_2(y,\lambda) & \xi_3(y,\lambda)\\ \mu_1(y,\lambda) & \mu_2(y,\lambda) & \mu_3(y,\lambda)\\ \xi_1'(y,\lambda) & \xi_2'(y,\lambda) & \xi_3'(y,\lambda)\end{vmatrix} \mathrm{w}(n_1, n_2\, n_3)(\mathrm{y}, \lambda) \qquad \neq 0$$

But by assumption is not an Eigen value hence

D (λ) ≠ 0.

By applying the theorem

D (λ) = i$P_o(x)$. w (η_1, η_2, η_3) (x, λ) $x_{P_0}(x)$. w (η_1, η_2, $\overline{\eta_3}$) (x, λ).

We have w (η_1, η_2, η_3) (y, λ) ≠ 0.

With the help of Cramer's rule we can find $\alpha_1, \alpha_2, \alpha_3$ i.e. $\alpha_1, \alpha_2, \beta_2, \beta_3$.

□ G_1 (x, y; λ) satisfies the properties of theorem & it is unique.

Now, solve the equation with the help of Cramer's rule and therefore $\alpha_1, \alpha_2, \alpha_3$ are given by

$$\frac{\alpha_1}{\frac{1}{P_0}\begin{vmatrix}0 & \xi_2 & \xi_3\\ 0 & \mu_1 & \mu_2\\ 1 & \xi_2' & \xi_3'\end{vmatrix}} = \frac{\alpha_2}{\frac{1}{P_0}\begin{vmatrix}\xi_1 & 0 & \xi_3\\ \mu_1 & 0 & \mu_3\\ \xi_1' & 1 & \xi_3'\end{vmatrix}} = \frac{\alpha_3}{\frac{1}{P_0}\begin{vmatrix}\xi_1 & \xi_2 & 0\\ \mu_1 & \mu_2 & 0\\ \xi_1' & \xi_2' & 1\end{vmatrix}} = \frac{1}{\begin{vmatrix}\xi_1 & \xi_2 & \xi_3\\ \mu_1 & \mu_2 & \mu_3\\ \xi_1' & \xi_2' & \xi_3'\end{vmatrix}}$$

From this equation we get,

$\alpha_1 = \begin{vmatrix}\xi_2 & \xi_3\\ \mu_2 & \mu_3\end{vmatrix}$/ P_0 (y) w(y, λ)

$\alpha_2 = \begin{vmatrix}\xi_3 & \xi_1\\ \mu_3 & \mu_1\end{vmatrix}$/ P_0 (y) w(y, λ) = - β_2

$\alpha_3 = \begin{vmatrix}\xi_1 & \xi_2\\ \mu_1 & \mu_2\end{vmatrix}$/ P_0 (y) w(y, λ) = - β_3

Now equation 13 is equivalent to

G_{21} (x, y: λ) = $A_1 u_1$ (a/x, λ) + $A_2 u_2$ (a/x, λ)

G_{22} (x, y: λ) = $A_1 v_1$ (a/x, λ) + $A_2 v_2$ (a/x, λ) a<x<b

…… (23)

G_{21} (x, y: λ) = $B_2 u_2$ (b/x, λ) + $B_3 u_3$ (b/x, λ)

G_{22} (x, y: λ) = $B_2 v_2$ (b/x, λ) + $B_3 v_3$ (b/x, λ) a<x<b

……. (24)

By differentiation yield of Equation 22 & 23,

G_{21}' (x, y: λ) =$A_1 u_1'$(a/x, λ) +$A_2 u_2'$(a/x, λ),

a < x < y < b

G_{22}' (x, y: λ) =$\beta_2 u_2'$(b/x, λ) +$\beta_3 u_3'$(b/x, λ),

a < x < y < b ……. (24)

From condition (b), 11, 22 and 23 we take limit as $x \to y$ *(h →0)*

$A_1 u_1$ (a /y, λ) + $A_2 u_2$ (a/y, λ) - $B_2 u_2$ (b/y, λ) - $B_3 u_3$ (b/y, λ) = 0

$A_1 v_1$ (a /y, λ) + $A_2 v_2$ (a/y, λ) - $B_2 v_2$ (b/y, λ) - $B_3 v_3$ (b/y, λ) = -1/1

$A_1 u_1'$ (a /y, λ) + $A_2 u_2'$(a /y, λ) -$B_2 u_3'$ (b/y, λ) - $B_3 u_3$ (b/y, λ) = 0 …… (25)

Put A_2 = - B_2 & A_3 = - B_3 in equation 25 then we get

$A_1 u_1$ (a /y, λ) + $A_2 u_2$ (a/y, λ) + u_2 (b/y, λ) + $A_3 u_3$ (b/y, λ) = 0

$A_1 v_1$ (a /y, λ) + $A_2 v_2$ (a/y, λ) + v_2 (b/y, λ) + $A_3 v_3$ (b/y, λ) = -1/1

$A_1 u_1'$ (a /y, λ) + $A_2 u_2'$(a /y, λ) +u_3' (b/y, λ) + $A_3 u_3'$ (b/y, λ) = 0 ……… (26)

Here, ϕ_3 and x_1 are trivial solutions.

$\eta_r = \begin{pmatrix}\xi_r\\ \mu_r\end{pmatrix} = \phi_r$ (a /x, λ) + x_r (a /x, λ),

Equation 26 may be expressed as

$A_1 \xi_1$ (y, λ) + $A_2 \xi_2$ (y, λ) + $A_3 \xi_3$ (y, λ) = 0

$A_1 \mu_1$ (y, λ) + $A_2 \mu_2$ (y, λ) + $A_3 \mu_3$ (y, λ) = -1/1

$A_1 \xi_1'$ (y, λ) + $A_2 \xi_2'$ (y, λ) + $A_3 \xi_3'$ (y, λ) = 0 ………………. (27)

The determinant formed by the coefficients of A_1, A_2, A_3 is given below

$\begin{vmatrix}\xi_1(y,\lambda) & \xi_2(y,\lambda) & \xi_3(y,\lambda)\\ \mu_1(y,\lambda) & \mu_2(y,\lambda) & \mu_3(y,\lambda)\\ \xi_1'(y,\lambda) & \xi_2'(y,\lambda) & \xi_3'(y,\lambda)\end{vmatrix}$ = W (n_1, n_2, n_3) (y, λ) = W(y, λ)

We assume that it is not an Eigen value, hence D (λ) ≠ 0.

□ W (n_1, n_2, n_3) (y, λ) ≠ 0. Thus we can find A_1, A_2, A_3 .

$$\frac{A_1}{\begin{vmatrix}0 & \xi_2 & \xi_3\\ 1 & \mu_2 & \mu_3\\ 1 & \xi_2' & \xi_3'\end{vmatrix}} = \frac{A_2}{\begin{vmatrix}\xi_1 & 0 & \xi_3\\ \mu_1 & 1 & \mu_3\\ \xi_1' & 0 & \xi_3'\end{vmatrix}} = \frac{A_3}{\begin{vmatrix}\xi_1 & \xi_2 & 0\\ \mu_1 & \mu_2 & 1\\ \xi_1' & \xi_2' & 0\end{vmatrix}} = \frac{1}{\begin{vmatrix}\xi_1 & \xi_2 & \xi_3\\ \mu_1 & \mu_2 & \mu_3\\ \xi_1' & \xi_2' & \xi_3'\end{vmatrix}} = \frac{1}{W(y,\lambda)},$$

which is equivalent to

A_1 = -1 $\begin{vmatrix}\xi_2 & \xi_3\\ \xi_2' & \xi_3'\end{vmatrix}$ /W(y, λ) , A_2 = -1 $\begin{vmatrix}\xi_1 & \xi_3\\ \xi_1' & \xi_3'\end{vmatrix}$ /W(y, λ) = - B_2 ,

A_3 = -1 $\begin{vmatrix}\xi_1 & \xi_2\\ \xi_1' & \xi_2'\end{vmatrix}$ /W(y, λ) = -B_3

Therefore,

G_1 (x, y:λ) = $\frac{\begin{vmatrix}\xi_2 & \xi_3\\ \mu_2 & \mu_3\end{vmatrix}}{P_0(y)W(y,\lambda)}\phi_1$ (a/x, λ) + $\frac{\begin{vmatrix}\xi_3 & \xi_1\\ \mu_3 & \mu_1\end{vmatrix}}{P_0(y)W(y,\lambda)}\phi_2$ (a/x, λ); x ϵ [a, y [

G_1 (x, y: λ) = $\frac{\begin{vmatrix}\xi_3 & \xi_1\\ \mu_3 & \mu_1\end{vmatrix}}{P_0(y)W(y,\lambda)} x_2$ (b/x, λ) + $\frac{\begin{vmatrix}\xi_1 & \xi_2\\ \mu_1 & \mu_2\end{vmatrix}}{P_0(y)W(y,\lambda)} x_3$ (b/x, λ); x ϵ] y, b [….. (28)

G_2 (x, y: λ) = $\frac{-1\begin{vmatrix}\xi_2 & \xi_3\\ \xi_2' & \xi_3'\end{vmatrix}}{W(y,\lambda)}\phi_1$ (a/x, λ) + $\frac{\begin{vmatrix}\xi_1 & \xi_3\\ \xi_1' & \xi_3'\end{vmatrix}}{W(y,\lambda)}\phi_2$ (a/x, λ); x ϵ] a, y]

G_2 (x, y: λ) = $\frac{-1\begin{vmatrix}\xi_1 & \xi_3\\ \xi_1' & \xi_3'\end{vmatrix}}{W(y,\lambda)} x_2$ (b/x, λ) + $\frac{\begin{vmatrix}\xi_1 & \xi_2\\ \xi_1' & \xi_2'\end{vmatrix}}{W(y,\lambda)} x_3$

(b/x, λ); x ϵ] y, b] …… (29)

Where (2×2) determinants are evaluated at (y, λ).

International Journal of Applied Engineering Research ISSN 0973-4562 Volume 14, Number 13 (2019) pp. 3119-3122
 https://dx.doi.org/10.37622/IJAER/14.13.2019.3119-3122

CONCLUSION:

Here, a partial differential equation problem is taken. The solution of the equation is very difficult. We convert the partial differential equation into ordinary differential equation for finding the solution. By the help of crammer's rule we findα_1, α_2, and α_3. After finding these value and use of boundary value condition we find Green matrix G_1 &G_2 where (2×2) determinants are evaluated at (y, λ). After this reduction of Green matrix in suitable form is performed and then we find a singular solution of the equation.

REFERENCES:

[1] Everitt, W.N., Some properties of Gram matrices and determinants, Quart. J. Math (oxford), (2), 9, 1958

[2] Bergmann, S., Integral operators in the theory of linear partial differential equations Springer, 1961.

[3] S.R. Bernfeld and V. Lakshmikantham*:* An Introduction to Nonlinear Boundary Value Problems. Academic Press, Inc., New York and London, 1974.

[4] Chakravarty, N.K., on the distribution of the eigen value of a differential system, J. Pure Math, Vol.2 PP: 71-90, (1982)

[5] Ramadhar Prasad and Devnandan Sharma, Existence and Uniqueness of Greens Matrix associated with a n × n matrix differential operator: Bulletine of Pure and Applied sciences. Vol. 16E (No.1) 1997; P. 55-59.

[6] Lallu Prasad, Fourier coefficients in a partial differential equation of fourth order, Ideal Research review Vol. 22, no. I, 11 June 2009.

[7] Himanshu Shekhar and Dr. Brijendra Kumar Singh, Fourier Co-Efficient in a partial Differential Equation of Even order. U.G.C. Sponsored National Seminar on Primary Education, Vedic Mathematics & Special Function (2011).