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Primitive Idempotent in FC_{8p^n} and Corresponding Codes

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Abstract:
In semi-simple ring $R_{8p^n} \equiv \frac{GF(q)[x]}{<x^{8p^n}-1>}$, where p and q (of type $8k+1$) are distinct odd primes, n is a positive integer and order of q modulo $8p^n$ is $\frac{\phi(p^n)}{2}$, expression for primitive idempotents are obtained.
Generating polynomials, dimensions and minimum distance bounds for the cyclic codes generated by these idempotents are also calculated.
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1. INTRODUCTION

The group algebra FC_{8p^n}, F is field of order q and C_{8p^n} is cyclic group of order $8p^n$ such that $g.c.d.(q, 8p) = 1$, is semi-simple having finite cardinality of collection of primitive idempotents which equals the cardinality of collection of q-cyclotomic cosets modulo $8p^n$[11]. The primitive idempotents of minimal cyclic codes of length m in case, when order of q modulo m is $\phi(m)$ for $m = 2, 4, p^n, 2p^n$ were computed in [6, 9]. The primitive idempotents of length p^n with order of q modulo p^n is $\frac{\phi(p^n)}{2}$ were obtained in [10] and minimal quadratic residue codes of length p^n in [7]. Cyclic codes of length $2p^n$ over F, where order of q modulo $2p^n$ is $\frac{\phi(2p^n)}{2}$ were discussed in [8]. Minimal cyclic codes of length p^nq, where p and q are distinct odd primes were derived in [1, 3]. Further, when order of q modulo p^n is $\phi(p^n)$, the minimal cyclic codes of length $8p^n$ were discussed in [4, 5]. Irreducible cyclic codes of length $4p^n$ and $8p^n$, where $q \equiv 3(mod\, 8)$ and $p/(q-1)$ were obtained in [2].
In present paper, we obtained cyclic codes of length $8p^n$ over F where q is of the form $8k+1$ and order of q modulo p^n is $\frac{\phi(p^n)}{2}$. The q-cyclotomic cosets modulo $8p^n$ are obtained in section 2 and corresponding primitive idempotents in section 3. In section 4, we discussed generating polynomials and dimensions for the corresponding cyclic codes of length $8p^n$. The minimum distance or the bounds for minimum distance of these codes are obtained in section 5. At the end, an example is discussed to illustrate the various parameters for these codes.

2. CYCLOTOMIC COSETS

Let $S = \{1, 2, ..., 8p^n\}$. For $a, b \in S$, consider $a \sim b$ iff $a \equiv bq^i(mod\, 8p^n)$ for some integer $i \geq 0$. This is an equivalence relation on S. The equivalence classes due to this relation are called q-cyclotomic cosets modulo $8p^n$. The q-cyclotomic coset containing $s \in S$ is $\Omega_s = \{s, sq, sq^2, ..., sq^{t_s-1}\}$, where t_s is the smallest positive integer such that $sq^{t_s} \equiv s(mod\, 8p^n)$.

Lemma 2.1. *[[8],Theorem 2.5] If $\frac{\phi(p^n)}{2}$ is the order of q modulo p^n, then the order of q modulo p^{n-i} is $\frac{\phi(p^{n-i})}{2}$, $0 \leq i \leq n-1$.*

Lemma 2.2. *If $\frac{\phi(p^n)}{2}$ is the order of q modulo p^n, then for $0 \leq i \leq n-1$, order of q modulo $2p^{n-i}$, $4p^{n-i}$ and $8p^{n-i}$ is $\frac{\phi(p^{n-i})}{2}$.*

Proof. Since $\frac{\phi(p^n)}{2}$ is the order of q modulo p^n, therefore by lemma 2.1, order of q modulo p^{n-i} is $\frac{\phi(p^{n-i})}{2}, 1 \leq i \leq n-1$. Hence

$$q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\, p^{n-i}) \tag{2.1}$$

Since q is of the form $8k+1$, therefore $q \equiv 1(mod\, 2)$. Hence, $q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\, 2)$. As $gcd(2, p^{n-i}) = 1$ and order of q modulo p^{n-i} is $\frac{\phi(p^{n-i})}{2}$, so $q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\, 2p^{n-i})$. This implies that $\frac{\phi(p^{n-i})}{2}$ is the smallest integer for which (2.1) holds. Hence order of q modulo $2p^{n-i}$ is $\frac{\phi(p^{n-i})}{2}$. Similar, result holds for $4p^{n-i}$ and $8p^{n-i}$. □

Lemma 2.3. *For $0 \leq i \leq n-1$ and $0 \leq k \leq \frac{\phi(p^{n-i})}{2} - 1$, $T \not\equiv q^k(mod\, 8p^{n-i})$, where $T = \lambda = (1+2p^n)$ or $T = \mu = 2(1+2p^n)$ or $T = \nu = (1+4p^n)$ or $T = \chi = (1+6p^n)$.*

Proof. Proof can be obtained by using lemma 2.1 and lemma 2.2. □

Lemma 2.4. *Let p be an odd prime. Then there exists an integer g , $1 < g < 8p$ and is primitive root modulo p. Further when*

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p is of the form $4k+1$ then order of g modulo 4 and modulo 8 is 2, and when p is of the form $4k+3$ then order of g modulo 4 is 1 and modulo 8 is 2. Also, if q is any prime power and $g.c.d.(q,p)=1$, then $g \notin \{1,q,q^2,...,q^{\frac{\phi(p)}{2}-1}\}$.

Proof. Consider the complete residue systems, $S_p = \{0,1,2,...,p-1\}$ modulo p, $S_2 = \{0,1\}$ modulo 2, and $S_{2p} = \{0,1,2,...,2p-1\}$ modulo $2p$. Since $g.c.d.(2,p)=1$, so there exist an integer $v \in S_p$ such that $2v-p=1$. Let a be any primitive root mod p in S_p. For $p \equiv 1(mod\ 4)$, let $g \equiv 2av+tp+6ap(mod\ 8p)$ where t is a prime of the form $8k_1+3$ implies $g \equiv a(mod\ p)$. Hence g is primitive root modulo p. Now, $g \equiv 2av+tp+6ap(mod\ 8)$ where t is a prime of the form $8k_1+3$, so $g \equiv 3(mod\ 4)$ as p is of the form $4k+1$. Hence order of g modulo 4 and modulo 8 is 2. Now for $p \equiv 3(mod\ 4)$, let $g \equiv 2av+tp+4ap(mod\ 8p)$ where t is a prime of the form $8k_2+7$ implies g is primitive root modulo p and order of g modulo 4 is 1 and modulo 8 is 2. Let $g \in \{1,q,q^2,...,q^{\frac{\phi(p)}{2}-1}\}$, so $g=q^i$ for some $1 \le i \le \frac{\phi(p)}{2}-1$ equivalently $o(g)=o(q^i)$. As order of q modulo $8p$ is $\frac{\phi(p)}{2}$, so $o(q^i) \le \frac{\phi(p)}{2}$ modulo $8p$. This implies $o(g) \le \frac{\phi(p)}{2}$ modulo $8p$, but order of g mod $8p$ is $\phi(p)$, hence $g \notin \{1,q,q^2,...,q^{\frac{\phi(p)}{2}-1}\}$. □

Lemma 2.5. *There exist a fixed integer g satisfying $gcd(g,2pq)=1, 1<g<8p, g \not\equiv q^k(mod p)$ where $0 \le k \le \frac{\phi(p)}{2}-1$ such that for $0 \le j \le n-1$, the set $\{1,q,q^2,...,q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ forms a reduced residue system modulo p^{n-j} and the set $\{1,q,q^2,...,q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}, \lambda, \lambda q, ..., \lambda q^{\frac{\phi(p^{n-j})}{2}-1}, \lambda g, \lambda gq, ..., \lambda gq^{\frac{\phi(p^{n-j})}{2}-1}, \nu, \nu q, ..., \nu q^{\frac{\phi(p^{n-j})}{2}-1}, \nu g, \nu gq, ..., \nu gq^{\frac{\phi(p^{n-j})}{2}-1}, \chi, \chi q, ..., \chi q^{\frac{\phi(p^{n-j})}{2}-1}, \chi g, \chi gq, ..., \chi gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ forms a reduced residue system modulo $8p^{n-j}$.*

Proof. By lemma 2.1, order of q modulo p is $\frac{\phi(p)}{2}$. Therefore the numbers $1,q,q^2,...,q^{\frac{\phi(p)}{2}-1}$ are incongruent modulo p. As there are exactly $\phi(p)$ numbers in the reduced residue system modulo p. Therefore there exist a number g satisfying $gcd(g,2pq)=1, 1<g<8p, g \not\equiv q^k(mod p)$ for $0 \le k \le \frac{\phi(p)}{2}-1$. Then the set $\{1,q,q^2,...,q^{\frac{\phi(p)}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p)}{2}-1}\}$ forms a reduced residue system modulo p. Since for $0 \le k \le \frac{\phi(p)}{2}-1$, $g \not\equiv q^k(mod p)$. It follows that $g \not\equiv q^k(mod p^{n-j})$ for $0 \le k \le \frac{\phi(p^{n-j})}{2}-1$. Hence the set $\{1,q,q^2,...,q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ forms a reduced residue system modulo p^{n-j}.

Similar result holds to show that the set $\{1,q,...,q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}, \lambda, \lambda q, ..., \lambda q^{\frac{\phi(p^{n-j})}{2}-1}, \lambda g, \lambda gq, ..., \lambda gq^{\frac{\phi(p^{n-j})}{2}-1}, \nu, \nu q, ..., \nu q^{\frac{\phi(p^{n-j})}{2}-1}, \nu g, \nu gq, ..., \nu gq^{\frac{\phi(p^{n-j})}{2}-1}, \chi, \chi q, ..., \chi q^{\frac{\phi(p^{n-j})}{2}-1}, \chi g, \chi gq, \chi gq^2, .., \chi gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ forms a reduced residue system modulo $8p^{n-j}$. □

Theorem 2.6. *The $(16n+8)$ q-cyclotomic cosets modulo $8p^n$ are $\Omega_{ap^n} = \{ap^n\}$, $a \in \mathbb{A} = \{0,1,2,...,7\}$ and for $0 \le i \le n-1$, $\Omega_{bp^i}=\{bp^i, bp^iq, bp^iq^2, ..., bp^iq^{\frac{\phi(p^{n-i})}{2}-1}\}$, $b \in \mathbb{B} = \{1,2,4,8,\lambda,\mu,\nu,\chi,g,2g,...,\chi g\}$.*

Proof. $\Omega_0 = \{0\}$ is trivial. Since q is of the form $8k+1$, so $p^nq \equiv p^n(mod\ 8p^n)$ and hence $\Omega_{p^n} = \{p^n\}$.
Similarly, $\Omega_{ap^n} = \{ap^n\}$ for $a \in \mathbb{A}$.
By lemma 2.2, $q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\ 8p^{n-i})$, equivalently $p^iq^{\frac{\phi(p^{n-i})}{2}} \equiv p^i(mod\ 8p^n)$.
Therefore, $\Omega_{p^i}=\{p^i, p^iq, p^iq^2, ..., p^iq^{\frac{\phi(p^{n-i})}{2}-1}\}$.
Similarly, $\Omega_{bp^i}=\{bp^i, bp^iq, bp^iq^2, ..., bp^iq^{\frac{\phi(p^{n-i})}{2}-1}\}$ for $b \in \mathbb{B}$.
Obviously, $|\Omega_0|=1$. Also, $|\Omega_{ap^n}|=1$ and $|\Omega_{bp^i}| = \frac{\phi(p^{n-i})}{2}$.
Therefore, $\sum_{i=0}^{n-1}|\Omega_{p^i}| = \sum_{i=0}^{n-1}\frac{\phi(p^{n-i})}{2} = \frac{\phi(p^n)}{2} + \frac{\phi(p^{n-1})}{2} + \frac{\phi(p^{n-2})}{2} + ... + \frac{\phi(p)}{2} = \frac{p^n-1}{2}$.
Hence, $|\sum_a \Omega_{ap^n}| + \sum_{i=0}^{n-1}|\sum_b \Omega_{bp^i}| = 8 + 16(\frac{p^n-1}{2}) = 8p^n$. □

3. PRIMITIVE IDEMPOTENTS

Throughout this paper, we consider that α is $8p^nth$ root of unity in some extension field of F. Let M_s be the minimal ideal in $R_{8p^n} = \frac{F[x]}{<x^{8p^n}-1>} \equiv FC_{8p^n}$, generated by $\frac{(x^{8p^n}-1)}{m_s(x)}$, where $m_s(x)$ is the minimal polynomial for α^s, $s \in \Omega_s$. We denote $P_s(x)$, the primitive idempotent in R_{8p^n}, corresponding to the minimal ideal M_s, given by $P_s(x) = \frac{1}{8p^n}\sum_{t=0}^{8p^n-1}\rho_i^s x^s$ where $\rho_i^s = \sum_{s\in\Omega_s}\alpha^{-is}$ and $\overline{Z}_s = \sum_{s\in\Omega_s}x^s$.
Then,

$$P_s(x) = \frac{1}{8p^n}[\sum_{a\in\mathbb{A}}\rho_{ap^n}^s\overline{Z}_{ap^n} + \sum_{i=0}^{n-1}\sum_{b\in\mathbb{B}}\rho_{bp^i}^s\overline{Z}_{bp^i}][4] \quad (3.1)$$

Lemma 3.1. *For any odd prime p and a positive integer k, if β is primitive p^kth root of unity in some extension field of F, then $\sum_{t=0}^{\frac{\phi(p^k)}{2}-1}(\beta^{q^t}+\beta^{gq^t}) = \begin{cases} -1, & k=1 \\ 0, & k\ge 2 \end{cases}$, when q is quadratic* *residue modulo* p^k *and* $\sum_{t=0}^{\phi(p^k)-1} \beta^{q^t} = \begin{cases} -1, & k=1 \\ 0, & k \geq 2 \end{cases}$, *when* q *is primitive root modulo* p^k.

Proof. By lemma 2.5, the set $\{1, q, q^2, ..., q^{\frac{\phi(p^k)}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^k)}{2}-1}\}$ is a reduced residue system $(mod\, p^k)$. So, $\sum_{t=0}^{\frac{\phi(p^k)}{2}-1} (\beta^{q^t} + \beta^{gq^t}) = \sum_{t=0}^{p^k-1} \beta^t - \sum_{t=1,p/t}^{p^k} \beta^t = -\sum_{t=1}^{p^{k-1}} \beta^{pt}$.

If $k = 1$, then $-\beta^p = -1$. If $k \geq 2$, then $\beta^p \neq 1$, therefore, $\sum_{t=1}^{p^{k-1}} \beta^{pt} = \beta^p(1 + \beta^p + ... + \beta^{pk-1}) = \beta^p \frac{(\beta^{pk} - 1)}{\beta^p - 1} = 0$. For the remaining see [[3], lemma 4]. □

Lemma 3.2. *For* $0 \leq i \leq n-1$, $\lambda^2\Omega_{p^i} = \nu^2\Omega_{p^i} = \chi^2\Omega_{p^i} = \Omega_{p^i} = \lambda\Omega_{\lambda p^i} = \nu\Omega_{\nu p^i} = \chi\Omega_{\chi p^i}$ *and* $\mu^2\Omega_{p^i} = 4\Omega_{p^i} = \mu\Omega_{\mu p^i} = 2\Omega_{2p^i} = \Omega_{4p^i}$.

Proof. Since $\lambda^2, \nu^2, \chi^2 \equiv 1(mod\ 8p^n)$ and $\mu^2 \equiv 4(mod\, 8p^n)$, the required result holds. □

Lemma 3.3. *For* Ω_{p^n}, Ω_{2p^n}, Ω_{3p^n}, Ω_{4p^n}, Ω_{5p^n}, Ω_{6p^n} *and* Ω_{7p^n}, $\Omega_{p^n} = -\Omega_{7p^n}$, $\Omega_{2p^n} = -\Omega_{6p^n}$, $\Omega_{3p^n} = -\Omega_{5p^n}$ *and* $\Omega_{4p^n} = -\Omega_{4p^n}$.

Proof. Proof of these are trivial. □

Lemma 3.4. *For* $p \equiv 1(mod\ 4)$, $\Omega_\chi = -\Omega_1$ *and for* $p \equiv 3(mod\ 4)$, $\Omega_{\chi g} = -\Omega_1$ *or* $\Omega_{\lambda g} = -\Omega_1$ *according as n is odd or even.*

Proof. For $p \equiv 1(mod\ 4)$, $\chi \equiv -1(mod\ 8)$ and $q^{\frac{\phi(p^n)}{4}} \equiv -1(mod\ 8)$. So, $\chi q^{\frac{\phi(p^n)}{4}} \equiv 1(mod\ 8)$. Also, $\chi \equiv 1(mod\ p^n)$ and $q^{\frac{\phi(p^n)}{4}} \equiv -1(mod\, p^n)$, thus $\chi q^{\frac{\phi(p^n)}{4}} \equiv -1(mod\, p^n)$.

As $(8, p^n) = 1$ from above relations, $\chi q^{\frac{\phi(p^n)}{4}} \equiv -1(mod\, 8p^n)$. Hence, $\Omega_\chi = -\Omega_1$.

If $p \equiv 3(mod\ 4)$ and n is odd then clearly $\chi \equiv 3(mod\ 8)$ and $g \equiv -3(mod\, 8)$, then $\chi g \equiv -1(mod\, 8)$. Also, $q^k \equiv 1(mod\, 8)$, so $\chi g q^k \equiv -1(mod\ 8)$.

Now $q^{\frac{\phi(p^n)}{2}} \equiv 1(mod\ p^n)$ and $\frac{\phi(p^n)}{2}$ is odd, thus $q^k \not\equiv -1(mod\, p^n)$ for $0 \leq k \leq \frac{\phi(p^n)}{2} - 1$.

The set $\{1, q, q^2, ..., q^{\frac{\phi(p^n)}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^n)}{2}-1}\}$ forms a reduced residue system modulo p^n, so $gq^k \equiv 1(mod\, p^n)$ and $\chi \equiv 1(mod\ p^n)$. Since $(8, p^n) = 1$, therefore $\chi g q^{\frac{\phi(p^n)}{4}} \equiv -1(mod\ 4p^n)$ and hence $\Omega_{\chi g} = -\Omega_1$.

If $p \equiv 3(mod\ 4)$ and n is even, then clearly $\lambda \equiv 3(mod\ 8)$ and $g \equiv -3(mod\ 8)$ so $\lambda g \equiv -1(mod\ 8)$.

As $q^k \equiv 1(mod\ 8)$, so $\lambda g q^k \equiv -1(mod\ 8)$. Similarly as above $gq^k \equiv 1(mod\, p^n)$ and $\lambda \equiv 1(mod\, p^n)$.

Thus, $\lambda g q^{\frac{\phi(p^n)}{4}} \equiv -1(mod\ 4p^n)$ and hence the result holds. □

Notation 3.5. For $0 \leq j \leq n-1$, define
$A_j = p^j \sum_{s\varepsilon\Omega_{gp^j}} \alpha^s$, $B_j = p^j \sum_{s\varepsilon\Omega_{\lambda gp^j}} \alpha^s$, $C_j = p^j \sum_{s\varepsilon\Omega_{p^j}} \alpha^s$, $D_j = p^j \sum_{s\varepsilon\Omega_{\lambda p^j}} \alpha^s$, $E_j = p^j \sum_{s\varepsilon\Omega_{2gp^j}} \alpha^s$, $F_j = p^j \sum_{s\varepsilon\Omega_{2p^j}} \alpha^s$, $G_j = p^j \sum_{s\varepsilon\Omega_{4gp^j}} \alpha^s$, $H_j = p^j \sum_{s\varepsilon\Omega_{4p^j}} \alpha^s$, $I_j = p^j \sum_{s\varepsilon\Omega_{8gp^j}} \alpha^s$, $J_j = p^j \sum_{s\varepsilon\Omega_{8p^j}} \alpha^s$.

Here, $A_j^q = A_j$, so $A_j \in F$. Similarly B_j, C_j, D_j, E_j, F_j, G_j, H_j, I_j and J_j all are in F.

Theorem 3.6. *The explicit expressions for primitive idempotents corresponding to cyclotomic cosets* Ω_0, Ω_{p^n}, Ω_{2p^n}, Ω_{3p^n}, Ω_{4p^n}, Ω_{5p^n}, Ω_{6p^n} *and* Ω_{7p^n} *in* R_{8p^n} *are given by*

$$P_0(x) = \tfrac{1}{8p^n}[\overline{Z}_0 + \overline{Z}_{p^n} + \overline{Z}_{2p^n} + \overline{Z}_{3p^n} + \overline{Z}_{4p^n} + \overline{Z}_{5p^n} + \overline{Z}_{6p^n} + \overline{Z}_{7p^n} + \sum_{i=0}^{n-1}\{\overline{Z}_{p^i} + \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} + \overline{Z}_{\lambda p^i}$$
$$+ \overline{Z}_{\mu p^i} + \overline{Z}_{\nu p^i} + \overline{Z}_{\chi p^i} + \overline{Z}_{gp^i} + \overline{Z}_{2gp^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \overline{Z}_{\lambda gp^i} + \overline{Z}_{\mu gp^i} + \overline{Z}_{\nu gp^i} + \overline{Z}_{\chi gp^i}\}]$$

$$P_{p^n}(x) = \tfrac{1}{8p^n}[\overline{Z}_0 - \alpha^{3p^{2n}}\overline{Z}_{p^n} - \alpha^{2p^{2n}}\overline{Z}_{2p^n} - \alpha^{p^{2n}}\overline{Z}_{3p^n} - \overline{Z}_{4p^n} + \alpha^{3p^{2n}}\overline{Z}_{5p^n} + \alpha^{2p^{2n}}\overline{Z}_{6p^n} + \alpha^{p^{2n}}\overline{Z}_{7p^n}$$
$$+ \sum_{i=0}^{n-1}\{-\alpha^{3p^{n+i}}\overline{Z}_{p^i} - \alpha^{2p^{n+i}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \alpha^{3\lambda p^{n+i}}\overline{Z}_{\lambda p^i} + \alpha^{2p^{n+i}}\overline{Z}_{\mu p^i} + \alpha^{3p^{n+i}}\overline{Z}_{\nu p^i} - \alpha^{3\chi p^{n+i}}\overline{Z}_{\chi p^i}$$
$$+ \alpha^{3p^{n+i}}\overline{Z}_{gp^i} + \alpha^{2p^{n+i}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \alpha^{3\lambda p^{n+i}}\overline{Z}_{\lambda gp^i} - \alpha^{2p^{n+i}}\overline{Z}_{\mu gp^i} - \alpha^{3p^{n+i}}\overline{Z}_{\nu gp^i} - \alpha^{3\lambda p^{n+i}}\overline{Z}_{\chi gp^i}\}]$$

$$P_{2p^n}(x) = \tfrac{1}{8p^n}[\overline{Z}_0 - \alpha^{2p^{2n}}\overline{Z}_{p^n} - \overline{Z}_{2p^n} + \alpha^{2p^{2n}}\overline{Z}_{3p^n} + \overline{Z}_{4p^n} - \alpha^{2p^{2n}}\overline{Z}_{5p^n} - \overline{Z}_{6p^n} + \alpha^{2p^{2n}}\overline{Z}_{7p^n} + \sum_{i=0}^{n-1}\{-\alpha^{2p^{n+i}}\overline{Z}_{p^i}$$
$$- \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} + \alpha^{2p^{n+i}}\overline{Z}_{\lambda p^i} - \overline{Z}_{\mu p^i} - \alpha^{2p^{n+i}}\overline{Z}_{\nu p^i} + \alpha^{2p^{n+i}}\overline{Z}_{\chi p^i} + \alpha^{2p^{n+i}}\overline{Z}_{gp^i} + \overline{Z}_{2p^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i}$$
$$- \alpha^{2p^{n+i}}\overline{Z}_{\lambda gp^i} - \overline{Z}_{\mu gp^i} + \alpha^{2p^{n+i}}\overline{Z}_{\nu gp^i} - \alpha^{2p^{n+i}}\overline{Z}_{\chi gp^i}\}]$$

$P_{3p^n}(x) = \frac{1}{8p^n}[\overline{Z}_0 - \alpha^{p^{2n}}\overline{Z}_{p^n} + \alpha^{2p^{2n}}\overline{Z}_{2p^n} - \alpha^{3p^{2n}}\overline{Z}_{3p^n} - \overline{Z}_{4p^n} + \alpha^{p^{2n}}\overline{Z}_{5p^n} - \alpha^{2p^{2n}}\overline{Z}_{6p^n} + \alpha^{3p^{2n}}\overline{Z}_{7p^n}$
$+ \sum_{i=0}^{n-1}\{-\alpha^{p^{n+i}}\overline{Z}_{p^i} + \alpha^{2p^{n+i}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \alpha^{\lambda p^{n+i}}\overline{Z}_{\lambda p^i} - \alpha^{2p^{n+i}}\overline{Z}_{\mu p^i} + \alpha^{p^{n+i}}\overline{Z}_{\nu p^i} - \alpha^{\chi p^{n+i}}\overline{Z}_{\chi p^i}$
$+ \alpha^{p^{n+i}}\overline{Z}_{gp^i} - \alpha^{2p^{n+i}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \alpha^{\lambda p^{n+i}}\overline{Z}_{\lambda gp^i} + \alpha^{2p^{n+i}}\overline{Z}_{\mu gp^i} - \alpha^{p^{n+i}}\overline{Z}_{\nu gp^i} - \alpha^{\lambda p^{n+i}}\overline{Z}_{\chi gp^i}\}]$

$P_{4p^n}(x) = \frac{1}{8p^n}[\overline{Z}_0 - \overline{Z}_{p^n} + \overline{Z}_{2p^n} - \overline{Z}_{3p^n} + \overline{Z}_{4p^n} - \overline{Z}_{5p^n} + \overline{Z}_{6p^n} - \overline{Z}_{7p^n} + \sum_{i=0}^{n-1}\{-\overline{Z}_{p^i} + \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \overline{Z}_{\lambda p^i}$
$+ \overline{Z}_{\mu p^i} - \overline{Z}_{\nu p^i} - \overline{Z}_{\chi p^i} - \overline{Z}_{gp^i} + \overline{Z}_{2gp^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} - \overline{Z}_{\lambda gp^i} + \overline{Z}_{\mu gp^i} - \overline{Z}_{\nu gp^i} - \overline{Z}_{\chi gp^i}\}]$

$P_{5p^n}(x) = \frac{1}{8p^n}[\overline{Z}_0 + \alpha^{3p^{2n}}\overline{Z}_{p^n} - \alpha^{2p^{2n}}\overline{Z}_{2p^n} + \alpha^{p^{2n}}\overline{Z}_{3p^n} - \overline{Z}_{4p^n} - \alpha^{3p^{2n}}\overline{Z}_{5p^n} + \alpha^{2p^{2n}}\overline{Z}_{6p^n} - \alpha^{p^{2n}}\overline{Z}_{7p^n}$
$+ \sum_{i=0}^{n-1}\{\alpha^{3p^{n+i}}\overline{Z}_{p^i} - \alpha^{2p^{n+i}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} + \alpha^{3\lambda p^{n+i}}\overline{Z}_{\lambda p^i} + \alpha^{2p^{n+i}}\overline{Z}_{\mu p^i} - \alpha^{3p^{n+i}}\overline{Z}_{\nu p^i} + \alpha^{3\chi p^{n+i}}\overline{Z}_{\chi p^i}$
$- \alpha^{3p^{n+i}}\overline{Z}_{gp^i} + \alpha^{2p^{n+i}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} - \alpha^{3\lambda p^{n+i}}\overline{Z}_{\lambda gp^i} - \alpha^{2p^{n+i}}\overline{Z}_{\mu gp^i} + \alpha^{3p^{n+i}}\overline{Z}_{\nu gp^i} + \alpha^{3\lambda p^{n+i}}\overline{Z}_{\chi gp^i}\}]$

$P_{6p^n}(x) = \frac{1}{8p^n}[\overline{Z}_0 + \alpha^{2p^{2n}}\overline{Z}_{p^n} - \overline{Z}_{2p^n} - \alpha^{2p^{2n}}\overline{Z}_{3p^n} + \overline{Z}_{4p^n} + \alpha^{2p^{2n}}\overline{Z}_{5p^n} - \overline{Z}_{6p^n} - \alpha^{2p^{2n}}\overline{Z}_{7p^n} + \sum_{i=0}^{n-1}\{\alpha^{2p^{n+i}}\overline{Z}_{p^i}$
$- \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \alpha^{2p^{n+i}}\overline{Z}_{\lambda p^i} - \overline{Z}_{\mu p^i} + \alpha^{2p^{n+i}}\overline{Z}_{\nu p^i} - \alpha^{2p^{n+i}}\overline{Z}_{\chi p^i} - \alpha^{2p^{n+i}}\overline{Z}_{gp^i} + \overline{Z}_{2p^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i}$
$+ \alpha^{2p^{n+i}}\overline{Z}_{\lambda gp^i} - \overline{Z}_{\mu gp^i} - \alpha^{2p^{n+i}}\overline{Z}_{\nu gp^i} + \alpha^{2p^{n+i}}\overline{Z}_{\chi gp^i}\}]$

$P_{7p^n}(x) = \frac{1}{8p^n}[\overline{Z}_0 + \alpha^{p^{2n}}\overline{Z}_{p^n} + \alpha^{2p^{2n}}\overline{Z}_{2p^n} + \alpha^{3p^{2n}}\overline{Z}_{3p^n} - \overline{Z}_{4p^n} - \alpha^{p^{2n}}\overline{Z}_{5p^n} - \alpha^{2p^{2n}}\overline{Z}_{6p^n} - \alpha^{3p^{2n}}\overline{Z}_{7p^n}$
$+ \sum_{i=0}^{n-1}\{\alpha^{p^{n+i}}\overline{Z}_{p^i} + \alpha^{2p^{n+i}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} + \alpha^{\lambda p^{n+i}}\overline{Z}_{\lambda p^i} - \alpha^{2p^{n+i}}\overline{Z}_{\mu p^i} - \alpha^{p^{n+i}}\overline{Z}_{\nu p^i} - \alpha^{\lambda p^{n+i}}\overline{Z}_{\chi p^i} - \alpha^{p^{n+i}}\overline{Z}_{gp^i}$
$- \alpha^{2p^{n+i}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} - \alpha^{\lambda p^{n+i}}\overline{Z}_{\lambda gp^i} + \alpha^{2p^{n+i}}\overline{Z}_{\mu gp^i} + \alpha^{p^{n+i}}\overline{Z}_{\nu gp^i} + \alpha^{\lambda p^{n+i}}\overline{Z}_{\chi gp^i}\}]$

Proof. To evaluate $P_0(x)$, take $s = 0$ in (3.1), then $\rho_k^0 = \sum_{s\epsilon\Omega_0}\alpha^0 = 1$ for all $0 \le k \le 8p^n - 1$. Therefore, $P_0(x) = \frac{1}{8p^n}[\overline{Z}_0 +$
$\overline{Z}_{p^n} + \overline{Z}_{2p^n} + \overline{Z}_{3p^n} + \overline{Z}_{4p^n} + \overline{Z}_{5p^n} + \overline{Z}_{6p^n} + \overline{Z}_{7p^n} + \sum_{i=0}^{n-1}\{\overline{Z}_{p^i} + \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} + \overline{Z}_{\lambda p^i}$
$+ \overline{Z}_{\mu p^i} + \overline{Z}_{\nu p^i} + \overline{Z}_{\chi p^i} + \overline{Z}_{gp^i} + \overline{Z}_{2gp^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \overline{Z}_{\lambda gp^i} + \overline{Z}_{\mu gp^i} + \overline{Z}_{\nu gp^i} + \overline{Z}_{\chi gp^i}\}]$
For evaluation of $P_{p^n}(x)$, take $s = p^n$ in (3.1), so we have to compute $\rho_k^{p^n}$ for $k = 0, p^n, 2p^n, 3p^n, 4p^n, 5p^n, 6p^n,$
$7p^n, p^i, 2p^i, 4p^i, 8p^i, \lambda p^i, \mu p^i, \nu p^i, \chi p^i, gp^i, 2gp^i, 4gp^i, 8gp^i, \lambda gp^i, \mu gp^i, \nu gp^i, \chi gp^i$.
Here, $\rho_k^{p^n} = \sum_{s\epsilon\Omega_{p^n}}\alpha^{-ks} = \alpha^{-p^n k} = \alpha^{7p^n k}$, using lemma 3.3.
Therefore, $\rho_0^{p^n} = -\rho_{4p^n}^{p^n} = -\rho_{4p^i}^{p^n} = \rho_{8p^i}^{p^n} = -\rho_{4gp^i}^{p^n} = \rho_{8gp^i}^{p^n} = 1,$
$\rho_{p^n}^{p^n} = -\rho_{5p^n}^{p^n} = -\alpha^{3p^{2n}}, \rho_{2p^n}^{p^n} = -\rho_{6p^n}^{p^n} = -\alpha^{2p^{2n}}, \rho_{3p^n}^{p^n} = -\rho_{7p^n}^{p^n} = -\alpha^{p^{2n}},$
$\rho_{p^i}^{p^n} = -\rho_{\nu p^i}^{p^n} = -\rho_{gp^i}^{p^n} = -\rho_{\nu gp^i}^{p^n} = -\alpha^{p^{n+i}},$
$\rho_{2p^i}^{p^n} = -\rho_{\mu p^i}^{p^n} = -\rho_{2gp^i}^{p^n} = -\rho_{\mu gp^i}^{p^n} = -\alpha^{2p^{n+i}},$
$\rho_{\lambda p^i}^{p^n} = -\rho_{\chi p^i}^{p^n} = -\rho_{\lambda gp^i}^{p^n} = -\rho_{\chi gp^i}^{p^n} = -\alpha^{3\lambda p^{n+i}}.$
Hence, $P_{p^n}(x) = \frac{1}{8p^n}[\overline{Z}_0 - \alpha^{3p^{2n}}\overline{Z}_{p^n} - \alpha^{2p^{2n}}\overline{Z}_{2p^n} - \alpha^{p^{2n}}\overline{Z}_{3p^n} - \overline{Z}_{4p^n} + \alpha^{3p^{2n}}\overline{Z}_{5p^n} + \alpha^{2p^{2n}}\overline{Z}_{6p^n} + \alpha^{p^{2n}}\overline{Z}_{7p^n}$
$+ \sum_{i=0}^{n-1}\{-\alpha^{3p^{n+i}}\overline{Z}_{p^i} - \alpha^{2p^{n+i}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \alpha^{3\lambda p^{n+i}}\overline{Z}_{\lambda p^i} + \alpha^{2p^{n+i}}\overline{Z}_{\mu p^i} + \alpha^{3p^{n+i}}\overline{Z}_{\nu p^i} - \alpha^{3\chi p^{n+i}}\overline{Z}_{\chi p^i}$
$+ \alpha^{3p^{n+i}}\overline{Z}_{gp^i} + \alpha^{2p^{n+i}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \alpha^{3\lambda p^{n+i}}\overline{Z}_{\lambda gp^i} - \alpha^{2p^{n+i}}\overline{Z}_{\mu gp^i} - \alpha^{3p^{n+i}}\overline{Z}_{\nu gp^i} - \alpha^{3\lambda p^{n+i}}\overline{Z}_{\chi gp^i}\}]$
Similarly, $P_{2p^n}(x)$,$P_{3p^n}(x)$,$P_{4p^n}(x)$,$P_{5p^n}(x)$,$P_{6p^n}(x)$ and $P_{7p^n}(x)$ can be obtained using lemma 3.3. □

Lemma 3.7. *For* $0 \le i \le n$ *and* $0 \le j \le n-1$*,*

$$\sum_{s\epsilon\Omega_{p^j}} \alpha^{4gp^is} = \sum_{s\epsilon\Omega_{\lambda p^j}} \alpha^{4gp^is} = \sum_{s\epsilon\Omega_{\nu p^j}} \alpha^{4gp^is} = \sum_{s\epsilon\Omega_{\chi p^j}} \alpha^{4gp^is} = \begin{cases} -\frac{\phi(p^{n-j})}{2}, \ if\ i+j \geq n, \\ \frac{1}{p^j}G_{i+j}, \ if\ i+j \leq n-1, \ \ g \neq 1, \\ \frac{1}{p^j}H_{i+j}, \ if\ i+j \leq n-1, \ \ g = 1. \end{cases}$$

$$\sum_{s\epsilon\Omega_{p^j}} \alpha^{8gp^is} = \sum_{s\epsilon\Omega_{\lambda p^j}} \alpha^{8gp^is} = \sum_{s\epsilon\Omega_{\nu p^j}} \alpha^{8gp^is} = \sum_{s\epsilon\Omega_{\chi p^j}} \alpha^{8gp^is} = \begin{cases} \frac{\phi(p^{n-j})}{2}, \ if\ i+j \geq n, \\ \frac{1}{p^j}I_{i+j}, \ if\ i+j \leq n-1, \ \ g \neq 1, \\ \frac{1}{p^j}J_{i+j}, \ if\ i+j \leq n-1, \ \ g = 1. \end{cases}$$

Proof. Here $\sum_{s\varepsilon\Omega_{p^j}} \alpha^{4gp^is} = \sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1} \alpha^{4(1+2p^n)gp^{i+j}q^t} = \sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1} \alpha^{4gp^{i+j}q^t} = \sum_{s\varepsilon\Omega_{p^j}} \alpha^{4gp^is}$

Let $\beta = \alpha^{4p^{i+j}}$. Then, $\sum_{s\varepsilon\Omega_{p^j}} \alpha^{4gp^is} = \sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1} \beta^{gq^t}$.

If $i+j \geq n$, then β is $2nd$ root of unity, therefore $\sum_{s\varepsilon\Omega_{p^j}} \alpha^{4gp^is} = \sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1} \alpha^{4gp^{i+j}q^t} = -\frac{\phi(p^{n-j})}{2}$.

If $i+j \leq n-1$, then β is $2p^{n-i-j}th$ root of unity, then $\beta^{gq^l} \equiv \beta^{gq^r}$ if and only if $gq^l \equiv gq^r (mod\ 2p^{n-i-j})$ if and only if $l \equiv r(mod\ \frac{\phi(p^{n-i-j})}{2})$, therefore $\sum_{s\varepsilon\Omega_{p^j}} \alpha^{4gp^is} = \sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1} \beta^{gq^t} = \frac{p^{i+j}}{p^j} \sum_{t=0}^{\frac{\phi(p^{n-i-j})}{2}-1} \beta^{gq^t} = \frac{1}{p^j}G_{i+j}$.

Similarly, using lemma 3.2, the result holds for other expression. □

Theorem 3.8. *For $p \equiv 1(mod\ 4)$, the expressions for primitive idempotents corresponding to Ω_{4p^j} and Ω_{8p^j} are given by*

$$P_{4p^j}(x) = \tfrac{1}{8p^n}[\tfrac{\phi(p^{n-j})}{2}\{\overline{Z}_0 - \overline{Z}_{p^n} + \overline{Z}_{2p^n} - \overline{Z}_{3p^n} + \overline{Z}_{4p^n} - \overline{Z}_{5p^n} + \overline{Z}_{6p^n} - \overline{Z}_{7p^n}\} + \tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{-\overline{Z}_{p^i}$$
$$+ \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \overline{Z}_{\lambda p^i} + \overline{Z}_{\mu p^i} - \overline{Z}_{\nu p^i} - \overline{Z}_{\chi p^i} - \overline{Z}_{gp^i} + \overline{Z}_{2gp^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} - \overline{Z}_{\lambda gp^i} + \overline{Z}_{\mu gp^i} - \overline{Z}_{\nu gp^i}$$
$$- \overline{Z}_{\chi gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{H_{i+j}\overline{Z}_{p^i} + J_{i+j}\overline{Z}_{2p^i} + J_{i+j}\overline{Z}_{4p^i} + J_{i+j}\overline{Z}_{8p^i} + H_{i+j}\overline{Z}_{\lambda p^i} + J_{i+j}\overline{Z}_{\mu p^i} + H_{i+j}\overline{Z}_{\nu p^i} + H_{i+j}\overline{Z}_{\chi p^i}$$
$$+ G_{i+j}\overline{Z}_{gp^i} + I_{i+j}\overline{Z}_{2gp^i} + I_{i+j}\overline{Z}_{4gp^i} + I_{i+j}\overline{Z}_{8gp^i} + G_{i+j}\overline{Z}_{\lambda gp^i} + I_{i+j}\overline{Z}_{\mu gp^i} + G_{i+j}\overline{Z}_{\nu gp^i} + G_{i+j}\overline{Z}_{\chi gp^i}\}]$$

$$P_{8p^j}(x) = \tfrac{1}{8p^n}[\tfrac{\phi(p^{n-j})}{2}\{\overline{Z}_0 + \overline{Z}_{p^n} + \overline{Z}_{2p^n} + \overline{Z}_{3p^n} + \overline{Z}_{4p^n} + \overline{Z}_{5p^n} + \overline{Z}_{6p^n} + \overline{Z}_{7p^n}\} + \tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{Z}_{p^i}$$
$$+ \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} + \overline{Z}_{\lambda p^i} + \overline{Z}_{\mu p^i} + \overline{Z}_{\nu p^i} + \overline{Z}_{\chi p^i} + \overline{Z}_{gp^i} + \overline{Z}_{2gp^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \overline{Z}_{\lambda gp^i} + \overline{Z}_{\mu gp^i} + \overline{Z}_{\nu gp^i}$$
$$+ \overline{Z}_{\chi gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{J_{i+j}\overline{Z}_{p^i} + J_{i+j}\overline{Z}_{2p^i} + J_{i+j}\overline{Z}_{4p^i} + J_{i+j}\overline{Z}_{8p^i} + J_{i+j}\overline{Z}_{\lambda p^i} + J_{i+j}\overline{Z}_{\mu p^i} + J_{i+j}\overline{Z}_{\nu p^i} + J_{i+j}\overline{Z}_{\chi p^i}$$
$$+ I_{i+j}\overline{Z}_{gp^i} + I_{i+j}\overline{Z}_{2gp^i} + I_{i+j}\overline{Z}_{4gp^i} + I_{i+j}\overline{Z}_{8gp^i} + I_{i+j}\overline{Z}_{\lambda gp^i} + I_{i+j}\overline{Z}_{\mu gp^i} + I_{i+j}\overline{Z}_{\nu gp^i} + I_{i+j}\overline{Z}_{\chi gp^i}\}].$$

where $G_{n-1} = \frac{1}{2}p^{n-1}(\sqrt{p}+1)$, $H_{n-1} = \frac{1}{2}p^{n-1}(\sqrt{p}-1)$, $I_{n-1} = \frac{1}{2}p^{n-1}(\sqrt{p}-1)$, $J_{n-1} = \frac{1}{2}p^{n-1}(-\sqrt{p}-1)$ for $p \equiv 1(mod\ 4)$ and $G_{n-1} = \frac{1}{2}p^{n-1}(1+\sqrt{-p})$, $H_{n-1} = \frac{1}{2}p^{n-1}(1-\sqrt{-p})$, $I_{n-1} = \frac{1}{2}p^{n-1}(\sqrt{-p}-1)$, $J_{n-1} = -\frac{1}{2}p^{n-1}(\sqrt{-p}+1)$ for $p \equiv 3(mod\ 4)$ and for all $j \leq n-2$, $G_j = H_j = I_j = J_j = 0$.

Proof. To evaluate $P_{4p^j}(x)$, take $s = 4p^j$ in (3.1), so we have to compute $\rho_k^{4p^j}$ for $k = 0, p^n, 2p^n, 3p^n, 4p^n,$ $5p^n, 6p^n, 7p^n, p^i, 2p^i, 4p^i, 8p^i, \lambda p^i, \mu p^i, \nu p^i, \chi p^i, gp^i, 2gp^i, 4gp^i, 8gp^i, \lambda gp^i, \mu gp^i, \nu gp^i, \chi gp^i$.

Since in this case $\Omega_{4p^j} = -\Omega_{4p^j}$, using lemma 3.4. So, $\rho_k^{4p^j} = \sum_{s\epsilon\Omega_{4p^j}} \alpha^{-sk} = \sum_{s\epsilon\Omega_{p^j}} \alpha^{4ks}$.

Therefore, using lemma 3.7, we have

$$\rho_0^{4p^j} = -\rho_{p^n}^{4p^j} = \rho_{2p^n}^{4p^j} = -\rho_{3p^n}^{4p^j} = \rho_{4p^n}^{4p^j} = -\rho_{5p^n}^{4p^j} = \rho_{6p^n}^{4p^j} = -\rho_{7p^n}^{4p^j} = \tfrac{\phi(p^{n-j})}{2},$$
$$\rho_{p^i}^{4p^j} = \rho_{\lambda p^i}^{4p^j} = \rho_{\nu p^i}^{4p^j} = \rho_{\chi p^i}^{4p^j} = \begin{cases} -\frac{\phi(p^{n-j})}{2}, \ if\ i+j \geq n, \\ \frac{1}{p^j}H_{i+j}, \ if\ i+j \leq n-1, \end{cases}$$
$$\rho_{2p^i}^{4p^j} = \rho_{4p^i}^{4p^j} = \rho_{8p^i}^{4p^j} = \rho_{\mu p^i}^{4p^j} = \begin{cases} \frac{\phi(p^{n-j})}{2}, \ if\ i+j \geq n, \\ \frac{1}{p^j}J_{i+j}, \ if\ i+j \leq n-1, \end{cases}$$

$$\rho_{gp^i}^{4p^j} = \rho_{\lambda gp^i}^{4p^j} = \rho_{\nu gp^i}^{4p^j} = \rho_{\chi gp^i}^{4p^j} = \begin{cases} -\frac{\phi(p^{n-j})}{2}, \ if \ i+j \geq n, \\ \frac{1}{p^j} G_{i+j}, \ if \ i+j \leq n-1, \end{cases}$$

$$\rho_{2gp^i}^{4p^j} = \rho_{4gp^i}^{4p^j} = \rho_{8gp^i}^{4p^j} = \rho_{\mu gp^i}^{4p^j} = \begin{cases} \frac{\phi(p^{n-j})}{2}, \ if \ i+j \geq n, \\ \frac{1}{p^j} I_{i+j}, \ if \ i+j \leq n-1, \end{cases}$$

So, $P_{4p^j}(x) = \frac{1}{8p^n}[\frac{\phi(p^{n-j})}{2}\{\overline{Z}_0 - \overline{Z}_{p^n} + \overline{Z}_{2p^n} - \overline{Z}_{3p^n} + \overline{Z}_{4p^n} - \overline{Z}_{5p^n} + \overline{Z}_{6p^n} - \overline{Z}_{7p^n}\} + \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{-\overline{Z}_{p^i}$
$+ \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \overline{Z}_{\lambda p^i} + \overline{Z}_{\mu p^i} - \overline{Z}_{\nu p^i} - \overline{Z}_{\chi p^i} - \overline{Z}_{gp^i} + \overline{Z}_{2gp^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} - \overline{Z}_{\lambda gp^i} + \overline{Z}_{\mu gp^i} - \overline{Z}_{\nu gp^i}$
$- \overline{Z}_{\chi gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{H_{i+j}\overline{Z}_{p^i} + J_{i+j}\overline{Z}_{2p^i} + J_{i+j}\overline{Z}_{4p^i} + J_{i+j}\overline{Z}_{8p^i} + H_{i+j}\overline{Z}_{\lambda p^i} + J_{i+j}\overline{Z}_{\mu p^i} + H_{i+j}\overline{Z}_{\nu p^i} + H_{i+j}\overline{Z}_{\chi p^i}$
$+ G_{i+j}\overline{Z}_{gp^i} + I_{i+j}\overline{Z}_{2gp^i} + I_{i+j}\overline{Z}_{4gp^i} + I_{i+j}\overline{Z}_{8gp^i} + G_{i+j}\overline{Z}_{\lambda gp^i} + I_{i+j}\overline{Z}_{\mu gp^i} + G_{i+j}\overline{Z}_{\nu gp^i} + G_{i+j}\overline{Z}_{\chi gp^i}\}]$
Similarly, using lemma 3.7, we can evaluate $P_{8p^j}(x)$. □

We can obtain the expressions for $P_{4gp^j}(x), P_{8gp^j}(x)$ by interchanging G and I by H and J respectively in the expression of $P_{4p^j}(x), P_{8p^j}(x)$ for $p \equiv 1(mod\ 4)$. The expressions for $P_{4p^j}(x)$,$P_{8p^j}(x)$, $P_{4gp^j}(x)$ and $P_{8gp^j}(x)$ above also represents $P_{4gp^j}(x)$,$P_{8gp^j}(x)$, $P_{4p^j}(x)$ and $P_{8p^j}(x)$ respectively in case when $p \equiv 3(mod\ 4)$.

Lemma 3.9. *For $0 \leq i \leq n$ and $0 \leq j \leq n-1$,*

$$\sum_{s\epsilon\Omega_{p^j}} \alpha^{2gp^is} = \sum_{s\epsilon\Omega_{\nu p^j}} \alpha^{2gp^is} = \sum_{s\epsilon\Omega_{\lambda p^j}} \alpha^{\mu gp^is} = \begin{cases} -\frac{\phi(p^{n-j})}{2}\alpha^{2p^{i+j}}, \ if \ i+j \geq n, \\ \frac{1}{p^j}E_{i+j}, \ if \ i+j \leq n-1. \end{cases}$$

$$\sum_{s\epsilon\Omega_{p^j}} \alpha^{2p^is} = \sum_{s\epsilon\Omega_{\nu p^j}} \alpha^{2p^is} = \sum_{s\epsilon\Omega_{\lambda p^j}} \alpha^{\mu p^is} = \begin{cases} \frac{\phi(p^{n-j})}{2}\alpha^{2p^{i+j}}, \ if \ i+j \geq n, \\ \frac{1}{p^j}F_{i+j}, \ if \ i+j \leq n-1. \end{cases}$$

Proof. Proof can be obtained on similar lines as that of lemma 3.7 and using lemma 3.4. □

Theorem 3.10. *For $p \equiv 1(mod\ 4)$, the expressions for primitive idempotents corresponding to Ω_{2p^j} and $\Omega_{\mu p^j}$ are given by*

$P_{2p^j}(x) = \frac{1}{8p^n}[\frac{\phi(p^{n-j})}{2}\{\overline{Z}_0 - \alpha^{2p^{n+j}}\overline{Z}_{p^n} - \overline{Z}_{2p^n} + \alpha^{2p^{n+j}}\overline{Z}_{3p^n} + \overline{Z}_{4p^n} - \alpha^{2p^{n+j}}\overline{Z}_{5p^n} - \overline{Z}_{6p^n} + \alpha^{2p^{n+j}}\overline{Z}_{7p^n}\}$
$+ \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{-\alpha^{2p^{i+j}}\overline{Z}_{p^i} - \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} + \alpha^{2p^{i+j}}\overline{Z}_{\lambda p^i} - \overline{Z}_{\mu p^i} - \alpha^{2p^{i+j}}\overline{Z}_{\nu p^i} + \alpha^{2p^{i+j}}\overline{Z}_{\chi p^i} + \alpha^{2p^{i+j}}\overline{Z}_{gp^i}$
$+ \overline{Z}_{2gp^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} - \alpha^{2p^{i+j}}\overline{Z}_{\lambda gp^i} - \overline{Z}_{\mu gp^i} + \alpha^{2p^{i+j}}\overline{Z}_{\nu gp^i} - \alpha^{2p^{i+j}}\overline{Z}_{\chi gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{-F_{i+j}\overline{Z}_{p^i} + H_{i+j}\overline{Z}_{2p^i}$
$+ J_{i+j}\overline{Z}_{4p^i} + J_{i+j}\overline{Z}_{8p^i} + F_{i+j}\overline{Z}_{\lambda p^i} + H_{i+j}\overline{Z}_{\mu p^i} - F_{i+j}\overline{Z}_{\nu p^i} + F_{i+j}\overline{Z}_{\chi p^i} - E_{i+j}\overline{Z}_{gp^i} + G_{i+j}\overline{Z}_{2gp^i} + I_{i+j}\overline{Z}_{4gp^i}$
$+ I_{i+j}\overline{Z}_{8gp^i} + E_{i+j}\overline{Z}_{\lambda gp^i} + G_{i+j}\overline{Z}_{\mu gp^i} - E_{i+j}\overline{Z}_{\nu gp^i} + E_{i+j}\overline{Z}_{\chi gp^i}\}]$

$P_{\mu p^j}(x) = \frac{1}{8p^n}[\frac{\phi(p^{n-j})}{2}\{\overline{Z}_0 + \alpha^{2p^{n+j}}\overline{Z}_{p^n} - \overline{Z}_{2p^n} - \alpha^{2p^{n+j}}\overline{Z}_{3p^n} + \overline{Z}_{4p^n} + \alpha^{2p^{n+j}}\overline{Z}_{5p^n} - \overline{Z}_{6p^n} - \alpha^{2p^{n+j}}\overline{Z}_{7p^n}\}$
$+ \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\alpha^{2p^{i+j}}\overline{Z}_{p^i} - \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \alpha^{2p^{i+j}}\overline{Z}_{\lambda p^i} - \overline{Z}_{\mu p^i} + \alpha^{2p^{i+j}}\overline{Z}_{\nu p^i} - \alpha^{2p^{i+j}}\overline{Z}_{\chi p^i} - \alpha^{2p^{i+j}}\overline{Z}_{gp^i}$
$- \overline{Z}_{2gp^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \alpha^{2p^{i+j}}\overline{Z}_{\lambda gp^i} - \overline{Z}_{\mu gp^i} - \alpha^{2p^{i+j}}\overline{Z}_{\nu gp^i} + \alpha^{2p^{i+j}}\overline{Z}_{\chi gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{F_{i+j}\overline{Z}_{p^i} + H_{i+j}\overline{Z}_{2p^i}$
$+ J_{i+j}\overline{Z}_{4p^i} + J_{i+j}\overline{Z}_{8p^i} - F_{i+j}\overline{Z}_{\lambda p^i} + H_{i+j}\overline{Z}_{\mu p^i} + F_{i+j}\overline{Z}_{\nu p^i} - F_{i+j}\overline{Z}_{\chi p^i} + E_{i+j}\overline{Z}_{gp^i} + G_{i+j}\overline{Z}_{2gp^i} + I_{i+j}\overline{Z}_{4gp^i}$
$+ I_{i+j}\overline{Z}_{8gp^i} + E_{i+j}\overline{Z}_{\lambda gp^i} + G_{i+j}\overline{Z}_{\mu gp^i} + E_{i+j}\overline{Z}_{\nu gp^i} - E_{i+j}\overline{Z}_{\chi gp^i}\}]$.
where $E_{n-1} = \frac{1}{2}\sqrt{-p^{2n-1} - 2p^{2(n-1)}} + \frac{1}{2}p^{n-1}$, $F_{n-1} = -\frac{1}{2}\sqrt{-p^{2n-1} - 2p^{2(n-1)}} + \frac{1}{2}p^{n-1}$ *for* $p \equiv 1(mod\ 4)$ *and* $E_{n-1} = \frac{1}{2}\sqrt{3p^{2n-1}} + \frac{1}{2}p^{n-1}$, $F_{n-1} = -\frac{1}{2}\sqrt{3p^{2n-1}} + \frac{1}{2}p^{n-1}$ *for* $p \equiv 3(mod\ 4)$ *and for all* $j \leq n-2$, $E_j = F_j = 0$.
Proof. Proof can be obtained on similar lines as that of theorem 3.8 using lemmas 3.7 and 3.9. □

We can obtain the expressions for $P_{\mu gp^j}(x), P_{2gp^j}(x)$ by interchanging E and G by $-F$ and H respectively in the expression of $P_{2p^j}(x), P_{\mu p^j}(x)$ for $p \equiv 1(mod\ 4)$. The expressions for $P_{2p^j}(x)$,$P_{\mu p^j}(x)$, $P_{2gp^j}(x)$ and $P_{\mu gp^j}(x)$ above also represents $P_{2gp^j}(x)$,$P_{\mu gp^j}(x)$, $P_{2p^j}(x)$ and $P_{\mu p^j}(x)$ respectively in case when $p \equiv 3(mod\ 4)$.

Lemma 3.11. *For* $0 \leq i \leq n$ *and* $0 \leq j \leq n-1$,

$$\sum_{s\epsilon\Omega_{p^j}} \alpha^{gp^i s} = \sum_{s\epsilon\Omega_{\lambda p^j}} \alpha^{\lambda gp^i s} = \sum_{s\epsilon\Omega_{\nu p^j}} \alpha^{\nu gp^i s} = \begin{cases} -\frac{\phi(p^{n-j})}{2}\alpha^{p^{i+j}}, \; if\ i+j \geq n, \\ \frac{1}{p^j}A_{i+j}\,, \quad if\ i+j \leq n-1. \end{cases}$$

$$\sum_{s\epsilon\Omega_{p^j}} \alpha^{\lambda gp^i s} = \sum_{s\epsilon\Omega_{\chi p^j}} \alpha^{\nu gp^i s} = -\sum_{s\epsilon\Omega_{\chi p^j}} \alpha^{gp^i s} = \begin{cases} -\frac{\phi(p^{n-j})}{2}\alpha^{\lambda p^{i+j}}, if\ i+j \geq n, \\ \frac{1}{p^j}B_{i+j}\,, \quad if\ i+j \leq n-1. \end{cases}$$

$$\sum_{s\epsilon\Omega_{p^j}} \alpha^{p^i s} = \sum_{s\epsilon\Omega_{\lambda p^j}} \alpha^{\lambda p^i s} = \sum_{s\epsilon\Omega_{\nu p^j}} \alpha^{\nu p^i s} = \begin{cases} \frac{\phi(p^{n-j})}{2}\alpha^{p^{i+j}}, \; if\ i+j \geq n, \\ \frac{1}{p^j}C_{i+j}\,, if\ i+j \leq n-1. \end{cases}$$

$$\sum_{s\epsilon\Omega_{p^j}} \alpha^{\lambda p^i s} = \sum_{s\epsilon\Omega_{\chi p^j}} \alpha^{\nu p^i s} = -\sum_{s\epsilon\Omega_{\lambda p^j}} \alpha^{\nu p^i s} = \begin{cases} \frac{\phi(p^{n-j})}{2}\alpha^{\lambda p^{i+j}}, \; if\ i+j \geq n, \\ \frac{1}{p^j}D_{i+j}\,, \quad if\ i+j \leq n-1. \end{cases}$$

Proof. Proof can be obtained on similar lines as that of lemma 3.7 and using lemma 3.4. □

Theorem 3.12. *For* $p \equiv 1(mod\ 4)$, *the expressions for primitive idempotents corresponding to* Ω_{p^j}, $\Omega_{\lambda p^j}$, Ω_{gp^j} *and* $\Omega_{\lambda gp^j}$ *are given by*

$$P_{p^j}(x) = \tfrac{1}{8p^n}\big[\tfrac{\phi(p^{n-j})}{2}\{\overline{Z}_0 - \alpha^{\lambda p^{n+j}}\overline{Z}_{p^n} - \alpha^{2p^{n+j}}\overline{Z}_{2p^n} - \alpha^{3\lambda p^{n+j}}\overline{Z}_{3p^n} - \overline{Z}_{4p^n} + \alpha^{\lambda p^{n+j}}\overline{Z}_{5p^n} + \alpha^{2p^{n+j}}\overline{Z}_{6p^n}$$
$$+ \alpha^{3\lambda p^{n+j}}\overline{Z}_{7p^n}\} + \tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{-\alpha^{\lambda p^{i+j}}\overline{Z}_{p^i} - \alpha^{2p^{i+j}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \alpha^{p^{i+j}}\overline{Z}_{\lambda p^i} + \alpha^{2p^{i+j}}\overline{Z}_{\mu p^i} + \alpha^{\lambda p^{i+j}}\overline{Z}_{\nu p^i}$$
$$+ \alpha^{p^{i+j}}\overline{Z}_{\chi p^i} + \alpha^{\lambda p^{i+j}}\overline{Z}_{gp^i} + \alpha^{2p^{i+j}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \alpha^{p^{i+j}}\overline{Z}_{\lambda gp^i} - \alpha^{2p^{i+j}}\overline{Z}_{\mu gp^i} - \alpha^{\lambda p^{i+j}}\overline{Z}_{\nu gp^i} - \alpha^{p^{i+j}}\overline{Z}_{\chi gp^i}\}$$
$$+ \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{-D_{i+j}\overline{Z}_{p^i} - F_{i+j}\overline{Z}_{2p^i} + H_{i+j}\overline{Z}_{4p^i} + J_{i+j}\overline{Z}_{8p^i} - C_{i+j}\overline{Z}_{\lambda p^i} + F_{i+j}\overline{Z}_{\mu p^i} + D_{i+j}\overline{Z}_{\nu p^i} + C_{i+j}\overline{Z}_{\chi p^i}$$
$$- B_{i+j}\overline{Z}_{gp^i} - E_{i+j}\overline{Z}_{2gp^i} + G_{i+j}\overline{Z}_{4gp^i} + I_{i+j}\overline{Z}_{8gp^i} - A_{i+j}\overline{Z}_{\lambda gp^i} + E_{i+j}\overline{Z}_{\mu gp^i} + B_{i+j}\overline{Z}_{\nu gp^i} + A_{i+j}\overline{Z}_{\chi gp^i}\}\big]$$

$$P_{\lambda p^j}(x) = \tfrac{1}{8p^n}\big[\tfrac{\phi(p^{n-j})}{2}\{\overline{Z}_0 - \alpha^{p^{n+j}}\overline{Z}_{p^n} + \alpha^{2p^{n+j}}\overline{Z}_{2p^n} - \alpha^{3p^{n+j}}\overline{Z}_{3p^n} - \overline{Z}_{4p^n} + \alpha^{p^{n+j}}\overline{Z}_{5p^n} - \alpha^{2p^{n+j}}\overline{Z}_{6p^n}$$
$$+ \alpha^{3p^{n+j}}\overline{Z}_{7p^n}\} + \tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{-\alpha^{p^{i+j}}\overline{Z}_{p^i} + \alpha^{2p^{i+j}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \alpha^{\lambda p^{i+j}}\overline{Z}_{\lambda p^i} - \alpha^{2p^{i+j}}\overline{Z}_{\mu p^i} + \alpha^{p^{i+j}}\overline{Z}_{\nu p^i}$$
$$+ \alpha^{\lambda p^{i+j}}\overline{Z}_{\chi p^i} + \alpha^{p^{i+j}}\overline{Z}_{gp^i} - \alpha^{2p^{i+j}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \alpha^{\lambda p^{i+j}}\overline{Z}_{\lambda gp^i} + \alpha^{2p^{i+j}}\overline{Z}_{\mu gp^i} - \alpha^{p^{i+j}}\overline{Z}_{\nu gp^i}$$
$$- \alpha^{\lambda p^{i+j}}\overline{Z}_{\chi gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{-C_{i+j}\overline{Z}_{p^i} + F_{i+j}\overline{Z}_{2p^i} + H_{i+j}\overline{Z}_{4p^i} + J_{i+j}\overline{Z}_{8p^i} - D_{i+j}\overline{Z}_{\lambda p^i} - F_{i+j}\overline{Z}_{\mu p^i} + C_{i+j}\overline{Z}_{\nu p^i}$$
$$+ D_{i+j}\overline{Z}_{\chi p^i} - A_{i+j}\overline{Z}_{gp^i} + E_{i+j}\overline{Z}_{2gp^i} + G_{i+j}\overline{Z}_{4gp^i} + I_{i+j}\overline{Z}_{8gp^i} - B_{i+j}\overline{Z}_{\lambda gp^i} - E_{i+j}\overline{Z}_{\mu gp^i} + A_{i+j}\overline{Z}_{\nu gp^i}$$
$$+ B_{i+j}\overline{Z}_{\chi gp^i}\}\big]$$

$$P_{gp^j}(x) = \tfrac{1}{8p^n}\big[\tfrac{\phi(p^{n-j})}{2}\{\overline{Z}_0 + \alpha^{\lambda p^{i+j}}\overline{Z}_{p^n} + \alpha^{2p^{n+j}}\overline{Z}_{2p^n} + \alpha^{3\lambda p^{n+j}}\overline{Z}_{3p^n} - \overline{Z}_{4p^n} - \alpha^{\lambda p^{n+j}}\overline{Z}_{5p^n} - \alpha^{2p^{n+j}}\overline{Z}_{6p^n}$$
$$- \alpha^{3\lambda p^{i+j}}\overline{Z}_{7p^n}\} + \tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\alpha^{\lambda p^{i+j}}\overline{Z}_{p^i} + \alpha^{2p^{i+j}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} + \alpha^{p^{i+j}}\overline{Z}_{\lambda p^i} - \alpha^{2p^{i+j}}\overline{Z}_{\mu p^i} - \alpha^{\lambda p^{i+j}}\overline{Z}_{\nu p^i}$$
$$- \alpha^{p^{i+j}}\overline{Z}_{\chi p^i} - \alpha^{\lambda p^{i+j}}\overline{Z}_{gp^i} - \alpha^{2p^{i+j}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} - \alpha^{p^{i+j}}\overline{Z}_{\lambda gp^i} + \alpha^{2p^{i+j}}\overline{Z}_{\mu gp^i} + \alpha^{\lambda p^{i+j}}\overline{Z}_{\nu gp^i} + \alpha^{p^{i+j}}\overline{Z}_{\chi gp^i}\}$$
$$+ \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{-B_{i+j}\overline{Z}_{p^i} - E_{i+j}\overline{Z}_{2p^i} + G_{i+j}\overline{Z}_{4p^i} + I_{i+j}\overline{Z}_{8p^i} - A_{i+j}\overline{Z}_{\lambda p^i} + E_{i+j}\overline{Z}_{\mu p^i} + B_{i+j}\overline{Z}_{\nu p^i} + A_{i+j}\overline{Z}_{\chi p^i}$$
$$- D_{i+j}\overline{Z}_{gp^i} - F_{i+j}\overline{Z}_{2gp^i} + H_{i+j}\overline{Z}_{4gp^i} + J_{i+j}\overline{Z}_{8gp^i} - C_{i+j}\overline{Z}_{\lambda gp^i} + F_{i+j}\overline{Z}_{\mu gp^i} + D_{i+j}\overline{Z}_{\nu gp^i} + C_{i+j}\overline{Z}_{\chi gp^i}\}\big]$$

$$P_{\lambda gp^j}(x) = \tfrac{1}{8p^n}\big[\tfrac{\phi(p^{n-j})}{2}\{\overline{Z}_0 + \alpha^{p^{n+j}}\overline{Z}_{p^n} - \alpha^{2p^{n+j}}\overline{Z}_{2p^n} + \alpha^{3p^{n+j}}\overline{Z}_{3p^n} - \overline{Z}_{4p^n} - \alpha^{p^{n+j}}\overline{Z}_{5p^n} + \alpha^{2p^{n+j}}\overline{Z}_{6p^n}$$
$$- \alpha^{3p^{n+j}}\overline{Z}_{7p^n}\} + \tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\alpha^{p^{i+j}}\overline{Z}_{p^i} - \alpha^{2p^{i+j}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} + \alpha^{\lambda p^{i+j}}\overline{Z}_{\lambda p^i} + \alpha^{2p^{i+j}}\overline{Z}_{\mu p^i} - \alpha^{p^{i+j}}\overline{Z}_{\nu p^i}$$
$$- \alpha^{\lambda p^{i+j}}\overline{Z}_{\chi p^i} - \alpha^{p^{i+j}}\overline{Z}_{gp^i} + \alpha^{2p^{i+j}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} - \alpha^{\lambda p^{i+j}}\overline{Z}_{\lambda gp^i} - \alpha^{2p^{i+j}}\overline{Z}_{\mu gp^i} + \alpha^{p^{i+j}}\overline{Z}_{\nu gp^i}$$
$$+ \alpha^{\lambda p^{i+j}}\overline{Z}_{\chi gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{-A_{i+j}\overline{Z}_{p^i} + E_{i+j}\overline{Z}_{2p^i} + G_{i+j}\overline{Z}_{4p^i} + I_{i+j}\overline{Z}_{8p^i} - B_{i+j}\overline{Z}_{\lambda p^i} - E_{i+j}\overline{Z}_{\mu p^i} + A_{i+j}\overline{Z}_{\nu p^i}$$

$+ B_{i+j}\overline{Z}_{\chi p^i} - C_{i+j}\overline{Z}_{gp^i} + F_{i+j}\overline{Z}_{2gp^i} + H_{i+j}\overline{Z}_{4gp^i} + J_{i+j}\overline{Z}_{8gp^i} - D_{i+j}\overline{Z}_{\lambda gp^i} - F_{i+j}\overline{Z}_{\mu gp^i} + C_{i+j}\overline{Z}_{\nu gp^i}$
$+ D_{i+j}\overline{Z}_{\chi gp^i}\}]$

where A_{i+j}, B_{i+j}, C_{i+j} and D_{i+j} are obtained by the following relations:

$$A_{n-1}B_{n-1} + C_{n-1}D_{n-1} = -\tfrac{1}{2}p^{(2n-1)} - \tfrac{1}{2}p^{2(n-1)}$$
$$A_{n-1}D_{n-1} + B_{n-1}C_{n-1} = -p^{(2n-1)}$$
$$A^2_{n-1} + B^2_{n-1} + C^2_{n-1} + D^2_{n-1} = 0$$
$$A_{n-1}C_{n-1} + B_{n-1}D_{n-1} = \tfrac{1}{2}p^{(2n-1)} + \tfrac{1}{2}p^{2(n-1)} \text{ for } p \equiv 1(mod\, 4)$$

and
$$A_{n-1}B_{n-1} + C_{n-1}D_{n-1} = -p^{(2n-1)}$$
$$A_{n-1}D_{n-1} + B_{n-1}C_{n-1} = -\tfrac{1}{2}p^{(2n-1)} - \tfrac{1}{2}p^{2(n-1)}$$
$$A^2_{n-1} + B^2_{n-1} + C^2_{n-1} + D^2_{n-1} = \tfrac{1}{2}p^{(2n-1)} + \tfrac{1}{2}p^{2(n-1)}$$
$$A_{n-1}C_{n-1} + B_{n-1}D_{n-1} = 0 \text{ for } p \equiv 3(mod\, 4), n \text{ is odd}$$

and
$$A_{n-1}B_{n-1} + C_{n-1}D_{n-1} = -2p^{(2n-1)} + p^{2(n-1)}$$
$$A_{n-1}D_{n-1} + B_{n-1}C_{n-1} = \tfrac{3}{2}p^{(2n-1)} - \tfrac{1}{2}p^{2(n-1)}$$
$$A^2_{n-1} + B^2_{n-1} + C^2_{n-1} + D^2_{n-1} = -p^{(2n-1)} - p^{2(n-1)}$$
$$A_{n-1}C_{n-1} + B_{n-1}D_{n-1} = 0 \text{ for } p \equiv 3(mod\, 4), n \text{ is even}$$

and for all $j \leq n-2$, $A_j = B_j = C_j = D_j = 0$.

Proof. Proof can be obtained on similar lines as that of theorem 3.8 and using lemmas 3.7, 3.9 and 3.11. □

The expressions for $P_{p^j}(x)$, $P_{\lambda p^j}(x)$, $P_{gp^j}(x)$ and $P_{\lambda gp^j}(x)$ above also represent $P_{gp^j}(x)$, $P_{\lambda gp^j}(x)$, $P_{p^j}(x)$ and $P_{\lambda p^j}(x)$ respectively in case when $p \equiv 3(mod\, 4)$, n is odd and $P_{\nu gp^j}(x)$, $P_{\chi gp^j}(x)$, $P_{\nu p^j}(x)$ and $P_{\chi p^j}(x)$ respectively in case when $p \equiv 3(mod\, 4)$, n is even.

Theorem 3.13. *The expressions for primitive idempotents corresponding to $\Omega_{\nu p^j}$, $\Omega_{\chi p^j}$, $\Omega_{\nu gp^j}$ and $\Omega_{\chi gp^j}$ are given by*

$$P_{\nu p^j}(x) = \tfrac{1}{8p^n}[\tfrac{\phi(p^{n-j})}{2}\{\overline{Z}_0 + \alpha^{\lambda p^{n+j}}\overline{Z}_{p^n} - \alpha^{2p^{n+j}}\overline{Z}_{2p^n} + \alpha^{3\lambda p^{n+j}}\overline{Z}_{3p^n} - \overline{Z}_{4p^n} - \alpha^{\lambda p^{n+j}}\overline{Z}_{5p^n} + \alpha^{2p^{n+j}}\overline{Z}_{6p^n}$$
$$- \alpha^{3\lambda p^{n+j}}\overline{Z}_{7p^n}\} + \tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\alpha^{\lambda p^{i+j}}\overline{Z}_{p^i} - \alpha^{2p^{i+j}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} + \alpha^{p^{i+j}}\overline{Z}_{\lambda p^i} + \alpha^{2p^{i+j}}\overline{Z}_{\mu p^i} - \alpha^{\lambda p^{i+j}}\overline{Z}_{\nu p^i}$$
$$- \alpha^{p^{i+j}}\overline{Z}_{\chi p^i} - \alpha^{\lambda p^{i+j}}\overline{Z}_{gp^i} + \alpha^{2p^{i+j}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} - \alpha^{p^{i+j}}\overline{Z}_{\lambda gp^i} - \alpha^{2p^{i+j}}\overline{Z}_{\mu gp^i} + \alpha^{\lambda p^{i+j}}\overline{Z}_{\nu gp^i} + \alpha^{p^{i+j}}\overline{Z}_{\chi gp^i}\}$$
$$+ \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{D_{i+j}\overline{Z}_{p^i} - F_{i+j}\overline{Z}_{2p^i} + H_{i+j}\overline{Z}_{4p^i} + J_{i+j}\overline{Z}_{8p^i} + C_{i+j}\overline{Z}_{\lambda p^i} + F_{i+j}\overline{Z}_{\mu p^i} - D_{i+j}\overline{Z}_{\nu p^i} - C_{i+j}\overline{Z}_{\chi p^i}$$
$$+ B_{i+j}\overline{Z}_{gp^i} - E_{i+j}\overline{Z}_{2gp^i} + G_{i+j}\overline{Z}_{4gp^i} + I_{i+j}\overline{Z}_{8gp^i} + A_{i+j}\overline{Z}_{\lambda gp^i} + E_{i+j}\overline{Z}_{\mu gp^i} - B_{i+j}\overline{Z}_{\nu gp^i} - A_{i+j}\overline{Z}_{\chi gp^i}\}]$$

$$P_{\chi p^j}(x) = \tfrac{1}{8p^n}[\tfrac{\phi(p^{n-j})}{2}\{\overline{Z}_0 + \alpha^{p^{n+j}}\overline{Z}_{p^n} + \alpha^{2p^{n+j}}\overline{Z}_{2p^n} + \alpha^{3p^{n+j}}\overline{Z}_{3p^n} - \overline{Z}_{4p^n} - \alpha^{p^{n+j}}\overline{Z}_{5p^n} - \alpha^{2p^{n+j}}\overline{Z}_{6p^n}$$
$$- \alpha^{3p^{n+j}}\overline{Z}_{7p^n}\} + \tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\alpha^{p^{i+j}}\overline{Z}_{p^i} + \alpha^{2p^{i+j}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} + \alpha^{\lambda p^{i+j}}\overline{Z}_{\lambda p^i} - \alpha^{2p^{i+j}}\overline{Z}_{\mu p^i} - \alpha^{p^{i+j}}\overline{Z}_{\nu p^i}$$
$$- \alpha^{\lambda p^{i+j}}\overline{Z}_{\chi p^i} + \alpha^{gp^{i+j}}\overline{Z}_{gp^i} + \alpha^{2gp^{i+j}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \alpha^{\lambda gp^{i+j}}\overline{Z}_{\lambda gp^i} - \alpha^{2gp^{i+j}}\overline{Z}_{\mu gp^i} - \alpha^{gp^{i+j}}\overline{Z}_{\nu gp^i}$$
$$- \alpha^{\lambda gp^{i+j}}\overline{Z}_{\chi gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{C_{i+j}\overline{Z}_{p^i} + F_{i+j}\overline{Z}_{2p^i} + H_{i+j}\overline{Z}_{4p^i} + J_{i+j}\overline{Z}_{8p^i} + D_{i+j}\overline{Z}_{\lambda p^i} - F_{i+j}\overline{Z}_{\mu p^i} - C_{i+j}\overline{Z}_{\nu p^i}$$
$$- D_{i+j}\overline{Z}_{\chi p^i} + A_{i+j}\overline{Z}_{gp^i} + E_{i+j}\overline{Z}_{2gp^i} + G_{i+j}\overline{Z}_{4gp^i} + I_{i+j}\overline{Z}_{8gp^i} + B_{i+j}\overline{Z}_{\lambda gp^i} - E_{i+j}\overline{Z}_{\mu gp^i} - A_{i+j}\overline{Z}_{\nu gp^i}$$
$$- B_{i+j}\overline{Z}_{\chi gp^i}\}]$$

$$P_{\nu gp^j}(x) = \tfrac{1}{8p^n}[\tfrac{\phi(p^{n-j})}{2}\{\overline{Z}_0 - \alpha^{\lambda p^{n+j}}\overline{Z}_{p^n} + \alpha^{2p^{n+j}}\overline{Z}_{2p^n} + \alpha^{3\lambda p^{n+j}}\overline{Z}_{3p^n} - \overline{Z}_{4p^n} + \alpha^{\lambda p^{n+j}}\overline{Z}_{5p^n} - \alpha^{2p^{n+j}}\overline{Z}_{6p^n}$$
$$- \alpha^{3\lambda p^{n+j}}\overline{Z}_{7p^n}\} + \tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{-\alpha^{\lambda p^{i+j}}\overline{Z}_{p^i} + \alpha^{2p^{i+j}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \alpha^{p^{i+j}}\overline{Z}_{\lambda p^i} - \alpha^{2p^{i+j}}\overline{Z}_{\mu p^i} + \alpha^{\lambda p^{i+j}}\overline{Z}_{\nu p^i}$$
$$+ \alpha^{p^{i+j}}\overline{Z}_{\chi p^i} + \alpha^{\lambda p^{i+j}}\overline{Z}_{gp^i} - \alpha^{2p^{i+j}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \alpha^{p^{i+j}}\overline{Z}_{\lambda gp^i} + \alpha^{2p^{i+j}}\overline{Z}_{\mu gp^i} - \alpha^{\lambda p^{i+j}}\overline{Z}_{\nu gp^i} - \alpha^{p^{i+j}}\overline{Z}_{\chi gp^i}\}$$
$$+ \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{B_{i+j}\overline{Z}_{p^i} - E_{i+j}\overline{Z}_{2p^i} + G_{i+j}\overline{Z}_{4p^i} + I_{i+j}\overline{Z}_{8p^i} + A_{i+j}\overline{Z}_{\lambda p^i} + E_{i+j}\overline{Z}_{\mu p^i} - B_{i+j}\overline{Z}_{\nu p^i} - A_{i+j}\overline{Z}_{\chi p^i}$$

$+ D_{i+j}\overline{Z}_{gp^i} - F_{i+j}\overline{Z}_{2gp^i} + H_{i+j}\overline{Z}_{4gp^i} + J_{i+j}\overline{Z}_{8gp^i} + C_{i+j}\overline{Z}_{\lambda gp^i} + F_{i+j}\overline{Z}_{\mu gp^i} - D_{i+j}\overline{Z}_{\nu gp^i} - C_{i+j}\overline{Z}_{\chi gp^i}\}]$

$$P_{\chi gp^j}(x) = \frac{1}{8p^n}[\frac{\phi(p^{n-j})}{2}\{\overline{Z}_0 - \alpha^{p^{n+j}}\overline{Z}_{p^n} - \alpha^{2p^{n+j}}\overline{Z}_{2p^n} - \alpha^{3p^{n+j}}\overline{Z}_{3p^n} - \overline{Z}_{4p^n} + \alpha^{p^{n+j}}\overline{Z}_{5p^n} + \alpha^{2p^{n+j}}\overline{Z}_{6p^n}$$
$$+ \alpha^{3p^{n+j}}\overline{Z}_{7p^n}\} + \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{-\alpha^{p^{i+j}}\overline{Z}_{p^i} - \alpha^{2p^{i+j}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \alpha^{\lambda p^{i+j}}\overline{Z}_{\lambda p^i} + \alpha^{2p^{i+j}}\overline{Z}_{\mu p^i} + \alpha^{p^{i+j}}\overline{Z}_{\nu p^i}$$
$$+ \alpha^{\lambda p^{i+j}}\overline{Z}_{\chi p^i} + \alpha^{p^{i+j}}\overline{Z}_{gp^i} + \alpha^{2p^{i+j}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \alpha^{\lambda p^{i+j}}\overline{Z}_{\lambda gp^i} - \alpha^{2p^{i+j}}\overline{Z}_{\mu gp^i} - \alpha^{p^{i+j}}\overline{Z}_{\nu gp^i}$$
$$- \alpha^{\lambda p^{i+j}}\overline{Z}_{\chi gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{A_{i+j}\overline{Z}_{p^i} + E_{i+j}\overline{Z}_{2p^i} + G_{i+j}\overline{Z}_{4p^i} + I_{i+j}\overline{Z}_{8p^i} + B_{i+j}\overline{Z}_{\lambda p^i} - E_{i+j}\overline{Z}_{\mu p^i} - A_{i+j}\overline{Z}_{\nu p^i}$$
$$- B_{i+j}\overline{Z}_{\chi p^i} + C_{i+j}\overline{Z}_{gp^i} + F_{i+j}\overline{Z}_{2gp^i} + H_{i+j}\overline{Z}_{4gp^i} + J_{i+j}\overline{Z}_{8gp^i} + D_{i+j}\overline{Z}_{\lambda gp^i} - F_{i+j}\overline{Z}_{\mu gp^i} - C_{i+j}\overline{Z}_{\nu gp^i}$$
$$- D_{i+j}\overline{Z}_{\chi gp^i}\}]$$

Proof. Proof can be obtained on similar lines as theorem 3.8 and using lemmas 3.7, 3.9 and 3.11. □

The expressions for $P_{\nu p^j}(x)$, $P_{\chi p^j}(x)$, $P_{\nu gp^j}(x)$ and $P_{\chi gp^j}(x)$ above also represent $P_{\nu gp^j}(x)$, $P_{\chi gp^j}(x)$, $P_{\nu p^j}(x)$ and $P_{\chi p^j}(x)$ respectively in case when $p \equiv 3(mod\ 4)$, n is odd and $P_{gp^j}(x)$, $P_{\lambda gp^j}(x)$, $P_{p^j}(x)$ and $P_{\lambda p^j}(x)$ respectively in case when $p \equiv 3(mod\ 4)$, n is even.

4. DIMENSION AND GENERATING POLYNOMIALS

If α is primitive $8p^n th$ root of unity in some extension field of F, then $m_s(x) = \prod_{s\epsilon\Omega_s}(x - \alpha^s)$ denote the minimal polynomial for α^s. Then the generating polynomial for cyclic code M_s of length $8p^n$ corresponding to the cyclotomic coset Ω_s is $\frac{x^{8p^n}-1}{m_s(x)}$ and the dimension of minimal cyclic code M_s is equal to the cardinality of the class Ω_s [5].

Theorem 4.1. (i) *The generating polynomial for the codes* $M_0, M_{p^n}, M_{2p^n}, M_{3p^n}, M_{4p^n}, M_{5p^n}, M_{6p^n}$ *and* M_{7p^n} *are* $(1 + x + x^2 + ... + x^{8p^n-1})$, $(x^4 - 1)(x^2 + \beta)(x + \delta)(1 + x^8 + ... + x^{8(p^n-1)})$, $(x^6 - x^4 + x^2 - 1)(x + \beta)(1 + x^8 + ... + x^{8(p^n-1)})$, $(x^4-1)(x^2+\beta)(x+\delta_1)(1+x^8+...+x^{8(p^n-1)})$, $(x^7-x^6+x^5-x^4+x^3-x^2+x-1)(1+x^8+...+x^{8(p^n-1)})$, $(x^4-1)(x^2+\beta)(x-\delta)(1+x^8+...+x^{8(p^n-1)})$, $(x^4-1)(x^2+\beta)(x-\beta)(1+x^8+...+x^{8(p^n-1)})$, *and* $(x^4-1)(x^2+\beta)(x-\delta_1)(1+x^8+...+x^{8(p^n-1)})$ *respectively, where* β *is* $4th$ *and* δ, δ_1 *are* $8th$ *root of unity.*

(ii) *The generating polynomial for* $M_{4p^i} \oplus M_{4gp^i}$, $M_{8p^i} \oplus M_{8gp^i}$ *and* $M_{p^i} \oplus M_{2p^i} \oplus M_{\lambda p^i} \oplus M_{\mu p^i} \oplus M_{\nu p^i} \oplus M_{\chi p^i} \oplus M_{gp^i} \oplus M_{2gp^i} \oplus M_{\lambda gp^i} \oplus M_{\mu gp^i} \oplus M_{\nu gp^i} \oplus M_{\chi gp^i}$ *are* $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(1+x^{8p^{n-i}}+...+x^{8p^{n-i(p^i-1)}})$, $(x^{p^{n-i-1}}-1)(x^{p^{n-i}}+1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(1+x^{8p^{n-i}}+...+x^{8p^{n-i(p^i-1)}})$ *and* $(x^{2p^{n-i-1}}+1)(x^{2p^{n-i}}-1)(x^{4p^{n-i-1}}+1)(1+x^{8p^{n-i}}+...+x^{8p^{n-i(p^i-1)}})$ *respectively.*

Proof. (i) The minimal polynomial for α^0, α^{p^n}, α^{2p^n}, α^{3p^n}, α^{4p^n}, α^{5p^n}, α^{6p^n} and α^{7p^n} are $(x-1)$, $(x-\delta)$, $(x-\beta)$, $(x-\delta_1)$, $(x+1)$, $(x+\delta)$, $(x+\beta)$, and $(x+\delta_1)$ respectively. The corresponding generating polynomials are $(1+x+x^2+...+x^{8p^n-1})$, $(x^4-1)(x^2+\beta)(x+\delta)(1+x^8+...+x^{8(p^n-1)})$, $(x^6-x^4+x^2-1)(x+\beta)(1+x^8+...+x^{8(p^n-1)})$, $(x^4-1)(x^2+\beta)(x+\delta_1)(1+x^8+...+x^{8(p^n-1)})$, $(x^7-x^6+x^5-x^4+x^3-x^2+x-1)(1+x^8+...+x^{8(p^n-1)})$, $(x^4-1)(x^2+\beta)(x-\delta)(1+x^8+...+x^{8(p^n-1)})$, $(x^4-1)(x^2+\beta)(x-\beta)(1+x^8+...+x^{8(p^n-1)})$ and $(x^4-1)(x^2+\beta)(x-\delta_1)(1+x^8+...+x^{8(p^n-1)})$.

(ii) The product of minimal polynomial satisfied by α^{4p^i} and α^{4gp^i} is $\frac{x^{p^{n-i}}+1}{x^{p^{n-i-1}}+1}$. Therefore, the generating polynomial for $M_{4p^i} \oplus M_{4gp^i}$ is $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(1+x^{8p^{n-i}}+...x^{8p^{n-i(p^i-1)}})$. The product of minimal polynomial satisfied by α^{8p^i} and α^{8gp^i} is $\frac{x^{p^{n-i}}-1}{x^{p^{n-i-1}}-1}$. Therefore, the generating polynomial for $M_{8p^i} \oplus M_{8gp^i}$ is $(x^{p^{n-i-1}}-1)(x^{p^{n-i}}+1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(1+x^{8p^{n-i}}+...x^{8p^{n-i(p^i-1)}})$. Also the product of minimal polynomial satisfied by α^{p^i}, α^{2p^i}, α^{gp^i}, α^{2gp^i}, $\alpha^{\lambda p^i}$, $\alpha^{\lambda gp^i}$, $\alpha^{\mu p^i}$, $\alpha^{\mu gp^i}$, $\alpha^{\nu p^i}$, $\alpha^{\nu gp^i}$, $\alpha^{\chi p^i}$ and $\alpha^{\chi gp^i}$ is $\frac{x^{2p^{n-i}}+1}{x^{2p^{n-i-1}}+1}$. Therefore, the generating polynomial for $M_{4p^i} \oplus M_{4gp^i}$, $M_{8p^i} \oplus M_{8gp^i}$ and $M_{p^i} \oplus M_{2p^i} \oplus M_{\lambda p^i} \oplus M_{\mu p^i} \oplus M_{\nu p^i} \oplus M_{\chi p^i} \oplus M_{gp^i} \oplus M_{2gp^i} \oplus M_{\lambda gp^i} \oplus M_{\mu gp^i} \oplus M_{\nu gp^i} \oplus M_{\chi gp^i}$ is $(x^{2p^{n-i-1}}+1)(x^{2p^{n-i}}-1)(x^{4p^{n-i-1}}+1)(1+x^{8p^{n-i}}+...x^{8p^{n-i(p^i-1)}})$. □

5. MINIMUM DISTANCE

If l is a cyclic code of length m generated by $g(x)$ and its minimum distance is d, then the code $\bar{l}$ of length mk generated by $g(x)(1+x^m+x^{2m}+...+x^{(k-1)m})$ is a repetition code of l repeated k times and its minimum distance is dk[3]. Here, we find the minimum distance of the minimal cyclic code M_s of length $8p^n$, generated by the primitive idempotent P_s.

Theorem 5.1. *Each of the codes $M_0, M_{p^n}, M_{2p^n}, M_{3p^n}, M_{4p^n}, M_{5p^n}, M_{6p^n}$ and M_{7p^n} are of minimum distance $8p^n$. For $0 \leq i \leq n-1$, the minimum distance of the cyclic codes $M_{4p^i}, M_{4gp^i}, M_{8p^i}$ and M_{8gp^i} are greater than or equal $16p^i$ and minimum distance for the codes $M_{p^i}, M_{gp^i}, M_{2p^i}, M_{2gp^i}, M_{\lambda p^i}, M_{\lambda gp^i}, M_{\mu p^i}$, $M_{\mu gp^i}, M_{\nu p^i}, M_{\nu gp^i}, M_{\chi p^i}$ and $M_{\chi gp^i}$ are greater than or equal to $8p^i$.*

Proof. Since generating polynomial for the code M_0 is $(1+x+x^2+...+x^{8p^n-1})$, which is itself a polynomial of length $8p^n$, hence its minimum distance is $8p^n$. Also, the generating polynomial for the cyclic code M_{p^n} is $(x^4-1)(x^2+\beta)(x+\delta)(1+x^8+...+x^{8(p^n-1)})$. If we take a cyclic code of length 8 generated by the polynomial $(x^4-1)(x^2+\beta)(x+\delta)$, then the minimal distance of this code is 8. Since the cyclic code of length $8p^n$ with generating polynomial $(x^4-1)(x^2+\beta)(x+\delta)(1+x^8+...+x^{8(p^n-1)})$ is a repetition code of the cyclic code of length 8 with generating polynomial $(x^4-1)(x^2+\beta)(x+\delta)$, repeated p^n times. Therefore its minimum distance is $8p^n$.

Similarly, the minimum distance of each of the cyclic codes $M_{2p^n}, M_{3p^n}, M_{4p^n}, M_{5p^n}, M_{6p^n}$ and M_{7p^n} is $8p^n$. Consider the cyclic codes M_{4p^i} and M_{4gp^i}, since the generating polynomial of the cyclic code of length $8p^{n-i}$ is $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(1+x^{8p^{n-i}}+...+x^{8p^{n-i(p^i-1)}})$. Therefore, if we take a cyclic code C of length p^{n-i} generated by the polynomial $(x^{p^{n-i-1}}+1)$, then the minimal distance of this code is 2. Now consider the cyclic code C_1 of length $2p^{n-i}$ generated by the polynomial $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)$ and then minimum distance of this code is 4, as it is 2 time repetition of the code C. Further, the minimum distance of the code C_2 of length $4p^{n-i}$ generated by the polynomial $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)$ is 8, as it is 2 time repetition of the code C_1. Hence, the minimum distance of the code C_3 of length $8p^{n-i}$ generated by the polynomial $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)$ and then minimum distance of this code is 16, as it is 2 time repetition of the code C_2. Since the cyclic code of length $8p^n$ generated by the polynomial $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(1+x^{8p^{n-i}}+...x^{8p^{n-i(p^i-1)}})$ is a repetition code of the code C_3, repeated p^i times. Hence its minimum distance is $16p^i$. The codes corresponding to M_{4p^i} and M_{4gp^i} are the sub codes of the above codes, so their minimum distances are greater than or equal to $16p^i$. Similarly, the minimum distance of the cyclic code M_{8p^i} and M_{8gp^i} of length $8p^n$ with generating polynomial $(x^{p^{n-i-1}}-1)(x^{p^{n-i}}+1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(1+x^{8p^{n-i}}+...+x^{8p^{n-i(p^i-1)}})$ is also greater than or equal to $16p^i$.

Now, the product of generating polynomial for the cyclic codes $M_{p^i}, M_{gp^i}, M_{2p^i}, M_{2gp^i}, M_{\lambda p^i}, M_{\lambda gp^i}, M_{\mu p^i}$, $M_{\mu gp^i}, M_{\nu p^i}, M_{\nu gp^i}, M_{\chi p^i}$ and $M_{\chi gp^i}$ is $(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{2p^{n-i}}-1)(1+x^{8p^{n-i}}+...+x^{8p^{n-i(p^i-1)}})$, therefore, if we take a code C of length $8p^{n-i}$ generated by the polynomial $(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{2p^{n-i}}-1)$, then the minimum distance of this code is 8.Since the cyclic code C_1 of length $8p^n$ generated by the polynomial $(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{2p^{n-i}}-1)(1+x^{8p^{n-i}}+...+x^{8p^{n-i(p^j-1)}})$ is a repetition code of the code C, repeated p^i times. Hence its minimum distance is $8p^i$.

The codes corresponding to Ω_{p^i}, Ω_{gp^i},Ω_{2p^i}, Ω_{2gp^i}, $\Omega_{\lambda p^i}$, $\Omega_{\lambda gp^i}$, $\Omega_{\mu p^i}$, $\Omega_{\mu gp^i}$, $\Omega_{\nu p^i}$, $\Omega_{\nu gp^i}$, $\Omega_{\chi p^i}$ and $\Omega_{\chi gp^i}$ are the subcodes of above codes so, their minimum distances are greater than or equal to $8p^i$. □

6. EXAMPLE

Example 6.1. Cyclic Codes of length 24.

Take $p=3$, $n=1$, $q=73$. Then the q-cyclotomic cosets are

$\Omega_0=\{0\}$, $\Omega_1=\{1\}$, $\Omega_2=\{2\}$, $\Omega_3=\{3\}$, $\Omega_4=\{4\}$, $\Omega_5=\{5\}$, $\Omega_6=\{6\}$, $\Omega_7=\{7\}$, $\Omega_8=\{8\}$, $\Omega_9=\{9\}$, $\Omega_{10}=\{10\}$, $\Omega_{11}=\{11\}$, $\Omega_{12}=\{12\}$, $\Omega_{13}=\{13\}$, $\Omega_{14}=\{14\}$, $\Omega_{15}=\{15\}$, $\Omega_{16}=\{16\}$, $\Omega_{17}=\{17\}$, $\Omega_{18}=\{18\}$, $\Omega_{19}=\{19\}$, $\Omega_{20}=\{20\}$,$\Omega_{21}=\{21\}$, $\Omega_{22}=\{22\}$,$\Omega_{23}=\{23\}$,

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and the corresponding primitive idempotents in $\frac{GF(73)[x]}{<x^{24}-1>}$ are

$$P_0(x)=\frac{1}{24}[\overline{Z}_0+\overline{Z}_3+\overline{Z}_6+\overline{Z}_9+\overline{Z}_{12}+\overline{Z}_{15}+\overline{Z}_{18}+\overline{Z}_{21}+\overline{Z}_1+\overline{Z}_2+\overline{Z}_4+\overline{Z}_8+\overline{Z}_7+\overline{Z}_{14}+\overline{Z}_{13}+\overline{Z}_{19} +\overline{Z}_5+\overline{Z}_{10}+\overline{Z}_{20}+\overline{Z}_{16}+\overline{Z}_{11}+\overline{Z}_{22}+\overline{Z}_{17}+\overline{Z}_{23}]$$

$$P_1(x)=\frac{1}{24}[\overline{Z}_0+51\overline{Z}_3+27\overline{Z}_6+10\overline{Z}_9-\overline{Z}_{12}-51\overline{Z}_{15}-27\overline{Z}_{18}-10\overline{Z}_{21}-2\overline{Z}_2+9\overline{Z}_4 8\overline{Z}_8+4\overline{Z}_7+2\overline{Z}_{14}- 4\overline{Z}_{19}-13\overline{Z}_5+\overline{Z}_{10}+8\overline{Z}_{20}-9\overline{Z}_{16}-24\overline{Z}_{11}-\overline{Z}_{22}+13\overline{Z}_{17}+24\overline{Z}_{23}]$$

$$P_2(x)=\frac{1}{24}[\overline{Z}_0+27\overline{Z}_3-\overline{Z}_6-27\overline{Z}_9+\overline{Z}_{12}+27\overline{Z}_{15}-\overline{Z}_{18}-27\overline{Z}_{21}-2\overline{Z}_1+9\overline{Z}_2+8\overline{Z}_4+8\overline{Z}_8-2\overline{Z}_7+ 9\overline{Z}_{14}-2\overline{Z}_{13}+2\overline{Z}_{19}+\overline{Z}_5+8\overline{Z}_{10}-9\overline{Z}_{20}-9\overline{Z}_{16}-\overline{Z}_{11}+8\overline{Z}_{22}+\overline{Z}_{17}-\overline{Z}_{23}]$$

$$P_3(x)=\frac{1}{24}[\overline{Z}_0+10\overline{Z}_3+27\overline{Z}_6+51\overline{Z}_9-\overline{Z}_{12}-10\overline{Z}_{15}-27\overline{Z}_{18}-51\overline{Z}_{21}+10\overline{Z}_1+27\overline{Z}_2-\overline{Z}_4+\overline{Z}_8-10\overline{Z}_7 +27\overline{Z}_{14}-51\overline{Z}_{13}+10\overline{Z}_{19}-51\overline{Z}_5+27\overline{Z}_{10}-\overline{Z}_{20}+\overline{Z}_{16}+10\overline{Z}_{11}-27\overline{Z}_{22}+51\overline{Z}_{17}-10\overline{Z}_{23}]$$

$$P_4(x)=\frac{1}{24}[\overline{Z}_0-\overline{Z}_3+\overline{Z}_6-\overline{Z}_9+\overline{Z}_{12}-\overline{Z}_{15}+\overline{Z}_{18}-\overline{Z}_{21}+9\overline{Z}_1+8\overline{Z}_2+8\overline{Z}_4+8\overline{Z}_8+9\overline{Z}_7+8\overline{Z}_{14}+9\overline{Z}_{13} +9\overline{Z}_{19}+8\overline{Z}_5-9\overline{Z}_{10}-9\overline{Z}_{20}-9\overline{Z}_{16}+8\overline{Z}_{11}-9\overline{Z}_{22}+8\overline{Z}_{17}+8\overline{Z}_{23}]$$

$$P_5(x)=\frac{1}{24}[\overline{Z}_0-51\overline{Z}_3-27\overline{Z}_6-10\overline{Z}_9-\overline{Z}_{12}+51\overline{Z}_{15}+27\overline{Z}_{18}+10\overline{Z}_{21}-13\overline{Z}_1+\overline{Z}_2+8\overline{Z}_4-9\overline{Z}_8-24\overline{Z}_7 -\overline{Z}_{14}+13\overline{Z}_{13}+24\overline{Z}_{19}-2\overline{Z}_{10}+9\overline{Z}_{20}+8\overline{Z}_{16}+4\overline{Z}_{11}+2\overline{Z}_{22}-4\overline{Z}_{23}]$$

$$P_6(x)=\frac{1}{24}[\overline{Z}_0+27\overline{Z}_3-\overline{Z}_6-27\overline{Z}_9+\overline{Z}_{12}+27\overline{Z}_{15}-\overline{Z}_{18}-27\overline{Z}_{21}-27\overline{Z}_1-\overline{Z}_2+\overline{Z}_4+\overline{Z}_8+27\overline{Z}_7-\overline{Z}_{14} -27\overline{Z}_{13}+27\overline{Z}_{19}+27\overline{Z}_5-\overline{Z}_{10}+\overline{Z}_{20}+\overline{Z}_{16}-27\overline{Z}_{11}-\overline{Z}_{22}+27\overline{Z}_{17}-27\overline{Z}_{23}]$$

$$P_7(x)=\frac{1}{24}[\overline{Z}_0-10\overline{Z}_3-27\overline{Z}_6-51\overline{Z}_9-\overline{Z}_{12}+10\overline{Z}_{15}+27\overline{Z}_{18}+51\overline{Z}_{21}+4\overline{Z}_1+2\overline{Z}_2+9\overline{Z}_4+8\overline{Z}_8-2\overline{Z}_{14} -4\overline{Z}_{13}-24\overline{Z}_5-\overline{Z}_{10}+8\overline{Z}_{20}-9\overline{Z}_{16}-13\overline{Z}_{11}+\overline{Z}_{22}+24\overline{Z}_{17}+13\overline{Z}_{23}]$$

$$P_8(x)=\frac{1}{24}[\overline{Z}_0+\overline{Z}_3+\overline{Z}_6+\overline{Z}_9+\overline{Z}_{12}+\overline{Z}_{15}+\overline{Z}_{18}+\overline{Z}_{21}+8\overline{Z}_1+8\overline{Z}_2+8\overline{Z}_4+8\overline{Z}_8+8\overline{Z}_7+8\overline{Z}_{14}+8\overline{Z}_{13} +8\overline{Z}_{19}-9\overline{Z}_5-9\overline{Z}_{10}-9\overline{Z}_{20}-9\overline{Z}_{16}-9\overline{Z}_{11}-9\overline{Z}_{22}-9\overline{Z}_{17}-9\overline{Z}_{23}]$$

$$P_9(x)=\frac{1}{24}[\overline{Z}_0+51\overline{Z}_3-27\overline{Z}_6-10\overline{Z}_9-\overline{Z}_{12}-51\overline{Z}_{15}-27\overline{Z}_{18}-10\overline{Z}_{21}+10\overline{Z}_1+27\overline{Z}_2-\overline{Z}_4+\overline{Z}_8-51\overline{Z}_7 -27\overline{Z}_{14}-10\overline{Z}_{13}+51\overline{Z}_{19}-10\overline{Z}_5-27\overline{Z}_{10}-\overline{Z}_{20}+\overline{Z}_{16}+51\overline{Z}_{11}+27\overline{Z}_{22}+10\overline{Z}_{17}-51\overline{Z}_{23}]$$

$$P_{10}(x)=\frac{1}{24}[\overline{Z}_0-27\overline{Z}_3-\overline{Z}_6+27\overline{Z}_9+\overline{Z}_{12}-27\overline{Z}_{15}-\overline{Z}_{18}+27\overline{Z}_{21}+\overline{Z}_1+8\overline{Z}_2-9\overline{Z}_4-9\overline{Z}_8-\overline{Z}_7+8\overline{Z}_{14} +\overline{Z}_{13}-\overline{Z}_{19}-2\overline{Z}_5+9\overline{Z}_{10}+8\overline{Z}_{20}+8\overline{Z}_{16}+2\overline{Z}_{11}+9\overline{Z}_{22}-2\overline{Z}_{17}+2\overline{Z}_{23}]$$

$$P_{11}(x)=\frac{1}{24}[\overline{Z}_0+10\overline{Z}_3+27\overline{Z}_6+51\overline{Z}_9-\overline{Z}_{12}-10\overline{Z}_{15}-27\overline{Z}_{18}-51\overline{Z}_{21}-24\overline{Z}_1-\overline{Z}_2+8\overline{Z}_4-9\overline{Z}_8-13\overline{Z}_7 +\overline{Z}_{14}+24\overline{Z}_{13}+13\overline{Z}_{19}-4\overline{Z}_5+2\overline{Z}_{10}+9\overline{Z}_{20}+8\overline{Z}_{16}-2\overline{Z}_{22}-4\overline{Z}_{17}]$$

$$P_{12}(x)=\frac{1}{24}[\overline{Z}_0-\overline{Z}_3+\overline{Z}_6-\overline{Z}_9+\overline{Z}_{12}-\overline{Z}_{15}+\overline{Z}_{18}-\overline{Z}_{21}-\overline{Z}_1+\overline{Z}_2+\overline{Z}_4+\overline{Z}_8-\overline{Z}_7+\overline{Z}_{14}-\overline{Z}_{13}-\overline{Z}_{19} -\overline{Z}_5+\overline{Z}_{10}+\overline{Z}_{20}+\overline{Z}_{16}-\overline{Z}_{11}+\overline{Z}_{22}-\overline{Z}_{17}-\overline{Z}_{23}]$$

$$P_{13}(x)=\frac{1}{24}[\overline{Z}_0-51\overline{Z}_3+27\overline{Z}_6-10\overline{Z}_9-\overline{Z}_{12}+51\overline{Z}_{15}-27\overline{Z}_{18}+10\overline{Z}_{21}-2\overline{Z}_2+9\overline{Z}_4+8\overline{Z}_8-4\overline{Z}_7+2\overline{Z}_{14} +4\overline{Z}_{19}+13\overline{Z}_5+\overline{Z}_{10}+8\overline{Z}_{20}-9\overline{Z}_{16}+24\overline{Z}_{11}-\overline{Z}_{22}-13\overline{Z}_{17}-24\overline{Z}_{23}]$$

$$P_{14}(x)=\frac{1}{24}[\overline{Z}_0-27\overline{Z}_3-\overline{Z}_6+27\overline{Z}_9+\overline{Z}_{12}-27\overline{Z}_{15}-\overline{Z}_{18}+27\overline{Z}_{21}+2\overline{Z}_1+9\overline{Z}_2+8\overline{Z}_4+8\overline{Z}_8-2\overline{Z}_7 +9\overline{Z}_{14}+2\overline{Z}_{13}-2\overline{Z}_{19}-\overline{Z}_5+8\overline{Z}_{10}-9\overline{Z}_{20}-9\overline{Z}_{16}+\overline{Z}_{11}+8\overline{Z}_{22}-\overline{Z}_{17}+\overline{Z}_{23}]$$

$$P_{15}(x)=\frac{1}{24}[\overline{Z}_0-10\overline{Z}_3-27\overline{Z}_6-51\overline{Z}_9-\overline{Z}_{12}+10\overline{Z}_{15}-27\overline{Z}_{18}+51\overline{Z}_{21}-51\overline{Z}_1-27\overline{Z}_2-\overline{Z}_4+\overline{Z}_8+10\overline{Z}_7 +27\overline{Z}_{14}+51\overline{Z}_{13}-10\overline{Z}_{19}+51\overline{Z}_5+27\overline{Z}_{10}-\overline{Z}_{20}+\overline{Z}_{16}-10\overline{Z}_{11}-27\overline{Z}_{22}-51\overline{Z}_{17}+10\overline{Z}_{23}]$$

$$P_{16}(x)=\frac{1}{24}[\overline{Z}_0+\overline{Z}_3+\overline{Z}_6+\overline{Z}_9+\overline{C}_{12}+\overline{C}_{15}+\overline{C}_{18}+\overline{C}_{21}-9\overline{Z}_1-9\overline{Z}_2-9\overline{Z}_4-9\overline{Z}_8-9\overline{Z}_7-9\overline{Z}_{14} -9\overline{Z}_{13}-9\overline{Z}_{19}+8\overline{Z}_5+8\overline{Z}_{10}+8\overline{Z}_{20}+8\overline{Z}_{16}+8\overline{Z}_{11}+8\overline{Z}_{22}+8\overline{Z}_{17}+8\overline{Z}_{23}]$$

$$P_{17}(x)=\frac{1}{24}[\overline{Z}_0+51\overline{Z}_3-27\overline{Z}_6+10\overline{Z}_9-\overline{Z}_{12}-51\overline{Z}_{15}+27\overline{Z}_{18}-10\overline{Z}_{21}+13\overline{Z}_1+\overline{Z}_2+8\overline{Z}_4-9\overline{Z}_8+24\overline{Z}_7 -\overline{Z}_{14}-13\overline{Z}_{13}-24\overline{Z}_{19}-2\overline{Z}_{10}+9\overline{Z}_{20}+8\overline{Z}_{16}-4\overline{Z}_{11}+2\overline{Z}_{22}+4\overline{Z}_{23}]$$

$$P_{18}(x)=\frac{1}{24}[\overline{Z}_0-27\overline{Z}_3-\overline{Z}_6+27\overline{Z}_9+\overline{Z}_{12}-27\overline{Z}_{15}-\overline{Z}_{18}-27\overline{Z}_{21}+27\overline{Z}_1-\overline{Z}_2+\overline{Z}_4+\overline{Z}_8-27\overline{Z}_7-\overline{Z}_{14} +27\overline{Z}_{13}-27\overline{Z}_{19}-27\overline{Z}_5-\overline{Z}_{10}+\overline{Z}_{20}+\overline{Z}_{16}+27\overline{Z}_{11}-\overline{Z}_{22}-27\overline{Z}_{17}+27\overline{Z}_{23}]$$

$$P_{19}(x)=\frac{1}{24}[\overline{Z}_0+10\overline{Z}_3-27\overline{Z}_6+51\overline{Z}_9-\overline{Z}_{12}-10\overline{Z}_{15}+27\overline{Z}_{18}-51\overline{Z}_{21}-4\overline{Z}_1+2\overline{Z}_2+9\overline{Z}_4+8\overline{Z}_8-2\overline{Z}_{14} +4\overline{Z}_{13}+24\overline{Z}_5-\overline{Z}_{10}+8\overline{Z}_{20}-9\overline{Z}_{16}+13\overline{Z}_{11}+\overline{Z}_{22}-24\overline{Z}_{17}-13\overline{Z}_{23}]$$

$$P_{20}(x)=\frac{1}{24}[\overline{Z}_0-\overline{Z}_3+\overline{Z}_6-\overline{Z}_9+\overline{Z}_{12}-\overline{Z}_{15}+\overline{Z}_{18}-\overline{Z}_{21}+8\overline{Z}_1-9\overline{Z}_2-9\overline{Z}_4-9\overline{Z}_8+8\overline{Z}_7-9\overline{Z}_{14}+8\overline{Z}_{13} +8\overline{Z}_{19}+9\overline{Z}_5+8\overline{Z}_{10}+8\overline{Z}_{20}+8\overline{Z}_{16}+9\overline{Z}_{11}+8\overline{Z}_{22}+9\overline{Z}_{17}+9\overline{Z}_{23}]$$

$$P_{21}(x)=\frac{1}{24}[\overline{Z}_0-51\overline{Z}_3-27\overline{Z}_6-10\overline{Z}_9-\overline{Z}_{12}+51\overline{Z}_{15}+27\overline{Z}_{18}+10\overline{Z}_{21}-10\overline{Z}_1+27\overline{Z}_2-\overline{Z}_4+\overline{Z}_8+51\overline{Z}_7 -27\overline{Z}_{14}+10\overline{Z}_{13}-51\overline{Z}_{19}-10\overline{Z}_5-27\overline{Z}_{10}-\overline{Z}_{20}+\overline{Z}_{16}-51\overline{Z}_{11}+27\overline{Z}_{22}-10\overline{Z}_{17}+51\overline{Z}_{23}]$$

$$P_{22}(x)=\frac{1}{24}[\overline{Z}_0+27\overline{Z}_3-\overline{Z}_6-27\overline{Z}_9+\overline{Z}_{12}+27\overline{Z}_{15}-\overline{Z}_{18}-27\overline{Z}_{21}-\overline{Z}_1+8\overline{Z}_2-9\overline{Z}_4-9\overline{Z}_8+\overline{Z}_7+8\overline{Z}_{14} -\overline{Z}_{13}+\overline{Z}_{19}+2\overline{Z}_5+9\overline{Z}_{10}+8\overline{Z}_{20}+8\overline{Z}_{16}-2\overline{Z}_{11}+9\overline{Z}_{22}+2\overline{Z}_{17}-2\overline{Z}_{23}]$$

$P_{23}(x) = \frac{1}{24}[\overline{Z}_0 - 10\overline{Z}_3 + 27\overline{Z}_6 - 51\overline{Z}_9 - \overline{Z}_{12} - 10\overline{Z}_{15} - 27\overline{Z}_{18} + 51\overline{Z}_{21} + 24\overline{Z}_1 - \overline{Z}_2 + 8\overline{Z}_4 - 9\overline{Z}_8 + 13\overline{Z}_7$
$+\overline{Z}_{14} - 24\overline{Z}_{13} - 13\overline{Z}_{19} - 4\overline{Z}_5 + 2\overline{Z}_{10} + 9\overline{Z}_{20} + 8\overline{Z}_{16} - 2\overline{Z}_{22} + 4\overline{Z}_{17}]$

Minimal polynomials for $\alpha^0, \alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7, \alpha^8, \alpha^9, \alpha^{10}, \alpha^{11}, \alpha^{12}, \alpha^{13}, \alpha^{14}, \alpha^{15}, \alpha^{16}, \alpha^{17}, \alpha^{18}, \alpha^{19}$, $\alpha^{20}, \alpha^{21}, \alpha^{22}$ and α^{23} are $x-1$, $x-30$, $x-24$, $x-63$, $x-65$, $x-52$, $x-27$, $x-7$, $x-64$, $x-22$, $x-3$, $x-17$, $x+1$, $x-43$, $x-49$, $x-10$, $x-8$, $x-21$, $x-46$, $x-66$, $x-9$, $x-51$, $x-70$ and $x-56$ respectively.

The minimal codes M_0, M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9, M_{10}, M_{11}, M_{12}, M_{13}, M_{14}, M_{15}, M_{16}, M_{17}, M_{18}, M_{19}, M_{20}, M_{21}, M_{22} and M_{23} of length 24 are as follows:

Code	Dim.	Min. Distance Bound	Generating Polynomial
M_0	1	24	$1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}+x^{12}+x^{13}+x^{14}+x^{15}+x^{16}+x^{17}+x^{18}+x^{19}+x^{20}+x^{21}+x^{22}+x^{23}$
M_1	1	$8 \leq d \leq 24$	$56+70x+51x^2+9x^3+66x^4+46x^5+21x^6+8x^7+10x^8+49x^9+43x^{10}+72x^{11}+17x^{12}+3x^{13}+22x^{14}+64x^{15}+7x^{16}+27x^{17}+52x^{18}+65x^{19}+63x^{20}+24x^{21}+30x^{22}+x^{23}$
M_2	1	$8 \leq d \leq 24$	$70+9x+46x^2+8x^3+49x^4+72x^5+3x^6+64x^7+27x^8+65x^9+24x^{10}+x^{11}+70x^{12}+9x^{13}+46x^{14}+8x^{15}+49x^{16}+72x^{17}+3x^{18}+64x^{19}+27x^{20}+65x^{21}+24x^{22}+x^{23}$
M_3	1	24	$51+46x+10x^2+72x^3+22x^4+27x^5+63x^6+x^7+51x^8+46x^9+10x^{10}+72x^{11}+22x^{12}+27x^{13}+63x^{14}+x^{15}+51x^{16}+46x^{17}+10x^{18}+72x^{19}+22x^{20}+27x^{21}+63x^{22}+x^{23}$
M_4	1	$16 \leq d \leq 24$	$9+8x+72x^2+64x^3+65x^4+x^5+9x^6+8x^7+72x^8+64x^9+65x^{10}+x^{11}+9x^{12}+8x^{13}+72x^{14}+64x^{15}+65x^{16}+x^{17}+9x^{18}+8x^{19}+72x^{20}+64x^{21}+65x^{22}+x^{23}$
M_5	1	$8 \leq d \leq 24$	$66+49x+22x^2+65x^3+56x^4+46x^5+43x^6+64x^7+63x^8+70x^9+21x^{10}+72x^{11}+7x^{12}+24x^{13}+51x^{14}+8x^{15}+17x^{16}+27x^{17}+30x^{18}+9x^{19}+10x^{20}+3x^{21}+52x^{22}+x^{23}$
M_6	1	24	$46+72x+27x^2+x^3+46x^4+72x^5+27x^6+x^7+46x^8+72x^9+27x^{10}+x^{11}+46x^{12}+72x^{13}+27x^{14}+x^{15}+46x^{16}+72x^{17}+27x^{18}+x^{19}+46x^{20}+72x^{21}+27x^{22}+x^{23}$
M_7	1	$8 \leq d \leq 24$	$21+3x+63x^2+9x^3+43x^4+27x^5+56x^6+8x^7+22x^8+24x^9+66x^{10}+72x^{11}+52x^{12}+70x^{13}+10x^{14}+64x^{15}+30x^{16}+46x^{17}+17x^{18}+65x^{19}+51x^{20}+49x^{21}+7x^{22}+x^{23}$
M_8	1	$16 \leq d \leq 24$	$8+64x+x^2+8x^3+64x^4+x^5+8x^6+64x^7+x^8+8x^9+64x^{10}+x^{11}+8x^{12}+64x^{13}+x^{14}+8x^{15}+64x^{16}+x^{17}+8x^{18}+64x^{19}+x^{20}+8x^{21}+64x^{22}+x^{23}$

Code	Dim.	Min. Distance Bound	Generating Polynomial
M_9	1	24	$10+27x+51x^2+72x^3+63x^4+46x^5+22x^6+x^7+10x^8+27x^9+51x^{10}+72x^{11}+63x^{12}+46x^{13}+22x^{14}+x^{15}+10x^{16}+27x^{17}+51x^{18}+72x^{19}+63x^{20}+46x^{21}+22x^{22}+x^{23}$
M_{10}	1	$8 \leq d \leq 24$	$49+65x+46x^2+64x^3+70x^4+72x^5+24x^6+8x^7+27x^8+9x^9+3x^{10}+x^{11}+49x^{12}+65x^{13}+46x^{14}+64x^{15}+70x^{16}+72x^{17}+24x^{18}+8x^{19}+27x^{20}+9x^{21}+3x^{22}+x^{23}$
M_{11}	1	$8 \leq d \leq 24$	$43+24x+10x^2+65x^3+21x^4+27x^5+66x^6+64x^7+51x^8+3x^9+56x^{10}+72x^{11}+30x^{12}+49x^{13}+63x^{14}+8x^{15}+52x^{16}+46x^{17}+7x^{18}+9x^{19}+22x^{20}+70x^{21}+17x^{22}+x^{23}$
M_{12}	1	24	$-1+x-x^2+x^3-x^4+x^5-x^6+x^7-x^8+x^9-x^{10}+x^{11}-x^{12}+x^{13}-x^{14}+x^{15}-x^{16}+x^{17}-x^{18}+x^{19}-x^{20}+x^{21}-x^{22}+x^{23}$
M_{13}	1	$8 \leq d \leq 24$	$17+70x+22x^2+9x^3+7x^4+46x^5+52x^6+8x^7+63x^8+49x^9+30x^{10}+72x^{11}+56x^{12}+3x^{13}+51x^{14}+64x^{15}+66x^{16}+27x^{17}+21x^{18}+65x^{19}+10x^{20}+24x^{21}+43x^{22}+x^{23}$
M_{14}	1	$8 \leq d \leq 24$	$3+9x+27x^2+8x^3+24x^4+72x^5+70x^6+64x^7+46x^8+65x^9+49x^{10}+x^{11}+3x^{12}+9x^{13}+27x^{14}+8x^{15}+24x^{16}+72x^{17}+70x^{18}+64x^{19}+46x^{20}+65x^{21}+49x^{22}+x^{23}$
M_{15}	1	24	$22+46x+63x^2+72x^3+51x^4+27x^5+10x^6+x^7+22x^8+46x^9+63x^{10}+72x^{11}+51x^{12}+27x^{13}+10x^{14}+x^{15}+22x^{16}+46x^{17}+63x^{18}+72x^{19}+51x^{20}+27x^{21}+10x^{22}+x^{23}$
M_{16}	1	$16 \leq d \leq 24$	$64+8x+x^2+64x^3+8x^4+x^5+64x^6+8x^7+x^8+64x^9+8x^{10}+x^{11}+64x^{12}+8x^{13}+x^{14}+64x^{15}+8x^{16}+x^{17}+64x^{18}+8x^{19}+x^{20}+64x^{21}+8x^{22}+x^{23}$
M_{17}	1	$8 \leq d \leq 24$	$7+49x+51x^2+65x^3+17x^4+46x^5+30x^6+64x^7+10x^8+70x^9+52x^{10}+72x^{11}+66x^{12}+24x^{13}+22x^{14}+8x^{15}+56x^{16}+27x^{17}+43x^{18}+9x^{19}+63x^{20}+3x^{21}+21x^{22}+x^{23}$
M_{18}	1	24	$27+72x+46x^2+x^3+27x^4+72x^5+46x^6+x^7+27x^8+72x^9+46x^{10}+x^{11}+27x^{12}+72x^{13}+46x^{14}+x^{15}+27x^{16}+72x^{17}+46x^{18}+x^{19}+27x^{20}+72x^{21}+46x^{22}+x^{23}$
M_{19}	1	$8 \leq d \leq 24$	$52+3x+10x^2+9x^3+30x^4+27x^5+17x^6+8x^7+51x^8+24x^9+7x^{10}+72x^{11}+21x^{12}+70x^{13}+63x^{14}+64x^{15}+43x^{16}+46x^{17}+56x^{18}+65x^{19}+22x^{20}+49x^{21}+66x^{22}+x^{23}$
M_{20}	1	$16 \leq d \leq 24$	$65+64x+72x^2+8x^3+9x^4+x^5+65x^6+64x^7+72x^8+8x^9+9x^{10}+x^{11}+65x^{12}+64x^{13}+72x^{14}+8x^{15}+9x^{16}+x^{17}+65x^{18}+64x^{19}+72x^{20}+8x^{21}+9x^{22}+x^{23}$
M_{21}	1	24	$63+27x+22x^2+72x^3+10x^4+46x^5+51x^6+x^7+63x^8+27x^9+22x^{10}+72x^{11}+10x^{12}+46x^{13}+51x^{14}+x^{15}+63x^{16}+27x^{17}+22x^{18}+72x^{19}+10x^{20}+46x^{21}+51x^{22}+x^{23}$
M_{22}	1	$8 \leq d \leq 24$	$24+65x+27x^2+64x^3+3x^4+72x^5+49x^6+8x^7+46x^8+9x^9+70x^{10}+x^{11}+24x^{12}+65x^{13}+27x^{14}+64x^{15}+3x^{16}+72x^{17}+49x^{18}+8x^{19}+46x^{20}+9x^{21}+70x^{22}+x^{23}$
M_{23}	1	$8 \leq d \leq 24$	$30+24x+63x^2+65x^3+52x^4+27x^5+7x^6+64x^7+22x^8+3x^9+17x^{10}+72x^{11}+43x^{12}+49x^{13}+10x^{14}+8x^{15}+21x^{16}+46x^{17}+66x^{18}+9x^{19}+51x^{20}+70x^{21}+56x^{22}+x^{23}$

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